

Bounds on the Ratio of Eigenvalues

Tracy Powell

Lesley A. Ward, Advisor

Estelle Basor, Reader

December 10, 2005

HARVEY MUDD
C O L L E G E

Department of Mathematics

Abstract

In this paper, we find lower and upper bounds for the leading and second leading eigenvalues (respectively) of particular symmetric matrices. This has applications to web search algorithms that are based on adjacency matrices of the links between web pages.

Chapter 1

Introduction

This paper will address the simpleness of leading eigenvalues of particular $n \times n$ matrices in the limit as $n \rightarrow \infty$. Chapter 1 contains an introduction to the problem, explain a specific application, and give some examples. In Chapter 2 are some key definitions for this project. Finally, Chapter 3 consists of new and previous results for this problem.

1.1 Problem

A graph G is made up of vertices, or nodes, and edges connecting them. The corresponding adjacency matrix $A = A(G)$ is an $n \times n$ matrix (n being the number of nodes) where the a_{ij} th entry is 1 if there is a directed edge, or link, from node i to node j and 0 otherwise. We will consider the matrix B defined by

$$B = e^A - I = A + A^2/2! + A^3/3! + \dots .$$

In [6], it is proven that the matrices BB^T and B^TB have simple, or non-repeated, dominant eigenvalues if the graph G is weakly connected. As n goes to infinity for a particular family of graphs (which will be discussed in further detail later on), it can be observed that the second leading eigenvalue of B , $\lambda_2 = \lambda_2(B)$, becomes increasingly close to the leading eigenvalue of B , $\lambda_1 = \lambda_1(B)$. If the ratio of λ_2 to λ_1 is not bounded above by a number less than one, then the simpleness of the dominant eigenvalue no longer holds, that is, "in the limit" the leading eigenvalue will be repeated.

There have been several papers written trying to find a "good" upper bound for leading eigenvalues of certain trees and graphs. Hofmeis-

ter finds upper bounds for the leading and second leading eigenvalues of an adjacency matrix based on the number of vertices of a tree in [9]. A lower bound on the leading eigenvalue and an upper bound on the second leading eigenvalue will put an upper bound on the ratio of the two. There are also several articles on improved bounds, bounds on the eigenvalues of other types of graphs (other than trees), Laplacian eigenvalues, etc. (See bibliography below.)

The particular family of trees that we consider in this paper are directed trees made up of one node of degree h that branches out to h nodes of degree 2. These nodes branch out to h more nodes of degree 2 ℓ times then terminate with the last set of h nodes having b branches coming out of them. (A definition of the degree of a node can be found below.) Examples from this family of trees are shown in Figures 1.1 and 1.2. The parameters h , ℓ , and b are taken directly from the structure of the graph. Respectively, these parameters correspond to the number of handles, the length, and the number of bristles of the graph. This family of graphs will be denoted $G_{h,\ell,b}$. When no third parameter is given, that is, when $G_{h,\ell}$ is written, it is assumed that $b = 1$.

For this family of trees, $G_{h,\ell,b}$, as $\ell \rightarrow \infty$, the leading and second leading eigenvalues of $B^T B$ seem to become increasingly close, that is, $\lambda_2/\lambda_1 \rightsquigarrow 1$. Specifically, numerical experiments [6] suggested that $\lambda_2/\lambda_1 \rightarrow 1$ for $b = 1, 2$, and 3. The initial goal of this project is to determine whether λ_2/λ_1 does in fact have limit 1, as $\ell \rightarrow \infty$, in these examples.

1.2 Applications

Internet web page search engines, such as Google and Teoma, use particular algorithms to rank web pages based on their relevancy to the search query. One specific search algorithm is Kleinberg's HITS (Hypertext Induced Topic Search) algorithm [10](used by Teoma and Ask Jeeves). This algorithm uses a form of the adjacency matrix of the graph of web pages, $A^T A$, to calculate the rankings of the pages in the graph. When the dominant eigenvalue of $A^T A$ is repeated, the ranking of these web pages is not unique. This implies that the same search query entered at different times, or on a different computer, could potentially produce web pages in different orders. To prevent the repetition of the dominant eigenvalue, the matrix $B^T B$ can be used where B is as defined above.

