Constructing a Matrix Representation of G_2

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Overview

1. Background
   - Differentiable Manifolds
   - Lie Groups
   - Lie Algebras
   - The Exponential Map
2. The Many Ways of Viewing G_2 and $\mathfrak{g}_2$
3. A Matrix Representation of G_2
   - The Goal
   - The Solution
   - The Geometric Characterization of $\mathfrak{g}_2$
   - A Basis for $\mathfrak{g}_2$
   - Our Matrix Representation of G_2(*Finally*!)
4. Summary
5. Acknowledgements

What is a Differentiable Manifold?

Working definition

- A *differentiable manifold* is the basic structure used in differential geometry. Essentially, every piece of a differentiable manifold looks like $\mathbb{R}^n$.
- It is a generalization of the idea of a smooth surface to any number of dimensions. We do not allow "kinks" and sharp corners in our manifolds.
- These are spaces on which it makes sense to do calculus.

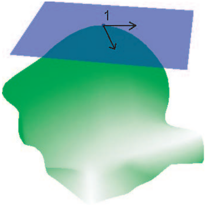
Examples

The circle, sphere, torus, Möbius strip, Klein bottle, ...

What is a Lie Group?

- A *Lie group* is a differentiable manifold as well as a group. Further, the group operations of multiplication and inversion are differentiable maps.
- Examples of Lie groups are some of the *matrix groups* such as $\mathrm{GL}(n)$,$\mathrm{O}(n)$, and $\mathrm{SO}(n)$. Another example is the circle S^1.
- Since Lie groups are geometric objects, topological properties such as connectedness, simple connectedness, compactness *may* apply to them.

What is a Lie Algebra?



Tangent Space Formulation

- The *Lie algebra* $\mathfrak{g}$ of a Lie group G is the tangent space to the identity element of G.
- Since a Lie algebra is a tangent space, it is also a vector space. So ideas like bases, linear independence, and orthogonality all apply.
- A Lie algebra comes equipped with the *bracket* operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. For matrix Lie algebras the bracket is the *commutator*, $[X, Y] = XY - YX$.

Moving from a Lie algebra to a Lie group

The Matrix Exponential

- The map $\exp : \mathfrak{g} \to G$ defined by

$$\exp(X) = 1 + X + X^2/2! + X^3/3! + \cdots$$

  takes a neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of the identity in G.
- The exponential map lets us take vectors in $\mathfrak{g}$ and generate points in G.

A Key Theorem

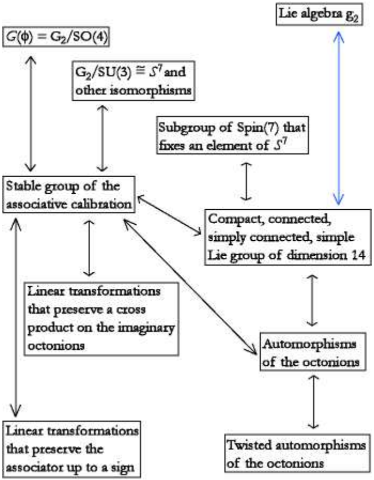
- The exponential map restricts to a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ onto some neighborhood of the identity in G.
- So if we have a basis for $\mathfrak{g}$ we can generate all of G.

The Lie Group G_2

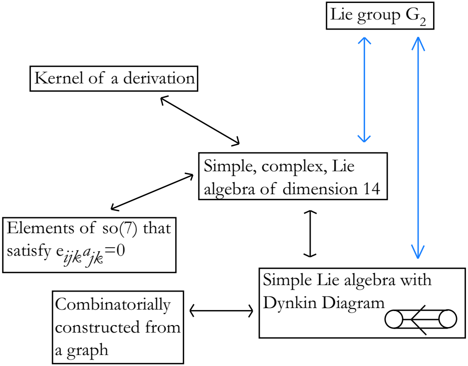
What do we know about G_2?

- G_2 is a 14-dimensional, connected, simply connected, simple, exceptional Lie group. It can be shown that G_2 is a subgroup of $\mathrm{SO}(7)$.
- There are two ways to study G_2 - either directly or through its Lie algebra $\mathfrak{g}_2$.

Ways of Viewing G_2



Ways of Viewing $\mathfrak{g}_2$



Our Goal

Our goal is to generate a complete description of G_2 in matrix form. This would be analogous to the 1-parameter matrix representation of $SO(2)$,

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}.$$

A Solution in 3 Steps

1. Characterize $\mathfrak{g}_2$ geometrically.
2. Construct a basis for $\mathfrak{g}_2$ using this geometric characterization.
3. Use the exponential map to generate G_2 from the basis for $\mathfrak{g}_2$.

Bryant's definition of $\mathfrak{g}_2$

- In a recent paper, Bryant shows the following: A matrix $A = (a_{ij}) \in \mathfrak{so}(7)$ is an element of $\mathfrak{g}_2$ if and only if $\epsilon_{ijk} a_{jk} = 0$ for $i, j, k = 1, \ldots, 7$.
- Here ϵ_{ijk} encapsulates 7^3 constants that come from the *associative calibration*. The associative calibration is related to one of the major definitions of G_2.

The inner product space of matrices

- How can we make $M(n, n)$, the set of n-by-n matrices, into an inner product space.
- What does this tell us about $\mathfrak{g}_2$?

Using the condition $\langle E_i, A\rangle = 0$ for $i = 1, \ldots, 7$ we can guess a basis for $\mathfrak{g}_2$. This involves finding 14 linearly independent matrices.

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & x_3 & -x_2 & x_5 & -x_4 & -x_7 & -x_6+y_6 \\ -x_3 & 0 & x_1 & x_6 & -x_7+y_7 & x_4-y_4 & x_5+y_5 \\ x_2 & -x_1 & 0 & -y_7 & y_6 & y_5 & y_4 \\ -x_5 & -x_6 & y_7 & 0 & -x_1+y_1 & -x_2+y_2 & -x_3+y_3 \\ x_4 & x_7-y_7 & -y_6 & x_1-y_1 & 0 & y_3 & -y_2 \\ x_7 & -x_4+y_4 & -y_5 & x_2-y_2 & -y_3 & 0 & y_1 \\ x_6-y_6 & -x_5-y_5 & -y_4 & x_3-y_3 & y_2 & -y_1 & 0 \end{pmatrix} \right\}.$$

Exponentiating Down

- The direct way of exponentiating to generate G_2 fails us computationally. We use the canonical coordinates of the second kind instead. In this case,

$$G_2 = \left\{ \left(\prod_{n=1}^{7} \exp(x_i X_i) \right) \cdot \left(\prod_{n=1}^{7} \exp(y_i Y_i) \right) \mid x_i, y_i \in \mathbb{R} \right\}.$$

- We have actually constructed a 14-paramater 7-by-7 matrix description of G_2. Unfortunately it is too large to display here.
- We find that individual elements of G_2 look like 2- and 3-tori.

What I did...

1. Developed a new geometric characterization of the Lie algebra $\mathfrak{g}_2$ with Professor Gu.
2. Found an explicit basis for $\mathfrak{g}_2$ using the geometric characterization.
3. Generated a 14-parameter matrix description of the Lie group G_2.
4. Identified the geometry of 1-parameter subgroups of G_2.


I'd like to thank ...

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