A Treatise on Subtropical Algebra: Midyear Report

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11 December 2005

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Chapter 1

Introduction

Subtropical arithmetic is an arithmetic system on the extended real line, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. The operations consist of the two binary operators $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$, which are defined for all $a, b \in \overline{\mathbb{R}}$ in the usual way. We shall refer to the abstract study of this arithmetic as *subtropical algebra*.

The reason for extending $\mathbb{R}$ to $\overline{\mathbb{R}}$ is completely technical. By adding ∞, we gain an identity with respect to the operation $\wedge$, since for all $a \in \overline{\mathbb{R}}$, $a \wedge \infty = \infty \wedge a = a$. Similarly, by appending $-\infty$ we gain an identity with respect to $\vee$. We note, however, that this structure does not afford any inverses with respect to $\wedge$ or $\vee$, since the only $a, b \in \overline{\mathbb{R}}$ such that $a \wedge b = \infty$ are $a = b = \infty$, and similarly $a \vee b = -\infty$ implies $a = b = -\infty$.

Example 1.1 $5 \vee 9 = 9,\ 3 \wedge 6 = 3.$

The adjective "subtropical" is derived from the use of "tropical" in the study of *tropical algebra*. Our motivation for studying the subtropical arithmetic system is primarily game-theoretic, as many of the most celebrated results in game theory come to us in the form of statements about minimums of maximums or maximums of minimums. However, it seems likely that subtropical research could prove useful to mathematicians working in algebraic geometry and other fields.

Since much of our study of subtropical algebra will greatly mirror that of tropical algebra, we first turn our focus there.

Chapter 2

Tropical Mathematics

2.1 Survey

The term "tropical" is an homage to the Brazilian mathematician, Imre Simon, one of the first mathematicians to work in the field (8). In recent years, the study of the tropical semiring has seen resurgence due to applications to the study of algebraic geometry, plane curves, combinatorics, phylogenetic trees, and various other fields. The tropical semiring $(\mathbb{T}, \oplus, \odot)$ has two equivalent variants, $(\mathbb{R} \cup \{\infty\}, \min, +)$ and $(\mathbb{R} \cup \{-\infty\}, \max, +)$, the inclusion of ∞ or $-\infty$ serving to provide an identity for the $\oplus$ operation min or max, respectively. In yet another version (3), the tropical ring is extended further to include an identical copy of $\mathbb{R}$ whose elements interact with those in the original copy under tropical operations. For the remainder of this paper, we will assume the unextended variation of tropical arithmetic where addition is defined as minimum.

Example 2.1 $5 \otimes 9 = 14$, $3 \oplus 6 = 3$.

Through the work of Bernd Sturmfels and many others, coherent notions of many traditional algebraic objects have been developed in a tropical setting. These include polynomials (8), (4), linear spaces (9), varieties (5), (3), ideals (3), matrices and their rank (2), (1), Nullstellensatz (6), (3), and Grassmannians (7). Perhaps most oddly of all, we have also seen the development of a tropical geometry (5), which has applications to the study of genomics.

As one of our goals will be mimicking some of these developments in a subtropical setting, we begin by examining tropical polynomials. Defined analogously to traditional algebraic polynomials, the tropical polynomials in one variable are of the form

$$p(x) = \bigoplus_{k=0}^{n} a_k \odot x^k, \text{ with } a_i \in \mathbb{T},$$

where exponentiation is defined tropically. For the sake of translation, in a traditional algebraic setting this corresponds to

$$p(x) = \min\{nx + a_n, (n-1)x + a_{n-1}, \ldots, x + a_1, a_0\}. \tag{2.1}$$

The graph of a tropical polynomial then looks like the minimum of a sequence of lines of decreasing, nonnegative integer slope.

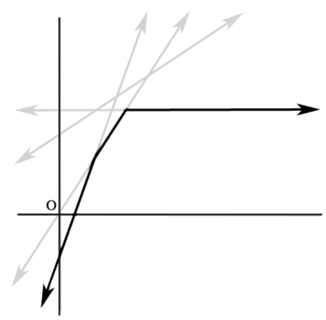

Figure 2.1: An example graph of a tropical polynomial.