Here is a more specific explanation of the HITS algorithm. A search term, or query, is fed into the algorithm by the user. A text-matching algorithm is used to retrieve a set S of the pages that include the text of the query. This set is then enlarged to the set T by adding in pages that link to or are linked to by a page in S . [10] calls S the root set and T the base set. Typically, T has 3000–5000 pages. (This is different from Google’s PageRank algorithm in that PageRank first ranks all of the pages in its database, about 8 billion pages, instead of looking at a subset of the pages.) HITS now ranks the pages of T in the following way. The pages of T and the links between them are represented by a graph, G , where the pages in T are the nodes of G and the hyperlinks between the pages are the (directed) edges of G . The nodes are numbered and the authority and hub vectors are initialized uniformly. The authority and hub vectors give the authority and hub weights, respectively, for each web page in G . The authority weight tells how often or how well a web page references the search query. A good authority would then have an authority weight close to 1 (since the vectors are normalized). The hub weight tells whether or not the web page points to good authorities. Thus, a hub weight close to one indicates a good hub. Therefore, the initial authority vector, \vec{a}_0 , and the initial hub vector, \vec{h}_0 , look like

$$\vec{a}_0 = \vec{h}_0 = \begin{bmatrix} 1/\sqrt{n} \\ 1/\sqrt{n} \\ \vdots \\ 1/\sqrt{n} \end{bmatrix}$$

where n is the number of nodes of G . Then the HITS algorithm updates the authority vector such that, for node i ,

$$\tilde{a}_1(i) = \sum_{j:j \rightarrow i} \vec{h}_0(j)$$

where $j : j \rightarrow i$ means that node j points directly to node i . The hub vector is updated similarly,

$$\tilde{h}_1(i) = \sum_{j:i \rightarrow j} \tilde{a}_1(j).$$

Both vectors are normalized so that

$$\vec{a}_1 = \frac{\tilde{a}_1}{\|\tilde{a}_1\|} \quad \text{and} \quad \vec{h}_1 = \frac{\tilde{h}_1}{\|\tilde{h}_1\|}.$$

This iteration is repeated until $\vec{h}_n = \vec{h}_{n+1}$ and $\vec{a}_n = \vec{a}_{n+1}$. The sequence of computation is then $a_0 = h_0, a_1, h_1, a_2, h_2, a_3, \dots$

Let A denote the adjacency matrix of the graph G . In terms of A , the algorithm described above can be rewritten as follows. At the k^{th} iteration,

$$\vec{a}_k = \phi_k A^T \vec{h}_{k-1} \quad (1.1)$$

$$\vec{h}_k = \psi_k A \vec{a}_k \quad (1.2)$$

where $\phi_k, \psi_k \in \mathbb{R}^{>0}$ are normalization constants and

$$\sum_{i=1}^n \vec{a}_k(i)^2 = \sum_{i=1}^n \vec{h}_k(i)^2 = 1.$$

Focusing on the authority vector, we can rewrite equation (1.1) and equation (1.2) to show

$$\vec{a}_k = \phi_k \psi_{k-1} A^T A \vec{a}_{k-1}. \quad (1.3)$$

Since $A^T A$ is a symmetric matrix ($(A^T A)^T = A^T (A^T)^T = A^T A$), the following theorems apply. (The proofs of these theorems can be found in [13].)

Theorem 1 (The Spectral Theorem). *Let A be an $n \times n$ matrix. Then A is symmetric if and only if it is orthogonally diagonalizable. In particular, a symmetric matrix A is diagonalizable.*

Theorem 2. *Let A be an $n \times n$ diagonalizable matrix with dominant eigenvalue λ_1 . Then there exists a nonzero vector \vec{x}_0 such that the sequence of vectors \vec{x}_k defined by*

$$\vec{x}_1 = A\vec{x}_0, \vec{x}_2 = A\vec{x}_1, \vec{x}_3 = A\vec{x}_2, \dots, \vec{x}_k = A\vec{x}_{k-1}, \dots$$

approaches a dominant eigenvector of A .

Thus, the sequence determined by equation (1.3) converges to a dominant eigenvector of $A^T A$. A similar conclusion can be made for the hub vectors such that the sequence of hub vectors will converge to a dominant eigenvector of AA^T . This iterative method is called the Power Method, and this form of the power method uses Rayleigh quotients [13: page 312].

Notice that equation (1.3) converges to an eigenvector of the dominant eigenvalue of $A^T A$. If the dominant eigenvalue is repeated, there can be

multiple eigenvectors associated with this eigenvalue. The Power Method will then converge to one of these dominant eigenvectors, but the one it converges to depends on the initial vector \vec{a}_0 . Therefore, for internet search rankings to be unique, it is imperative that the dominant eigenvalue of the matrix used in the iteration is simple.

1.3 Examples

The following is an example of a tree within the family of trees under consideration. This example gives an adjacency matrix that is 9×9 , since there are $n = 9$ nodes.

