The Phenomena of Mechanical Resonance, or "Think About Differential Equations the Next Time You Fly"

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Abstract

The purpose of this project is to develop a sense of mechanical resonance, given a simplified model of an aircraft wing. This Interdisciplinary Lively Applications Project was first used in "Introduction to Differential Equations," a freshman mathematics course in Spring 1994, United States Military Academy, developed by the Departments of Mathematical Sciences and Mechanical Engineering. It also appeared as a chapter in [1]. I have modified it to incorporate additional phenomena, such as beats.

Introduction

In this project we will study the phenomena of mechanical resonance in the context of a vibrating propeller on an airplane wing. Any structure or mechanical system is susceptible to damage by the forces of resonance. To set the stage, read the following excerpt from *Differential Equations with Applications*:

> *If you have ever looked out a window while in flight, you have probably observed that wings on an airplane are not perfectly rigid. A reasonable amount of flex or flutter is not only tolerated but necessary to prevent the wing from snapping like a piece of peppermint stick candy. In late 1959 and early 1960 two commercial plane crashes involving a relatively new model of propjet occurred, illustrating the destructive effects of large mechanical oscillations.... After a massive technical investigation, the problem was eventually traced in each case to an outboard engine and engine housing. Roughly, it was determined that when each plane surpassed a critical speed of approximately 400 mph, a propeller and engine began to wobble, causing a gyroscopic force, which could not be quelled or damped by the engine housing. This* external vibrational force *was then transferred to the already oscillating wing. This, in itself, need not have been destructively dangerous since aircraft wings are designed to withstand the stress of unusual and excessive forces.... But unfortunately, after a short period of time during which the engine wobbled rapidly,* the frequency of the impressed force actually slowed *to a point at which it approached and finally coincided with the maximum frequency of wing flutter. The amplitudes of wing flutter became large enough to snap the wing.*

Designers of mechanical systems must understand that such systems could be susceptible to mechanical resonance. The following excerpt from *Applied Differential Equations* presents a clear and interesting discussion of resonance:

> *When the frequency of a periodic external force applied to a mechanical system is related in a simple way to the natural frequency of the system, mechanical resonance may occur which builds up the oscillations to such tremendous magnitudes the system may fall apart. A company of soldiers marching in step across a bridge may in this manner cause the bridge to collapse even though the bridge would have been strong enough to carry many more soldiers had they marched out of step. For this reason soldiers [are] required to "break step" [when] crossing a bridge. In an analogous manner, it may be possible for a musical note of proper characteristic frequency to shatter a glass. Because of the great damages which may thus occur, mechanical resonance is in general something which needs to be avoided, especially by the engineer in designing structures or vibrating systems.*

Using this information, we examine how the forces developed by resonance can affect the flight of an airplane. Finally, an excerpt from *Differential Equations: A Modeling Perspective*:

> *There is an interesting interpretation when* $|\omega_0 - \omega|$ *is small compared to* $\omega_0 + \omega$*, the phenomenon of beats. The system's response is a* beat*, that is, a sinusoid of circular frequency* $(\omega_0 + \omega)/2$*, but with a slowly varying amplitude that is itself a sinusoid with the low circular* beat frequency $|\omega_0 - \omega|/2$*. One can hear this in the alternating swelling and fading sounds generated by a pair of tuning forks. Measuring time in seconds, say that one fork is tuned to middle C (258 Hz) and the other is slightly off pitch (260 Hz). Then the beat frequency is (260 - 258)/2 = 1 Hz, while the pitch you hear is (260 + 258)/2 = 259 Hz.*

Modeling the Wing's Motion

Lets use differential equations to construct a simple model of the wing flutter. Start with a picture of the propeller mounted on a wing (see Figure 1). The center-of-mass of an object is a point that can represent the entire mass of an object. We can build a simple model if we restrict our investigation to the fluttering motion that takes place at the wing's center-of-mass. By imposing this restriction we can then think of the wing as a spring-mass system. The spring is the wing-body joint that allows the wing's center-of-mass to move up and