In considering the roots of a tropical polynomial, it no longer makes much sense to consider the values of x where the expression achieves zero. First we remark that zero lacks the property $0 \odot a = 0$ for all $a \in \mathbb{T}$. This was the property that made roots of polynomials also roots of a product polynomial under multiplication in a traditional algebraic setting, but it no longer applies tropically. Instead we desire to define a root as something that propagates through tropical multiplication such that each tropically factorable polynomial has as its roots the roots of its divisor polynomials.

The distinguishing features of the tropical polynomial in Figure 2.1 are decidedly the "kinks" in the graph where the lines meet. Noting that if we were to add one graph of this form to another graph of this form in a traditional algebraic setting, the kinks of each graph would be preserved. This is due to the fact that the slopes of the lines defining the graph are strictly decreasing, since each kink is provided no means to "cancel" by tropical multiplication. At any given kink, multiplication by another tropical polynomial will only force the slopes on either side have the same or an even greater difference. Thus, the kinks of a tropical polynomial are preserved under tropical multiplication. It is with this in mind that the roots of the tropical polynomial $p(x) = \bigoplus_{k=0}^{n} a_k \odot x^k$ are defined as follows.

Definition 2.2 *The roots of a polynomial $p(x)$ are defined to be all $r \in \mathbb{T}$ where*

$$a_i \odot r^i = a_j \odot r^j = \min_{1 \leq k \leq n} \{a_k \odot r^k\} \quad \textit{for some } i \textit{ and } j \textit{ such that } 0 \leq i < j \leq n.$$

In other words, the roots are the values where the minimum in 2.1 are achieved at least twice. Since these values r are precisely the places where the lines $nx + a_n, (n-1)x + a_{n-1}, \ldots, x + a_1, a_0$ pairwise intersect and fall on the graph of $p(x)$, the roots are the kinks on the tropical polynomial, as hinted. The polynomials then have a delightfully familiar property.

Proposition 2.3 *Given any collection of n distinct roots $r_1 > \ldots > r_n$, there exists a tropical polynomial of degree n with these roots.*

Proof: Consider $p(x) = \bigodot_{i=1}^{n} (x \oplus r_i)$. ■
This is equivalent to expanding a bunch of linear terms, each of which we prepare in such a way that it gives the overall polynomial a particular desired root. The next statement is a bit stronger, but would require a stretch of the imagination to make analogous to traditional algebraic polynomials.

Proposition 2.4 *Given a collection of $n+1$ distinct natural numbers $k_0 < \ldots < k_n$ and n distinct roots $r_1 > \ldots > r_n$, we may construct a polynomial with root a_i occurring as the kink between line segments of slope k_{i-1} and k_i.*

Proof: Such a polynomial is given by

$$\bigoplus_{i=1}^{n} s_i \odot x^{k_i},$$

where

$$s_j = c + \sum_{i=1}^{j} a_i (k_{i-1} - k_i), \text{ with } c \in \mathbb{T}.$$

■

The constant c here merely shifts the graph vertically, leaving the roots and slopes untouched. This can be thought of as an analogue to multiplying a real polynomial by a nonzero constant.

It is not too much of a leap to see that this system of designating roots and slopes can be used to produce the curve defined by any polynomial, since such a curve is a piecewise combination of segments of decreasing nonnegative integer slope, intersecting at precisely the roots.

We next note that two tropical polynomials may describe precisely the same curve in the plane.

Example 2.5 *Consider the polynomials* $f(x) = x^2 \oplus 1 \odot x \oplus 2$ *and* $g(x) = x^2 \oplus 2 \odot x \oplus 2$*. These describe the same curve, since the* x^1 *term is greater than at least one of the* x^0 *or* x^2 *terms for all* x *in both* f *and* g*. Since the* x^0 *and* x^2 *terms then define the polynomials and are identical in* f *and* g*,* f *and* g *are identical.*

We can then create equivalence classes of polynomial expressions based on equality. We might note an interesting fact about our equivalent polynomials f and g in 2.5: f is easily factored while g is not. With a little tropical manipulation we can force $f(x) = x^2 \oplus 1 \odot x \oplus 2 = (x \oplus 1) \odot (x \oplus 1)$.

