HARVEY MUDD COLLEGE

Math 197: Senior Thesis

The Negs and Regs of Continued fractions

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Introduction

A finite continued fraction is an expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{\cdots + \cfrac{b_n}{a_n}}}. \quad (1)$$

Although this expression may look messy, there is an elementary, conceptually elegant way to understand continued fractions. In Chapter 3 of their award-winning book [1], Arthur T. Benjamin and Jennifer J. Quinn introduce a clever approach that relies only on counting! This approach works for the well-studied *regular* continued fractions, which have the form

$$a_0 + \cfrac{1}{\cdots + \cfrac{1}{a_n}} \quad (2)$$

where the a_i are positive integers. The main idea in this chapter is that (2) has a combinatorial counterpart: the *set of square-and-domino tilings of a length* $(n+1)$ *board with height conditions*, as shown in Figure 1.

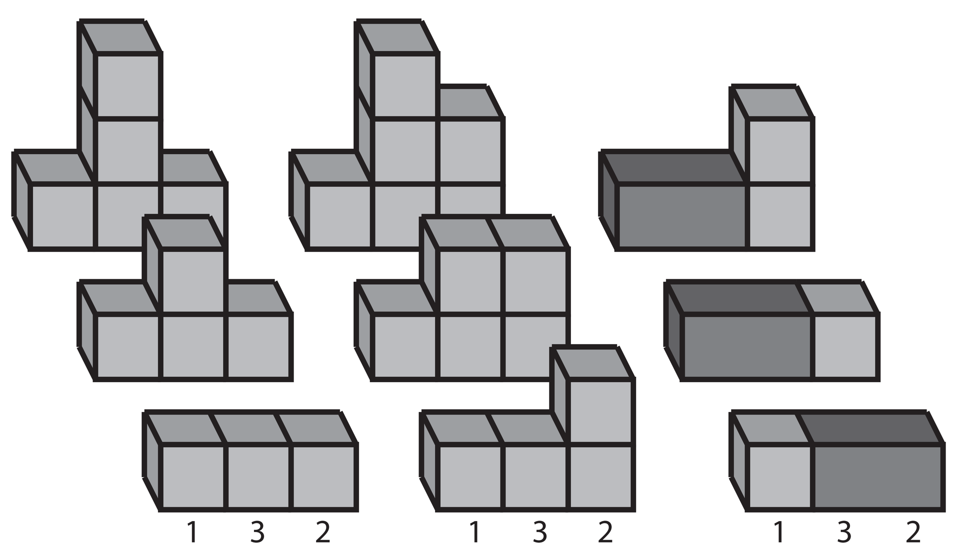

Figure 1: All nine ways to tile the length 3 board that do not exceed the height conditions $a_0 = 1$, $a_1 = 3$, $a_2 = 2$.

Research

Our original goal for the project was to relate regular continued fractions to the less-known *negative* continued fractions, which have the form

$$a_0 - \cfrac{1}{\cdots - \cfrac{1}{a_n}}. \quad (3)$$

This meant that we wanted to:

- Prove that the basic properties of regs (for instance, in [2]) hold for negs also
- Find a combinatorial interpretation for negs
- Describe how to convert between regs and negs

Over the course of the year, we accomplished these goals and more.

Results

Weighted Tilings

A continued fraction in general form (1), where a_i and b_i may be any real numbers, requires the concept of a *weighted* tiling to express combinatorially.

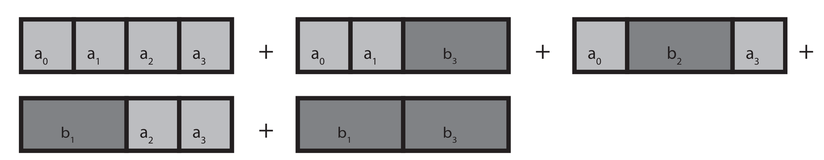

Figure 2: All the ways to tile the $|0:3|$ board.

We assign $a_0, \ldots, a_n$ to the squares and $b_1, \ldots, b_n$ to the dominoes, which gives each tile a weight. Then we define $|i:j|$ as the weighted sum of all possible tilings covering the cells from i to j. For instance

$$|0:3| = a_0a_1a_2a_3 + a_0a_1b_3 + a_0b_2a_3 + b_1a_2a_3 + b_1b_3$$

as shown in Figure 2. With this definition, the fundamental connection between continued fractions and combinatorics is given by

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{\cdots + \cfrac{b_n}{a_n}}} = \frac{|0:n|}{|1:n|}.$$

Mixed Tilings

For regular continued fractions, the weights aren't needed because having *height conditions* gave us a number of tilings equal the weighted sum $|0:n|$. This thesis shows how to do the same thing for negs, that is, when $b_i = -1$.