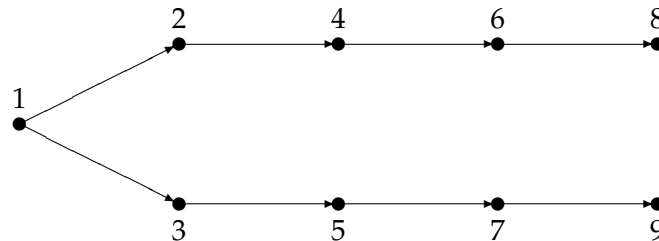


Figure 1.1: The graph $G_{2,3}$

This example is the graph with two handles, $h = 2$, length $\ell = 3$, and one bristle, $b = 1$. Notice that the graph is denoted $G_{2,3}$ as opposed to $G_{2,3,1}$ as described above. The associated adjacency and exponentiated adjacency matrices are

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 & 1/2 & 1/2 & 1/3! & 1/3! & 1/4! & 1/4! \\ 0 & 0 & 0 & 1 & 0 & 1/2 & 0 & 1/3! & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1/2 & 0 & 1/3! \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of $B^T B$, rounded to four decimal places, are 3.1421, 2.0625, 1.6674, 0.9604, 0.8002, 0.5048, 0.4771, 0, and 0. The authority and

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1/2 & 1/2 & 1/2 & 1/3! & 1/3! & 1/3! & 1/3! & 1/3! & 1/3! \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, the eigenvalues of $B^T B$ are 4.7014, 2.7808, 2.7808, 2.1105, 0.7192, 0.7192, 0.6047, and 0 (with algebraic multiplicity six). The authority and hub vectors are

$$\vec{a} = \vec{v}_1(B^T B) = \begin{bmatrix} 0 \\ 0.3915 \\ 0.3915 \\ 0.3915 \\ 0.3050 \\ 0.3050 \\ 0.3050 \\ 0.2086 \\ 0.2086 \\ 0.2086 \\ 0.2086 \\ 0.2086 \\ 0.2086 \\ 0.2086 \end{bmatrix} \quad \vec{h} = \vec{v}_1(BB^T) = \begin{bmatrix} 0.8489 \\ 0.2369 \\ 0.2369 \\ 0.2369 \\ 0.1924 \\ 0.1924 \\ 0.1924 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Again, the best authorities are nodes 2, 3, and 4 and the best hub is node 1. The ratio of the second largest eigenvalue to the largest eigenvalue, denoted $\alpha(h, \ell, b)$, is $\alpha(3, 2, 2) = 0.5915$.

Table 1.1 shows the ratios of $\lambda_2(B^T B)$ to $\lambda_1(B^T B)$ for the graphs $G_{h,\ell}$ where $b = 1$. It is interesting to note that the ratio tends to decrease along a row (that is, as the number of handles increases) and increase along a column (as the length of the handles increases). However, from $\ell = 2$ to $\ell = 9$, this is not the case as h increases from 1 to 2.

The next two examples are interesting to note. We will not consider them in much detail, however.

Figure 1.3 shows a binary tree. This tree fits into our family of trees and is the one corresponding to $G_{2,1,2}$.

	$h = 1$	2	3	4	5	6
$\ell = 1$	0.37162	0.35961	0.25000	0.19098	0.15436	0.12948
2	0.46565	0.53828	0.38680	0.29919	0.24334	0.20488
3	0.56377	0.65641	0.48033	0.37338	0.30431	0.25648
4	0.64665	0.73118	0.53991	0.42036	0.34276	0.28895
5	0.71296	0.77936	0.57824	0.45040	0.36430	0.30965
6	0.76434	0.81223	0.60371	0.47030	0.38353	0.32333
7	0.80427	0.83501	0.62128	0.48399	0.39470	0.33275
8	0.83549	0.85146	0.63383	0.49377	0.40267	0.33947
9	0.86020	0.86368	0.64309	0.50099	0.40856	0.34443
10	0.87994	0.87292	0.65005	0.50641	0.41298	0.34816
11	0.89595	0.88013	0.65546	0.51062	0.41641	0.35106
12	0.90905	0.88580	0.65970	0.51393	0.41911	0.35333

Table 1.1: The ratios $\alpha(h, \ell) = \lambda_2/\lambda_1$ of $B^T B$ for graphs $G_{h,\ell}$.

The associated matrices are

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & B &= \begin{bmatrix} 0 & 1 & 1 & .5 & .5 & .5 & .5 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 A^T A &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} & AA^T &= \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 B^T B &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & .5 & .5 & .5 & .5 \\ 0 & 1 & 1 & .5 & .5 & .5 & .5 \\ 0 & .5 & .5 & 1.25 & 1.25 & .25 & .25 \\ 0 & .5 & .5 & 1.25 & 1.25 & .25 & .25 \\ 0 & .5 & .5 & .25 & .25 & 1.25 & 1.25 \\ 0 & .5 & .5 & .25 & .25 & 1.25 & 1.25 \end{bmatrix} & BB^T &= \begin{bmatrix} 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 \lambda(A^T A) &= \{2, 2, 2, 0, 0, 0, 0\} & \lambda(B^T B) &= \{4, 2, 1, 0, 0, 0, 0\}
 \end{aligned}$$

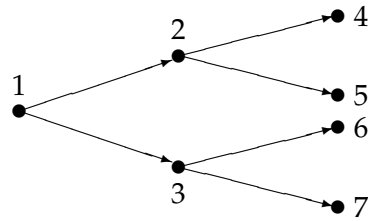


Figure 1.3: A Binary Tree

The ratio of the largest two eigenvalues of $A^T A$ is 1 while this ratio for $B^T B$ it is 0.5. This is an example of where the HITS algorithm could return improper weights for the ranked web pages using $A^T A$ but not for $B^T B$ since the eigenvalue is repeated in the first case and not in the second.