Chapter 3

Fundamentals of Subtropical Algebra

3.1 Commutativity, Associativity, and Bidistributivity

Perhaps the easiest observation to make about the operations $\wedge$ and $\vee$ is that each is commutative and associative. Furthermore, each is distributive over the other, a property we shall call bidistributivity. While the commutative and associative properties are straightforward to see, the proof of bidistributivity is a little less intuitive. In terms of our operations the proof is a little cumbersome, so in order to more elegantly demonstrate bidistributivity we shall reformulate $\wedge$ and $\vee$ in terms of an analogue to Dedekind cuts. For each $r \in \overline{\mathbb{R}}$, let the map $\varphi : \overline{\mathbb{R}} \to 2^{\overline{\mathbb{R}}}$ be defined by $\varphi(r) = [-\infty, r]$. We then note that for $a, b \in \overline{\mathbb{R}}$,

$$\varphi(a \wedge b) = \varphi(a) \cap \varphi(b) \quad \text{and} \quad \varphi(a \vee b) = \varphi(a) \cup \varphi(b).$$

Since $\varphi(a) \cap \varphi(b), \varphi(a) \cup \varphi(b) \in \varphi(\overline{\mathbb{R}})$ and φ is injective, we can then safely write

$$a \wedge b = \varphi^{-1}(\varphi(a) \cap \varphi(b)) \quad \text{and} \quad a \vee b = \varphi^{-1}(\varphi(a) \cup \varphi(b)).$$

Proposition 3.1 *The operations $\wedge$ and $\vee$ are bidistributive.*

Proof: Let $a, b, c \in \overline{\mathbb{R}}$. We then observe that since $\cap$ and $\cup$ are bidistributive,

$$\begin{aligned} a \wedge (b \vee c) &= \varphi^{-1}(\varphi(a) \cap (\varphi(b) \cup \varphi(c))) \\ &= \varphi^{-1}((\varphi(a) \cap \varphi(b)) \cup (\varphi(a) \cap \varphi(c))) \\ &= \varphi^{-1}(\varphi(a \wedge b) \cup \varphi(a \wedge c)) \\ &= (a \wedge b) \vee (a \wedge c). \end{aligned}$$

Thus, $\wedge$ distributes over $\vee$. The proof of distributivity of $\vee$ over $\wedge$ is similar. ■

It is interesting to note that we could have also defined $\psi : \overline{\mathbb{R}} \to \mathcal{P}(\overline{\mathbb{R}})$ by $\psi(r) = [r, \infty]$ and observed that

$$a \wedge b = \psi^{-1}(\psi(a) \cup \psi(b)) \quad \text{and} \quad a \vee b = \psi^{-1}(\psi(a) \cap \psi(b)).$$

Since φ allowed us to define $\wedge$ in terms of $\cap$ and $\vee$ in terms of $\cup$ while ψ associates $\wedge$ to $\cup$ and $\vee$ to $\cap$, there is an inherent symmetry to our operations that was not present in traditional or tropical algebra. This is a more heuristic way to see that distributivity of one operation over the other should imply bidistributivity.

3.2 Subtropical Polynomials

Consider a polynomial in $(\mathbb{R}, +, \cdot)$. It has the form $p(x) = \sum_{i=0}^{n} a_i \cdot x^i$. Polynomials in $(\mathbb{R}, +, \cdot)$ have many useful characteristics that we can ascribe to them such as degree, roots, and factorings. In tropical algebra, $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, there is the analogous form for a polynomial $p(x) = \bigoplus_{i=0}^{n} a_i \odot x^i$ that also has useful notions of degree, roots, and factorings defined for it. We now wish to define something similar in $(\overline{\mathbb{R}}, \wedge, \vee)$. Note that because of the bidistributivity of $\wedge$ and $\vee$, our allowing $\vee$ to fill the role of multiplication and $\wedge$ to fill the role of addition is arbitrary.

