Cesaro Limits of Analytically Perturbed Stochastic Matrices

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Overview

- Background definitions
- The basis for my thesis
- The problem I’m working on
- An example
- Results from summer
- Current work

Stochastic Matrices

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$|\lambda| \leq 1$ for any eigenvalue λ of a stochastic matrix.

Analytic Perturbations

Definition: An *analytic perturbation* of a matrix $T_0 \in M_n(\mathbb{C})$ is a power series

$$T(\varepsilon) = T_0 + A(\varepsilon) = T_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots$$

in which the “coefficients” A_1, A_2, $\ldots$ are in $M_n(\mathbb{C})$ as well.

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Example:

$$T(\varepsilon) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 + 2\varepsilon \\ 2 - \varepsilon & 1 + \varepsilon \end{pmatrix}$$

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We want $P(\varepsilon)$ to be stochastic for all sufficiently small positive ε.

Foundation for My Thesis

- In 2002, Filar, Krieger, and Syed characterized the Cesaro limit

$$\lim_{\varepsilon \downarrow 0} \frac{1}{N(\varepsilon)} \sum_{k=1}^{N(\varepsilon)} P^k(\varepsilon)$$

  for an analytically perturbed stochastic matrix $P(\varepsilon) = P_0 + A(\varepsilon)$

- Subject to the restriction that P_0 have no eigenvalues λ satisfying $|\lambda| = 1$ except for $\lambda = 1$.

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- Does the Cesaro limit still necessarily exist?
- If or when the limit does exist, how will such eigenvalues affect the limit?
- How does the rate at which $N(\varepsilon) \to \infty$ affect the existence or value of the limit?

An Example

Let

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- If $N(\varepsilon)\varepsilon \to \infty$ as $\varepsilon \downarrow 0$, the Cesaro limit is

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Example, cont.

- If $N(\varepsilon)\varepsilon \to 0$ as $\varepsilon \downarrow 0$, the limit is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example, cont.

- If $N(\varepsilon)\varepsilon \to 0$ as $\varepsilon \downarrow 0$, the limit is

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- If $N(\varepsilon)\varepsilon \to L$, with $0 < L < \infty$, as $\varepsilon \downarrow 0$, the limit is

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + \frac{1 - e^{-2L}}{2L} \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Summer Work

Showed, under fairly restrictive conditions, that the Cesaro limit still exists, and that eigenvalues on the unit circle other than 1 do not contribute to the value of the limit.

Current Work

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- Reading a paper that gives some information about the eigenvalue structure of $P(\varepsilon)$ connected with the eigenvalue $\lambda = 1$ of the unperturbed matrix P_0
- Trying to determine if methods can be adapted to give extra information about the eigenvalue structure of $P(\varepsilon)$ connected to other eigenvalues on the unit circle.