The next example is a slightly modified version of the previous one. We will call this tree the 3-2 tree for reasons that are clear from Figure 1.4.

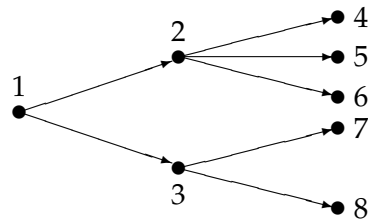


Figure 1.4: A 3-2 Tree

Again, the associated matrices are below:

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & B &= \begin{bmatrix} 0 & 1 & 1 & .5 & .5 & .5 & .5 & .5 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 A^T A &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} & AA^T &= \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 B^T B &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & .5 & .5 & .5 & .5 & .5 \\ 0 & 1 & 1 & .5 & .5 & .5 & .5 & .5 \\ 0 & .5 & .5 & 1.25 & 1.25 & 1.25 & .25 & .25 \\ 0 & .5 & .5 & 1.25 & 1.25 & 1.25 & .25 & .25 \\ 0 & .5 & .5 & 1.25 & 1.25 & 1.25 & .25 & .25 \\ 0 & .5 & .5 & .25 & .25 & .25 & 1.25 & 1.25 \\ 0 & .5 & .5 & .25 & .25 & .25 & 1.25 & 1.25 \end{bmatrix} & BB^T &= \begin{bmatrix} 3.25 & 1.5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1.5 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

The ratio of the largest two eigenvalues is 0.6667 for $A^T A$ and is 0.4907 for $B^T B$.

The last example is a diamond tree (Figure 1.5) with associated matrices.

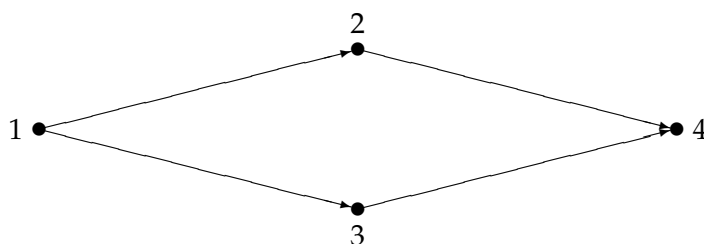


Figure 1.5: A Diamond Tree

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & B &= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 A^T A &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} & AA^T &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 B^T B &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix} & BB^T &= \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

It is interesting to note, in this last example, that the symmetries of the graph are also apparent in the symmetries of the matrices. For instance,

$$\begin{aligned}
 B^T B &= W(BB^T)W \\
 A^T A &= W(AA^T)W
 \end{aligned}$$

where

$$W = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Chapter 2

Definitions

Here are some of the key definitions we will need for our project. The following are from graph theory.

Definition 1. A *directed graph*, or *digraph*, is a graph with directed edges. A *simple directed graph* is a directed graph having no repeated edges.

Definition 2. The *adjacency matrix* of a simple directed graph is a matrix A where the a_{ij}^{th} entry is 1 if there is a path from vertex i to vertex j and 0 otherwise.

The following definitions are from linear algebra.

Definition 3. A matrix A is *symmetric* if $A = A^T$.

Definition 4. The *dominant* or *leading eigenvalue* of a matrix is the largest eigenvalue. Similarly, the *dominant eigenvector* is the eigenvector corresponding to the dominant eigenvalue.

Definition 5. A *simple eigenvalue* is one which is not repeated.

Definition 6. A *nilpotent* matrix is a square matrix A such that A^n is the zero matrix for some positive integer n .

Definition 7. Given $2n-1$ numbers a_k , where $k = -n+1, \dots, -1, 0, 1, \dots, n-1$, a *Toeplitz matrix* is a square matrix which has constant values along negative-sloping diagonals, i.e., a matrix of the form

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & & \\ a_2 & a_1 & a_0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{-1} \\ a_{n-1} & & \cdots & a_1 & a_0 \end{bmatrix}. \quad (2.1)$$

Definition 8. If there is a function $a \in L^\infty$ satisfying

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-in\theta} d\theta, \quad n \in \mathbb{Z},$$

then this function is unique. We therefore denote the matrix (2.1) and the operator it induces on l^2 by $T(a)$. The function a is in this context referred to as the *symbol* of the Toeplitz matrix or operator $T(a)$.