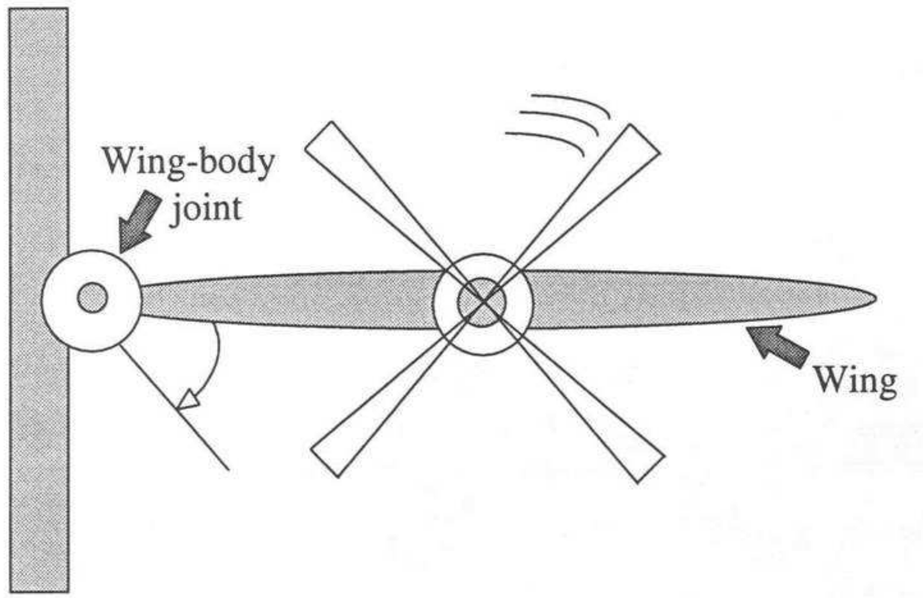

Figure 1: A simplified aircraft wing (from [1])

down in a fluttering motion. The forcing function (or driving function, as it is sometimes called) is the external vibrational force that comes from the motion of the propeller. Any motion at the wing's center-of-mass will be magnified out on the tip of the wing. Why?

For the sake of solving this simple model, we assume that the wing has a mass of 900 kg and that the wing-body joint will act like a spring with a spring constant of 8100 newtons/meter. The motion of the wing's center-of-mass is actually a curved arc; however, since we are working with relatively long wing spans, we can simplify the situation by assuming that the center-of-mass moves up and down in a straight line. Further assume that before the propeller begins to vibrate, the wing is at rest. Also, assume that any forces which tend to damp the motion of the wing are negligible.

The Project

Answering the following questions should help your understanding of mechanical resonance.

1. Starting at time $t = 0$, assume the propeller begins to vibrate with a force equal to $f_1(t) = 1800\ \sin(6t)$, measured in newtons.
a. Find the equation of motion for the wing's center-of-mass and plot it (print

this plot and include it in your project submission).
b. Describe the motion of the center-of-mass as t (time) grows large ($0 \leq t \leq 25$).

2. Just before wing failure, the frequency of the external force caused by the wobbling propeller actually slowed down. Let's simulate that slowing down by changing our forcing function $f(t)$ to $f_2(t) = 1800\ \sin(3.5t)$ newtons.
a. Plot the two forcing functions. What is the principle difference in the two functions?
b. Find the equation of motion using the new forcing function. Plot the motion. Describe what happens as t grows large. What consequences does this have for the wing?
c. Suppose that the new forcing function is $f_3(t) = 1800\ \sin(3t)$. Solve and plot the solution which models the motion. What behavior do you notice?

3. Think critically about the problems you just worked. What forces were at work on the real plane wing that we didn't include in our model? (Hint: Look at your differential equation. What type of motion did we model? For example, was it forced or free motion, damped or undamped motion?) What other factors or issues might an engineer want to analyze in his investigation of wing flutter? Can you somehow model "flex" in the wing?

Your report for this project is not just a list of answers to these questions - you must explain your analysis and support your conclusions, referring to your plots. The format for this report submission is described in your Course Study Guide. See your instructor if you have questions.

References

[1] D. C. Arney, Editor. 1997. *Interdisciplinary Lively Applications Projects (ILAPs)*. Washington, D.C.: The Mathematical Association of America.

[2] R. Borelli and C. Coleman. 2004. *Differential Equations: A Modeling Perspective*. 2d Edition. New York: John Wiley & Sons, Inc.

[3] M. R. Spiegel. 1981. *Applied Differential Equations*. Engelwood Cliffs, NJ: Prentice Hall.