We start by considering the basic form for a polynomial suggested by the other two considered systems,

$$p(x) = \bigwedge_{i=0}^{n} a_i \vee x^i \text{ for } a_i \in \overline{\mathbb{R}}, n \in \mathbb{N},$$

with exponentiation interpreted subtropically: $x^k = \bigvee_{i=1}^{k} x = x$ for $k \geq 1$. We shall use the natural definition $x^0 = -\infty$, since the $-\infty$ is the identity with respect to $\wedge$. Let P denote the set of all functions of the form $p(x)$

Proposition 3.2 *Any element of P may be expressed as $p(x) = (a \vee x) \wedge b$ for $a, b \in \overline{\mathbb{R}}$.*

Proof: Following the traditional order of operations allows us to reduce our polynomial to $p(x) = a_0 \wedge \bigwedge_{i=1}^{n}(a_i \vee x)$. By distributivity we then have $p(x) = a_0 \wedge (x \vee \bigwedge_{i=1}^{n} a_i)$. By letting $a = \bigwedge_{i=1}^{n} a_i$ and $b = a_0$, we then have $p(x) = (a \vee x) \wedge b$ for $a, b \in \overline{\mathbb{R}}$. ■

Using the form given for a polynomial by Proposition 3.2, we may then classify each polynomial $p(x)$ by considering the cases $a < b$, $a = b$, and $a > b$. Since we're clever, we can handle it in two cases:

- If $a < b$, then
$$p(x) = \begin{cases} a & \text{if} \quad x \in [-\infty, a] \\ x & \text{if} \quad x \in (a, b) \\ b & \text{if} \quad x \in [b, \infty] \end{cases}$$
- If $a \geq b$, then $p(x) = b$.

We graph the more interesting of the two forms in Figure 3.1.

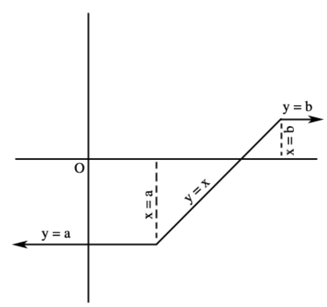

Figure 3.1: $p(x) = (a \vee x) \wedge b$ for $a < b$.

If we wish to develop any useful notion of a root, we would likely want roots to somehow correspond to the kinks in the graph. However, the above classification of all $p(x) \in P$ gives that each polynomial has either zero or two kinks. Since the product or sum of two polynomials again has zero or two kinks, there does not seem to be a useful notion of degree or root to be found this way, shedding doubt on this definition of a subtropical polynomial. Before we finish discussion them, however, we shall demonstrate the closure of P with respect to subtropical operations.

Proposition 3.3 *P is closed under $\wedge$.*

Proof: Let $f(x), g(x) \in P$. By Proposition 3.2 we are allowed to write $f(x) = (a \wedge x) \vee b, g(x) = (c \wedge x) \vee d$ for some $a, b, c, d \in \overline{\mathbb{R}}$. We then observe that

$$\begin{aligned} f(x) \wedge g(x) &= [(a \wedge x) \vee b] \wedge [(c \wedge x) \vee d] \\ &= [(a \wedge x) \wedge (c \wedge x)] \vee [(a \wedge x) \wedge d] \vee [(c \wedge x) \wedge b] \vee [b \wedge d] \\ &= [a \wedge c \wedge x] \vee [a \wedge d \wedge x] \vee [b \wedge c \wedge x] \vee [b \wedge d] \\ &= ([[a \wedge c] \vee [a \wedge d] \vee [b \wedge c]] \wedge x) \vee [b \wedge d]. \end{aligned}$$

Letting $r = [[a \wedge c] \vee [a \wedge d] \vee [b \wedge c]]$ and $s = [b \wedge d]$, we have that $f(x) \wedge g(x) = (r \wedge x) \vee s$ for $r, s \in \overline{\mathbb{R}}$, so $f(x) \wedge g(x) \in P$. ■

Proposition 3.4 *P is closed under* $\vee$.