Definition 9. The i th *singular value* of a matrix A is the square root of the i th eigenvalue of $A^T A$. That is,

$$\sigma_i(A) = \sqrt{\lambda_i(A^T A)}.$$

Definition 10. The *operator norm* of a linear operator $T : V \rightarrow W$ is the largest value by which T stretches an element of V ,

$$\|T\| = \sup_{\|v\|=1} \|T(v)\|.$$

When T is given by a matrix, $T(v) = Av$, and $\|v\|$ is the L_2 norm, then $\|T\|$ is the largest singular value of A . That is,

$$\sigma_1(A) = \sqrt{\lambda_1(A^T A)} = \|A\|.$$

Definition 11. Let A_n be an $n \times n$ matrix. A *principal submatrix* of A_n , A_{n-1} , is obtained by omitting one row and the corresponding column of A_n .

Definition 12. For $j \in \{0, 1, \dots, n\}$, let $\mathcal{F}_j^{(n)}$ denote the collection of all $n \times n$ matrices of rank at most j . The j th *approximation number* $a_j(A_n)$ of an $n \times n$ matrix A_n is defined by

$$a_j(A_n) := \text{dist}(A, \mathcal{F}_j^{(n)}) := \min\{\|A_n - F_n\| : F_n \in \mathcal{F}_j^{(n)}\}.$$

The following are theorems used in the proof of Theorem 5.

Theorem 3. [Cauchy's Interlace Theorem] If A is a Hermitian matrix and B is a principal submatrix of A , then the eigenvalues of B interlace the eigenvalues of A . That is, if $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are the eigenvalues of A and $\mu_2 \geq \mu_3 \geq \dots \geq \mu_n$ are the eigenvalues of B , then

$$\lambda_1 \geq \mu_2 \geq \lambda_2 \geq \dots \geq \mu_{n-1} \geq \lambda_{n-1} \geq \mu_n \geq \lambda_n.$$

Theorem 4. If A_n is an $n \times n$ matrix, then

$$\sigma_j(A_n) = a_{n-j}(A_n)$$

for every $j \in \{0, 1, \dots, n\}$.

Chapter 3

Results

3.1 New Results

The following lemmas are results that aid in the calculation of the types of matrices we are interested in.

Lemma 1. For an $n \times m$ matrix M ,

$$(M^T M)_{ij} = \text{col } M_i \cdot \text{col } M_j$$
$$(M M^T)_{ij} = \text{row } M_i \cdot \text{row } M_j$$

Proof. Let M be represented by column vectors. Then M^T is represented by row vectors. That is,

$$M = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_m]$$
$$M^T = \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_m^T \end{bmatrix}$$

Therefore, by simple matrix multiplication, $(M^T M)_{ij} = \vec{v}_i^T \vec{v}_j$ and, from linear algebra, we conclude that $(M^T M)_{ij} = \vec{v}_i \cdot \vec{v}_j$ where \vec{v}_i is the i th column of M . The second part of the proof is identical except the row and column vectors are reversed. \square

Lemma 2. For the graph $G_{h,l}$ (h handles of length l), $b = 1$, AA^T is diagonal, and so the eigenvalues of A are on its diagonal.

Proof. Using Lemma 1, the entries of AA^T are the dot products of the rows. Since there is no more than one 1 in each column of A , $(\text{row } i) \cdot (\text{row } j) = 0$ when $i \neq j$. Therefore, the only possible nonzero entries are when $i = j$ which are exactly the entries on the main diagonal of AA^T . So AA^T is diagonal. (The fact that the eigenvalues are on the diagonal is a result of basic linear algebra.) \square

Lemma 3. For $G_{h,\ell}$, the entries of $A^T A$ are 0's and 1's, and $A^T A$ is block-diagonal of a specific form.

Proof. Using Lemma 1, we can construct the following formula for the entries of $A^T A$.

$$(A^T A)_{ij} = \begin{cases} 1, & \text{if column } i \text{ of } A = \text{column } j \text{ of } A \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, since columns 2 through $h+1$ are the same (1 in the first row and zeros everywhere else), there will be an $h \times h$ matrix whose entries are all ones in the position after the first row and column and ones will run down the rest of the diagonal of $A^T A$. That is, $A^T A$ will look like

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 0 \\ & \vdots & 1 & \vdots & \\ \vdots & 1 & \cdots & 1 & \vdots \\ & & & & 1 \\ & & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

\square

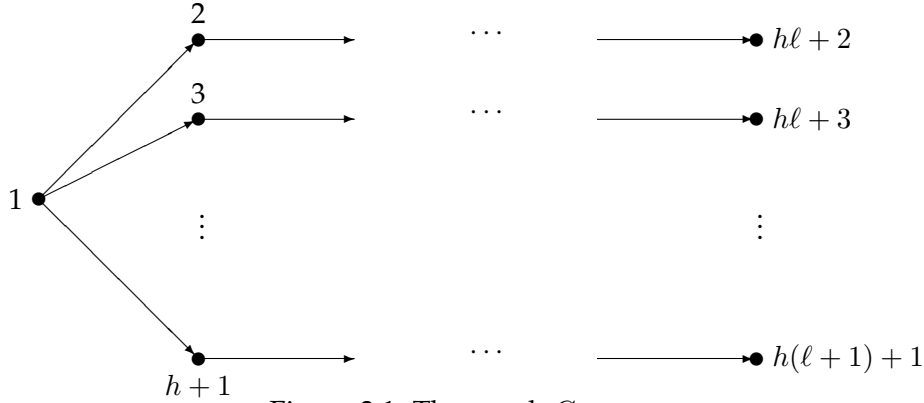
From observations made when calculating the ratios λ_2/λ_1 , I have proposed the following conjectures. I will attempt to prove these next semester.