[4] D. G. Zill. 1989. *A First Course in Differential Equations With Applications*. Boston: PWS-Kent.

Solutions

Requirement 1

Starting at time $t = 0$, assume the propeller begins to vibrate with a force equal to $f_1(t) = 1800\ \sin(6t)$, measured in newtons.
a. Find the equation of motion for the wing's center-of-mass and plot it (print this plot and include it in your project submission).
b. Describe the motion of the center-of-mass as t (time) grows large ($0 \le t \le 25$).

Solution:

We model the motion of the aircraft wing as a harmonic oscillator. Define $y(t)$ as the position of the wing's center of mass at time t. The differential equation of motion then becomes

$$900\, y''(t) + 0\, y'(t) + 8100\, y(t) = 1800\ \sin(6t),$$

since the motion is undamped. This simplifies to

$$y''(t) + 9\, y(t) = 2\ \sin(6t). \tag{1}$$

The initial conditions are $y(0) = 0$ and $y'(0) = 0$. We solve this in two steps. First, solving the associated homogeneous ODE, the characteristic equation is

$$r^2 + 9 = 0,$$

which yields $r = \pm 3i$. This gives a homogeneous general solution of

$$y_h(t) = c_1\ \cos(3t) + c_2\ \sin(3t).$$

Next, we solve the associated non-homogeneous ODE. Since the forcing function contains $\sin(6t)$, we conjecture a solution of the same form; i.e.,

$$y_n(t) = A\ \cos(6t) + B\ \sin(6t),$$

where A and B are constants. Differentiate $y_n(t)$ twice and substitute $y_n(t)$ and $y_n''(t)$ into the Equation 1, finding

$$-36A\ \cos(6t) - 36\ \sin(6t) + 9\,[A\ \cos(6t) + B\ \sin(6t)] = 2\ \sin(6t).$$

Equating coefficients for $\cos(6t)$ and $\sin(6t)$ gives $A = 0$ and $B = -\frac{2}{27}$. The general solution for the ODE is now

$$y(t) = y_h(t) + y_n(t) = c_1\ \cos(3t) + c_2\ \sin(3t) - \frac{2}{27}\ \sin(6t).$$

Applying the initial conditions of $y(0) = 0$ and $y'(0) = 0$ (since the wing is initially at rest), we determine that $c_1 = 0$ and $c_2 = \frac{4}{27}$, providing a particular solution of