Proof: Let $f(x), g(x) \in P$. We may again write $f(x) = (a \wedge x) \vee b$ and $g(x) = (c \wedge x) \vee d$ for some $a, b, c, d \in \overline{\mathbb{R}}$. It follows that

$$\begin{aligned} f(x) \wedge g(x) &= [(a \wedge x) \vee b] \vee [(c \wedge x) \vee d] \\ &= (a \wedge x) \vee (c \wedge x) \vee b \vee d \\ &= ([a \vee c] \wedge x) \vee [b \vee d]. \end{aligned}$$

Letting $r = [a \vee c]$ and $s = [b \vee d]$, we have that $f(x) \vee g(x) = (r \wedge x) \vee s$ for $r, s \in \overline{\mathbb{R}}$, so $f(x) \vee g(x) \in P$. ■

While closure under both operations is very much a desired property for any potential definition of a subtropical polynomial, we still cannot help but feel that the polynomials afforded by analogy to traditional and tropical algebras are in some sense "too weak," since useful notions of degree and root are very elusive. An alternative form we will study later takes the form

$$q(x) = a_0 \wedge \bigwedge_{i=1}^{n} a_i \vee (x + b_i) \text{ for } a_i, b_i \in \overline{\mathbb{R}}.$$

The reason for this form is purely because it proves more interesting under analysis.

Chapter 4

Subtropical Geometry

One of the things we wish to establish within the context of Subtropical Algebra is a suitable geometry. A natural place to start is with the question of what defines a subtropical space.

4.1 Subtropical Spaces

As noted earlier, there are no additive inverses in the subtropical semiring. Because of this, we cannot hope to have a subtropical vector space in any natural way. However, if we choose to ignore this, the subtropical space contructed in analogue to the vector space $\mathbb{R}^n$ may have some interesting properties.

We define the space $\overline{\mathbb{R}}^n$ to be all n-component vectors with entries chosen from $\overline{\mathbb{R}}$. We define addition and scaling as below.

Example 4.1

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 \wedge b_1 \\ \vdots \\ a_n \wedge b_n \end{pmatrix}, \qquad k \vee \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} k \vee a_1 \\ \vdots \\ k \vee a_n \end{pmatrix} \textit{ for all } a_i, b_i, k \in \overline{\mathbb{R}}$$

However, due to the symmetry of our space it might also make sense to define the vector operations

Example 4.2

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \vee \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 \vee b_1 \\ \vdots \\ a_n \vee b_n \end{pmatrix}, \qquad k \wedge \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} k \wedge a_1 \\ \vdots \\ k \wedge a_n \end{pmatrix} \textit{ for all } a_i, b_i, k \in \overline{\mathbb{R}}$$

We begin by considering the notion of an inner product. Constructing the inner product as an analogue to the standard inner product on $\mathbb{R}^n$, we might define

$$\vec{a} \curlyvee \vec{b} = \bigwedge_{i=1}^{n} (a_i \vee b_i).$$

We note that if $a, b \in \overline{\mathbb{R}}$, we simply have $a \curlyvee b = a \vee b$, giving some rationale for our notation.

We next verify that this is almost an inner product, with "almost" in the sense that $\overline{\mathbb{R}}^n$ is almost a vector space.

Proposition 4.3 *$\curlyvee$ is almost an inner product on $\overline{\mathbb{R}}^n$.*

Proof:

Let $\vec{a}, \vec{b}, \vec{c} \in \overline{\mathbb{R}}^n$.

(i) Clearly $\vec{a} \curlyvee \vec{b} = \vec{b} \curlyvee \vec{a}$.

(ii) Define $\vec{a} \wedge \vec{b}$ componentwise. We then have that

$$\begin{aligned}
(\vec{a} \wedge \vec{b}) \curlyvee \vec{c} &= \left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) \curlyvee \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\
&= \begin{pmatrix} a_1 \wedge b_1 \\ \vdots \\ a_n \wedge b_n \end{pmatrix} \curlyvee \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\
&= \bigwedge_{i=1}^{n} (a_i \wedge b_i) \vee c_i \\
&= \bigwedge_{i=1}^{n} ((a_i \vee c_i) \wedge (b_i \vee c_i)) \\
&= \left(\bigwedge_{i=1}^{n} (a_i \vee c_i) \right) \wedge \left(\bigwedge_{i=1}^{n} (b_i \vee c_i) \right) \\
&= (\vec{a} \curlyvee \vec{c}) \wedge (\vec{b} \curlyvee \vec{c}).
\end{aligned}$$