Conjecture 1. For the graph $G_{h,\ell}$, $b = 1$, fix ℓ . Then

$$\lambda_2(B^T B) = \lambda_1(\widetilde{B^T B}),$$

where $\widetilde{B^T B}$ is some submatrix of $B^T B$, and $\lambda_2(B^T B)$ is independent of h .

Conjecture 2. For the graph $G_{h,\ell}$, $b = 1$, fix h . Then, as ℓ increases, $\lambda_1 B^T B < \infty$.

Figure 3.1: The graph $G_{h,\ell}$

3.2 Previous Result

Theorem 5. [Basor and Morrison, 2004]

Let $G_{h,\ell}$ be the directed graph shown in Figure 3.1. Also let $\lambda_j(M)$ be the j th largest eigenvalue of an $n \times n$ matrix M and $B = e^A - I$ where A is the adjacency matrix corresponding to $G_{h,\ell}$. Then

$$\frac{\lambda_2(B^T B)}{\lambda_1(B^T B)} < 0.94430808.$$

Theorem 5 was proven by Estelle Basor and Kent Morrison of California Polytechnic State University, San Luis Obispo. The following proof is a slightly more generalized version of their proof. (The modification here is that the proof that they had written is specifically for the graph when $h = 2$. I have modified the proof slightly to extend to any positive h .)

Proof. The adjacency matrix for $G_{h,\ell}$ is

$$A = \begin{bmatrix} 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 & 1 & & 0 & 0 \\ \vdots & & & & & & \ddots & & \vdots \\ 0 & 0 & & 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & & 0 & 0 & 0 & & 0 & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The matrix $B_{h,\ell} = e^A - I$, where there are h ones in the first row, corresponding to the links from node 1 to the h handles, for this particular graph, is

$$B_{h,\ell} = \begin{bmatrix} 0 & 1 & \cdots & 1 & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{3!} & \cdots & \frac{1}{3!} & \cdots & \frac{1}{m!} & \cdots & \frac{1}{m!} \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & 0 & \frac{1}{2} & \cdots & 0 & \cdots & \frac{1}{(m-1)!} & \cdots & 0 \\ \vdots & & & & & 1 & \vdots & & \frac{1}{2} & \vdots & & & \frac{1}{(m-1)!} & \vdots \\ & & & & & & 1 & 0 & \cdots & \frac{1}{2} & & & & \frac{1}{(m-1)!} \\ \vdots & & & & & & & & & & & & & \vdots \\ 0 & \cdots & & & & & & & & & & \cdots & & 0 \end{bmatrix}.$$

Take a unit vector $x \in \mathbb{R}^n$, where $n = h(\ell + 1) + 1$ is the number of nodes of $G_{h,\ell}$, such that $\|B_{h,\ell}\| = \|B_{h,\ell}x\|$. Now let $y = [x_1, x_2, \dots, x_n, 0]^T \in \mathbb{R}^{n+1}$. Then $\|B_{h,\ell}x\| = \|B_{h,\ell+1}y\|$. Therefore, by the usual vector and matrix norm axioms, $\|B_{h,\ell}\| \leq \|B_{h,\ell+1}\|$. (It is also true in general that if D is a submatrix of C , then $\|D\| \leq \|C\|$ by a similar argument.) By this argument, $B_{h,\ell}$ is a submatrix of $B_{h+1,\ell}$, thus $\|B_{h,\ell}\| \leq \|B_{h+1,\ell}\|$. Using MATLAB, we find that $\|B_{2,3}\| \approx 1.77261 > 1.77$ and since the operator norm of a matrix is its largest singular value, $\sigma_1(B_{2,3}) = \sqrt{\lambda_1(B_{2,3}^T B_{2,3})} > 1.77$ for $h \geq 2$ and $\ell \geq 3$.