$$y(t) = \frac{4}{27}\ \sin(3t) - \frac{2}{27}\ \sin(6t). \tag{2}$$

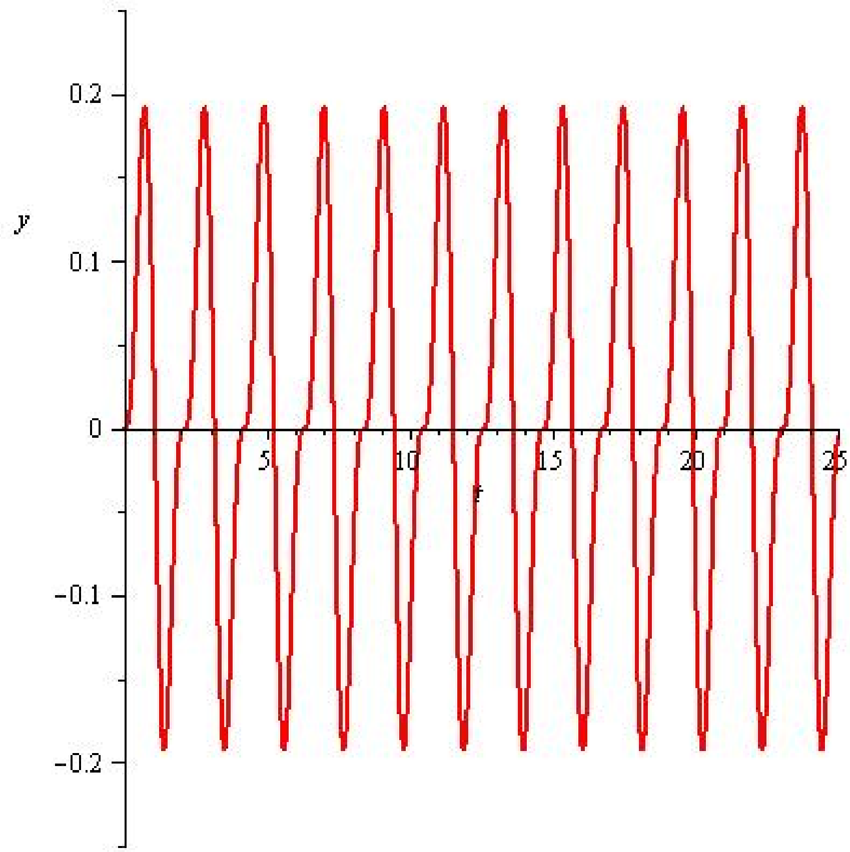

Figure 2: The aircraft wing's harmonic motion for $0 \leq t \leq 25$

A plot of the wing's motion (the solution from Equation 2 is shown in Figure 2. As t increases, the solution periodically oscillates, never reaching an amplitude greater than 0.2 meters.

Requirement 2

Just before wing failure, the frequency of the external force caused by the wobbling propeller actually slowed down. Let's simulate that slowing down by changing our forcing function $f(t)$ to $f_2(t) = 1800\ \sin(3.5t)$ newtons.
a. Plot the two forcing functions. What is the principle difference in the two functions?
b. Find the equation of motion using the new forcing function. Plot the motion. Describe what happens as t grows large. What consequences does this have for

the wing?
c. Suppose that the new forcing function is $f_3(t) = 1800\ \sin(3t)$. Solve and plot the solution which models the motion. What behavior do you notice?

Solution:

The two forcing functions are plotted in Figure 3. We can readily see the $f_2(t)$ has a longer period than $f_1(t)$. Both forcing functions have the same amplitude.

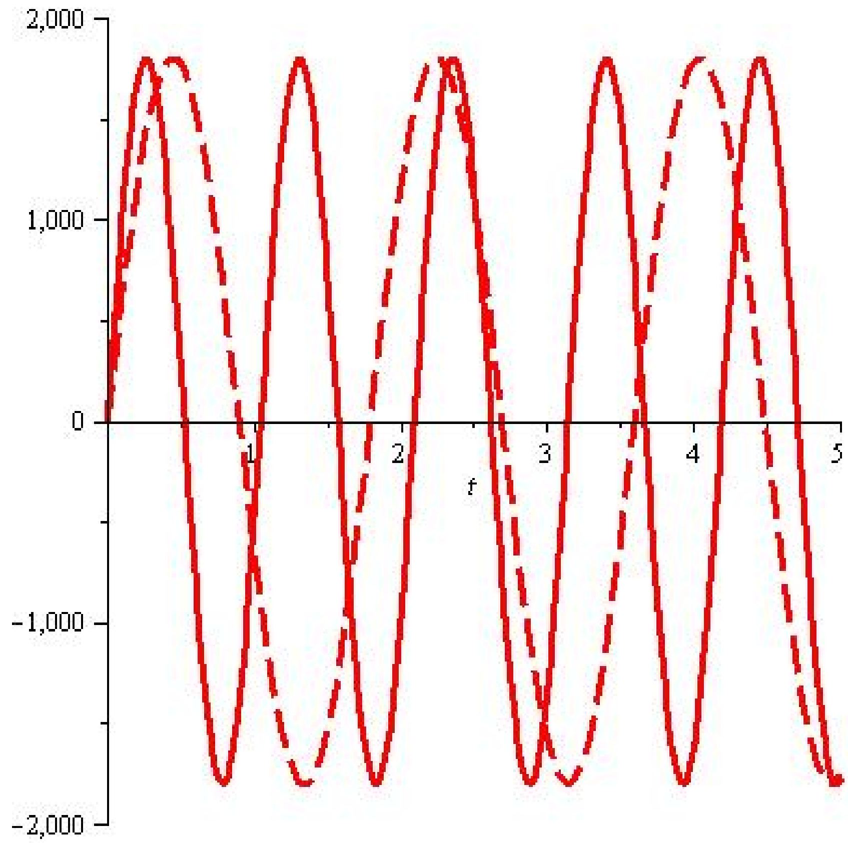

Figure 3: Two forcing functions (the dashed line is $f_2(t) = 1800\ \sin(3.5t)$)