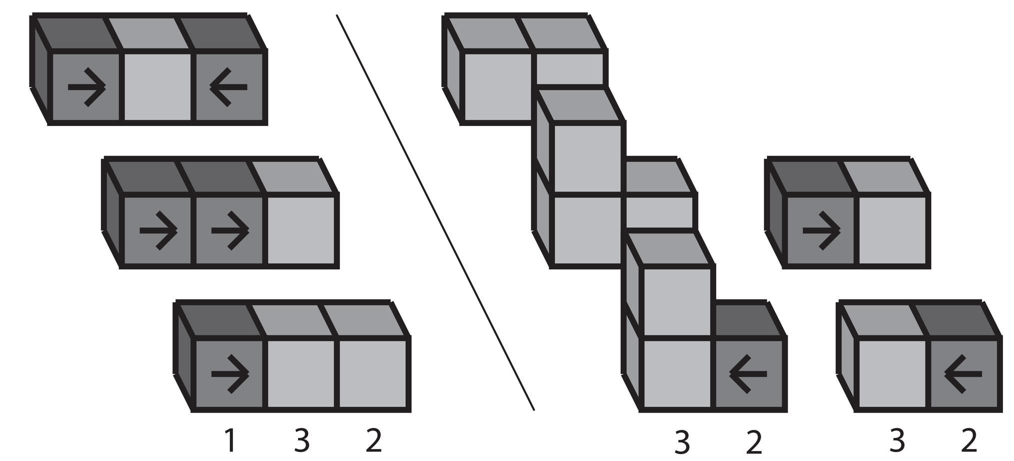

Figure 3: The board representing $1 - \frac{1}{3-\frac{1}{2}} = \frac{3}{5}$. We use the new objects $\rightarrow$ and $\leftarrow$ instead of dominoes, where $\rightarrow\leftarrow$ cannot appear consecutively..

Converting between Regs and Negs

Given a sequence $a_0, a_1, \ldots, a_n$ for a regular continued fraction,

- Increase the even-indexed numbers a_{2k} by 2 (except the one(s) on the ends, which are increased by 1), and
- Replace the odd-indexed numbers a_{2k+1} by a sequence of $(a_{2k-1} - 1)$ 2s

in order to obtain the negative continued fraction for the same number. For instance,

$$6 + \cfrac{1}{3 + \frac{1}{3}} = 7 - \cfrac{1}{2 - \cfrac{1}{2 - \frac{1}{4}}}.$$

Although this pattern is hard to guess, it's not hard to see combinatorially!

An Uncounted Identity

Another application of this framework is a new proof of the following identity:

Identity 1 *Let F_n denote the nth Fibonacci number and L_n denote the nth Lucas number. Then, for all $m, n \in \mathbb{Z}^+$,*

$$\frac{F_{(n+1)m}}{F_{nm}} = L_m - \cfrac{(-1)^m}{L_m - \cfrac{(-1)^m}{\cdots - \cfrac{(-1)^m}{L_m}}},$$

where the term L_m appears n times in the continued fraction.

In fact, by generalizing the Fibonacci and Lucas numbers, we obtain an interesting theorem about periodic continued fractions!

Conclusions

We focused on results and identities involving negative continued fractions, but a number of other topics related to continued fractions could surely be addressed using similar methods, such as,

- Generalizations of other Fibonacci identities
- Periodic continued fractions
- Two-squares theorems (from number theory)
- Class numbers of quadratic number fields.

It's always interesting to see the complex results that "fall out" as simple consequences of the combinatorial approach. Prof. Benjamin and I will be excited to see more such results from future thesis students! Moreover, only high school mathematics is really required to really dig into these subjects, making them ideal for educational purposes.

References

1. Arthur T. Benjamin and Jennifer J. Quinn. *Proofs That Really Count: The Art of Combinatorial Proof, Mathematical Association of America, Washington DC, 2003.*
2. *Andrew M. Rockett and Peter Szüsz. Continued fractions. World Scientific Publishing Co., New York, 1964.*

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For Further Information

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