Let the lower right principal submatrix of $B_{h,\ell}$ be denoted T_n . Note that T_n is a Toeplitz matrix of size $h(\ell + 1) \times h(\ell + 1)$. The matrix representation of the Toeplitz operator T is the infinite matrix attained by extending T_n indefinitely to the right and downward. The symbol for T is the function

$$z^{-h} + \frac{1}{2}z^{-2h} + \cdots + \frac{1}{m!}z^{-mh} + \cdots = e^{1/z^2} - 1 \quad (m = 1, 2, 3, \dots).$$

From Definition 12 and Theorem 4, we get the following relations:

$$\sigma_2(A_n) = a_1(A_n) := \text{dist}(A, \mathcal{F}_1^{(n)}) := \min\{\|A_n - F_n\| : F_n \in \mathcal{F}_1^{(n)}\}.$$

This says that $\sigma_2(B_{h,\ell})$ is the distance from $B_{h,\ell}$ to the set of matrices of rank at most 1. Now, we can write $B_{h,\ell}$ as the sum of a rank one matrix and an augmented Toeplitz matrix. The rank one matrix is the $n \times n$ matrix, where n is the number of nodes of $G_{h,\ell}$, that has the same first row as $B_{h,\ell}$ and zeros for all other entries. The augmented Toeplitz matrix, denoted C_n , is the Toeplitz matrix T_n with an additional row of zeros at the top and column of zeros on the left. Therefore,

$$B_{h,\ell} = \left[\begin{array}{c} \text{rank one} \\ \text{matrix} \end{array} \right] + \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & T_n & \\ 0 & & & \end{array} \right],$$

$$B_{h,\ell} - \left[\begin{array}{c} \text{rank one} \\ \text{matrix} \end{array} \right] = C_n.$$

Thus, $\|C_n\| \geq \sigma_2(B_{h,\ell})$. But $\|C_n\| = \|T_n\|$ since they differ only by a row and column of zeros. A result of Silbermann's gives

$$\lim_{n \rightarrow \infty} \|T_n\| = \|T\| = \max_{|z|=1} |e^{1/z^2} - 1| = e - 1,$$

the maximum occurring at $z = 1$. We can conclude that, for n sufficiently large, $\|T_n\| \leq e - 1 < 1.72$. Therefore, by Definition 10, $\sigma_2(B_{h,\ell}) \leq \|C_n\| = \|T_n\| < 1.72$.

These results imply that

$$\frac{\sigma_2(B_{h,\ell})}{\sigma_1(B_{h,\ell})} < \frac{1.72}{1.77} = 0.97175141\dots$$

and by Definition 9,

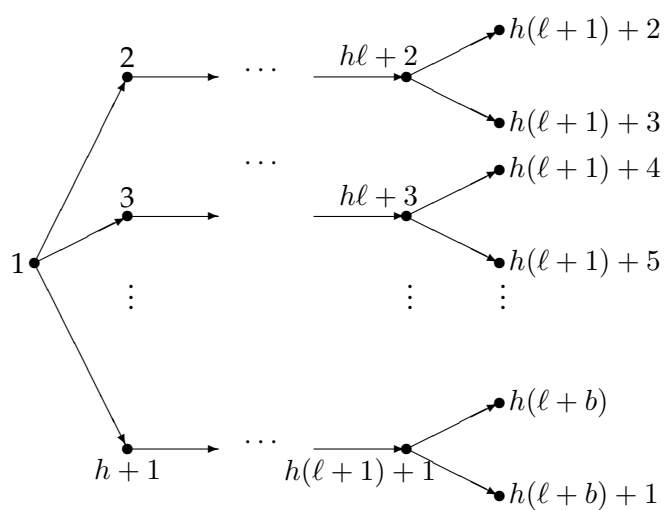
$$\frac{\lambda_2(B_{h,\ell}^T B_{h,\ell})}{\lambda_1(B_{h,\ell}^T B_{h,\ell})} < 0.944300808.$$

□

3.3 Extending Previous Results

The technique for finding a lower bound for the leading eigenvalue of $B^T B$ for the graph in Figure 3.2 is equivalent to the lower bound found in the proof of Theorem 5. I will show why the technique for finding an upper bound for the second leading eigenvalue used in that proof will not hold.

From the theory contained in the proof of Theorem 5, we can calculate the operator norm of any $B_{h,\ell,b}$ for large enough n , where n is the number of nodes of $G_{h,\ell,b}$. For example, $\|B_{2,3,2}\| = 1.870499 < 1.87$. It is, at this point, unclear as to whether or not these values are strictly increasing

Figure 3.2: The graph $G_{h,\ell,b}$

while the variables are increasing at different times.

The problem with the second part of the proof is that there is not a way to write $B_{h,\ell,b}$ as the sum of a Toeplitz matrix and a rank one matrix, for $b > 1$. It is possible, however, to write $B_{h,\ell,b}$ as the sum of a Toeplitz matrix and a rank three matrix. This direction would lead to a relation between λ_1 and λ_4 . We are considering another numerical approach involving the Lanczos iteration.