In obtaining a solution for requirement 2b, the left-hand side of Equation 1 remains unchanged, but the forcing function is now $1800\ \sin(3.5t)$. The differential equation of motion then becomes

$$y''(t) + 9\,y(t) = 2\ \sin(3.5t). \tag{3}$$

The solution methodology is the same as with Requirement 1, with only the non-homogeneous solution requiring adjustment. After following the methodolgy, we find that the particular solution is

$$y(t) = \frac{28}{39}\ \sin(3t) - \frac{8}{13}\ \sin(3.5t). \tag{4}$$

A plot of this motion (from Equation 4) is shown in Figure 4. Notice that the amplitude is greater for $f_2(t)$ than for $f_1(t)$, but there appears to be a periodic growth and decay of the rhythmic amplitude. As t increases, $y(t)$ tends to grow and then decrease while continuing to oscillate. This phenomenon is known as a "beat." The amplitude could potentially repeatedly reach a point where failure occurs (see the plot), depending on the properties of the wing material. We do not have sufficient information to predict the maximum allowable amplitude of the wing before failure occurs.

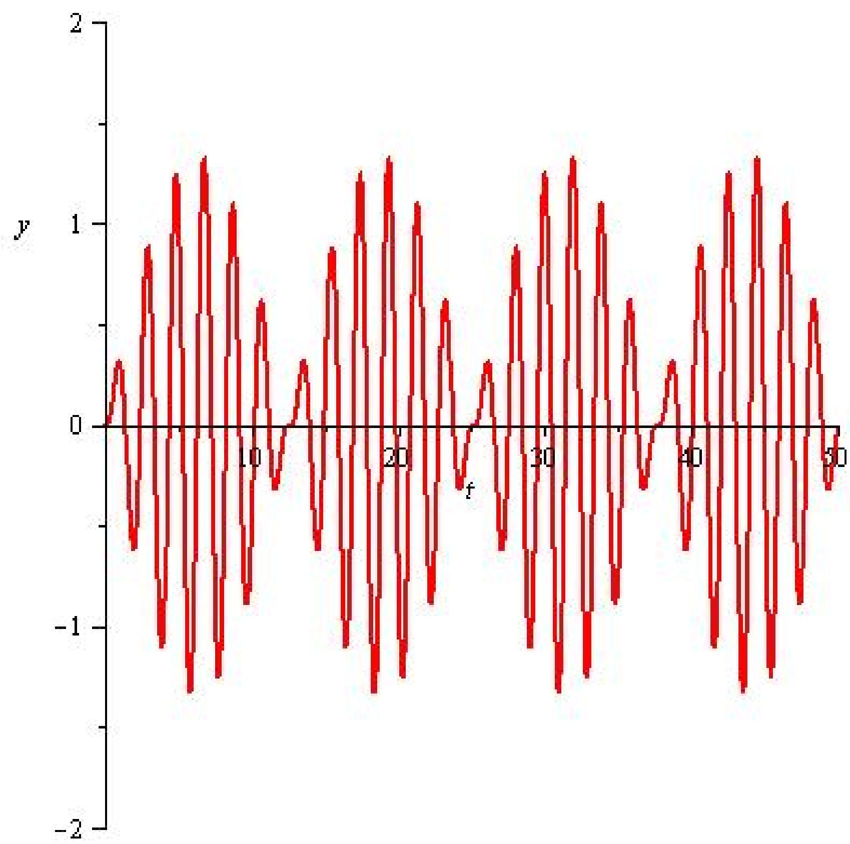

Figure 4: The aircraft wing's motion for $0 \leq t \leq 25$ due to $f_2(t)$

Finally, in Requirement 2c, we notice that the forcing function has the same frequency as the homogeneous solution to the ODE. Therefore, we must conjecture

$$y_n(t) = A\, t\, \cos(3t) + B\, t\, \sin(3t).$$

Again, take derivatives with respect to t (using the product rule) and substitute into Equation 3. Equating coefficients yields $A = -\frac{1}{3}$ and $B = 0$, giving a general solution of

$$y(t) = c_1\, \cos(3t) + c_2\, \sin(3t) - \frac{1}{3}\, t\, \cos(3t).$$

Applying the initial conditions of $y(0) = 0$ and $y'(0) = 0$, we determine that $c_1 = 0$ and $c_2 = \frac{1}{9}$, providing a particular solution of

$$y(t) = \frac{1}{9}\, \sin(3t) - \frac{1}{3}\, t\, \cos(3t). \tag{5}$$

A plot of the solution is shown in Figure 5. As t increases, $y(t)$ grows without bound. The phenomenon of “resonance” is caused by the factor of t in the second term of Equation 5. The wing will reach mechanical failure once the amplitude of the oscillations become too great.

Requirement 3

Think critically about the problems you just worked. What forces were at work on the real plane wing that we didn’t include in our model? (Hint: Look at your differential equation. What type of motion did we model? For example, was it forced or free motion, damped or undamped motion?) What other factors or issues might an engineer want to analyze in his investigation of wing flutter? Can you somehow model “flex” in the wing?

Solution:

As mentioned in Requirement 1, the motion is forced and undamped. Modeling the aircraft wing as an undamped harmonic oscillator is probably unrealistic; engineers would design some form of damping into the system. If the differential equation contains a damping force, the motion of the wing would remain bounded and resonance would not become an issue. In addition, as is the case on many aircraft, wings can be designed in sections to exhibit different resonant frequencies for each section, so that failure would be avoided if one forcing function matches a single piece’s resonant frequency. By designing the wing in separate sections, each with higher resonant frequencies than the engine frequency, the beat phenomenon will also be avoided.

Modeling this wing with a different center of mass might have an effect as well.

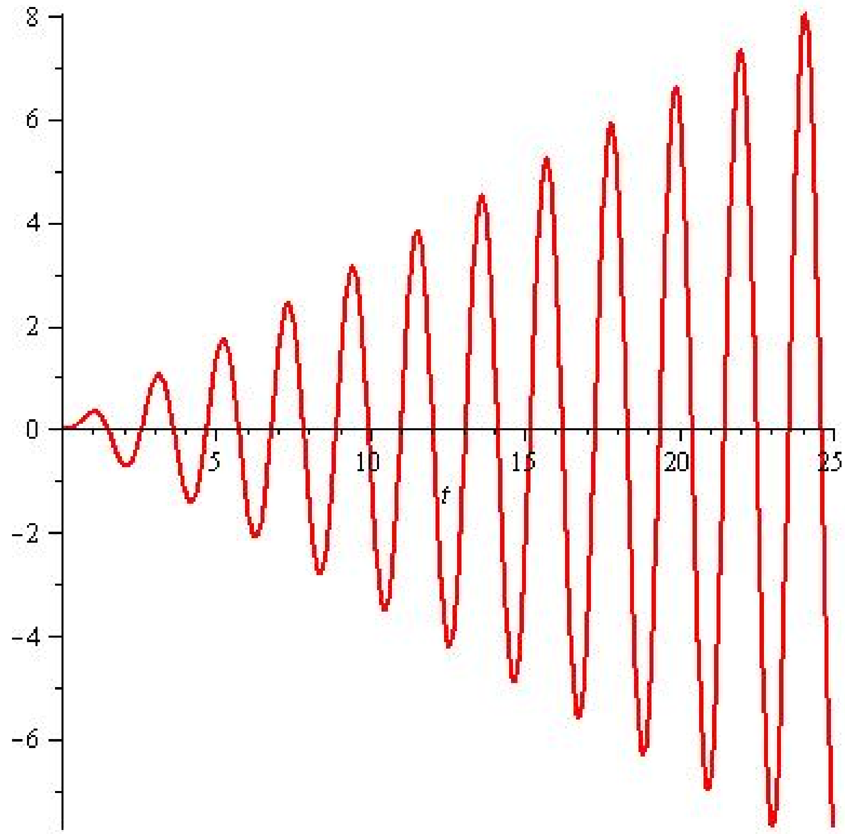

Figure 5: The aircraft wing's motion for $0 \leq t \leq 25$ due to resonance and $f_3(t)$

Many aircraft have fuel tanks in the wings, so the mass of the wing changes with the length of flight. This would have an effect on the differential equation. One way to offer flex in the wing is to create a design with a system of piecewise linear members. The spring constant of each would be varied, but there would have to be matching of the equations at joints of the separate sections.