Bibliography

- [1] Howard Anton and Robert Busby, *Contemporary linear algebra*, Wiley, Indianapolis, IN, 2002,
Section 5.4 goes into depth about the power method and its specific application to internet search algorithms.
- [2] Estelle L. Basor and Kent E. Morrison, *Singular values of an adjacency matrix*, (2004), 3,
The main source for the proof for $b = 1$.
- [3] Albrecht Böttcher and Sergei M. Grudsky, *Toeplitz matrices, asymptotic linear algebra, and functional analysis*, Birkhäuser Verlag, Basel, 2000,
This book contains a lot of background material, including some results used in the proof by Professors Basor and Morrison that the ratio of the second largest to the largest eigenvalues of $B^T B$ is bounded below 1. MR MR1772773 (2001g:47043)
- [4] Wai-Shun Cheung, Chi-Kwong Li, D. D. Olesky, and P. van den Driessche, *Optimizing quadratic forms of adjacency matrices of trees and related eigenvalue problems*, *Linear Algebra Appl.* **325** (2001), no. 1-3, 191–207,
Proposition 1.5 seems to be the only item that has much to do with eigenvalues. The section on optimal labeling of trees might be interesting to consider since it is different from our current labeling. MR MR1810104 (2001j:05081)
- [5] Chris H. Q. Ding, Hongyuan Zha, Xiaofeng He, Parry Husbands, and Horst D. Simon, *Link analysis: hubs and authorities on the World Wide Web*, *SIAM Rev.* **46** (2004), no. 2, 256–268 (electronic). MR MR2114454 (2005h:68165)
- [6] Ayman Farahat, Thomas LoFaro, Joel C. Miller, Gregory Rae, and Lesley A. Ward, *Authority rankings from HITS, PageRank, and SALSA: Existence, uniqueness, and effect of initialization*, To appear in the Journal of

- Scientific Computing, SIAM. (2005), 20,
Provides a useful theorem about the simpleness of the dominant eigenvalue of $B^T B$ for the exponentiated adjacency matrix B .
- [7] C. D. Godsil, *Eigenvalues of graphs and digraphs*, Linear Algebra Appl. **46** (1982), 43–50,
From the abstract, it does not look as if this article gives any kind of bounds for eigenvalues of digraphs. MR MR664694 (83k:05076)
- [8] Ji-Ming Guo and Shang-Wang Tan, *A note on the second largest eigenvalue of a tree with perfect matchings*, Linear Algebra Appl. **380** (2004), 125–134,
This paper finds an upper bound for the second largest eigenvalue of trees with $n = 2k = 4t$ ($t \geq 2$) nodes. This would only apply to particular trees within our family of trees. MR MR2038744 (2004k:05131)
- [9] M. Hofmeister, *On the two largest eigenvalues of trees*, Linear Algebra Appl. **260** (1997), 43–59,
The introduction and the last theorem might be particularly helpful. The proofs and bit about partial eigenvectors might be of less use. MR MR1448350 (99c:05134)
- [10] Jon M. Kleinberg, *Authoritative sources in a hyperlinked environment*, J. ACM **46** (1999), no. 5, 604–632,
The primary source for information on the HITS algorithm. MR MR1747649
- [11] Mikhail Klin, Akihiro Munemasa, Mikhail Muzychuk, and Paul-Hermann Zieschang, *Directed strongly regular graphs obtained from coherent algebras*, Linear Algebra Appl. **377** (2004), 83–109,
As can be seen from the title, this article has nothing to do with bounds for eigenvalues. However, it might make an interesting read some summer day. MR MR2021604 (2005a:05221)
- [12] Bo Lian Liu, Brendan D. McKay, Nicholas C. Wormald, and Ke Min Zhang, *The exponent set of symmetric primitive $(0, 1)$ matrices with zero trace*, Linear Algebra Appl. **133** (1990), 121–131,
This article might contain information about the primitiveness of the adjacency matrices of our graphs. MR MR1058109 (91h:15018)
- [13] David Poole, *Linear algebra: A modern introduction*, Brooks/Cole, Pacific Grove, CA, 2003,
A linear algebra text book with useful definitions and theorems.

- [14] Dragan Stevanović, *Bounding the largest eigenvalue of trees in terms of the largest vertex degree*, *Linear Algebra Appl.* **360** (2003), 35–42,
This article gives upper and lower bounds for the leading eigenvalue of the adjacency matrix and the Laplacian matrix of a graph G . MR MR1948472 (2004h:05082)
- [15] Hong Yuan, *The k th largest eigenvalue of a tree*, *Linear Algebra Appl.* **73** (1986), 151–155,
Yuan shows that, for the k th eigenvalue of the adjacency matrix of a tree T with n vertices, $\lambda_k(T) \leq \sqrt{\lfloor (n-2)/k \rfloor}$ for $2 \leq k \leq \lfloor n/2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer not greater than x . MR MR818898 (87b:05097)
- [16] Shuqin Zhao and Yuan Hong, *On the bounds of maximal entries in the principal eigenvector of symmetric nonnegative matrix*, *Linear Algebra Appl.* **340** (2002), 245–252,
This article is not directly related to eigenvalues as much as it is related to the entries of dominant eigenvectors. MR MR1869431 (2002i:15020)