(iii) Let $k \in \mathbb{T}$. Defining $k \max \vec{a}$ in the natural way one would define scalar action, we then observe

$$\begin{aligned}
(k \vee \vec{a}) \curlyvee \vec{b} &= \left(k \vee \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \curlyvee \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) \curlyvee \\
&= \left(\begin{pmatrix} k \vee a_1 \\ \vdots \\ k \vee a_n \end{pmatrix} \curlyvee \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) \curlyvee \\
&= \bigwedge_{i=1}^{n} (k \vee a_i) \vee b_i \\
&= k \vee \left(\bigwedge_{i=1}^{n} a_i \vee b_i \right) \\
&= k \vee (\vec{a} \curlyvee \vec{b}).
\end{aligned}$$

(iv) This is the axiom that slightly breaks down. Given that the subtropical additive identity is ∞, demanding $\vec{a} \curlyvee \vec{a} \geq \infty$ and $\vec{a} \curlyvee \vec{a} = \infty$ iff $\vec{a} = \vec{\infty}$ seems ridiculously. Instead, we might demand $\vec{a} \curlyvee \vec{a} \leq \infty$ and $\vec{a} \curlyvee \vec{a} = \infty$ iff $\vec{a} = \vec{\infty}$. While the former of these two requirements is trivially true, to see the latter we note that

$$\vec{a} \curlyvee \vec{a} = \bigwedge_{i=1}^{n} (a_i \vee a_i) = \bigwedge_{i=1}^{n} a_i,$$

which may achieve ∞ iff each a_i, and thus iff $\vec{a} = \vec{\infty}$.

■

The key hangup in this was that zero no longer holds the same significant in the subtropical setting that it did in traditional arithmetic. "Positive" and "negative" no longer have much significant meaning, since subtropically all numbers between $-\infty$ and ∞ are in some sense equivalent in their relations, up to translation.

Example 4.4

$$\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \curlyvee \begin{pmatrix} 7 \\ 1 \\ 9 \end{pmatrix} = (2 \vee 7) \wedge (5 \vee 1) \wedge (3 \vee 9) = 7 \wedge 5 \wedge 9 = 5.$$

Due to the symmetry of $\vee$ and $\wedge$, however, we note that it might make just as much sense to define the inner sum

$$\vec{a} \curlywedge \vec{b} = \bigvee_{i=1}^{n} (a_i \wedge b_i).$$

We note similarly that if $a, b \in \overline{\mathbb{R}}$, we have $a \curlywedge b = a \wedge b$. We may also prove that $\curlywedge$ has all the desired properties of an inner product if we were to reverse the roles of multiplication and addition. The proof would go as in Proposition 4.3, except we would demonstrate

(i) $\vec{a} \curlywedge \vec{b} = \vec{b} \curlywedge \vec{a}$,

(ii) $(\vec{a} \vee \vec{b}) \curlywedge \vec{c} = (\vec{a} \curlywedge \vec{c}) \vee (\vec{b} \curlywedge \vec{c})$,

(iii) $(k \wedge \vec{a}) \curlywedge \vec{b} = k \wedge (\vec{a} \curlywedge \vec{b})$,

(iv) $\vec{a} \curlywedge \vec{a} \geq -\infty$ and $\vec{a} \curlywedge \vec{a} = -\infty$ iff $\vec{a} = \vec{-\infty}$.

Since we must always have $a \wedge b \leq a \vee b$, we might suspect that $\vec{a} \curlywedge \vec{b} \leq \vec{a} \curlyvee \vec{b}$. For the case where $\vec{a}, \vec{b} \in \overline{\mathbb{R}}$, this obviously holds. However, the next example in $\overline{\mathbb{R}}^2$ shows that we may have $\vec{a} \curlywedge \vec{b} > \vec{a} \curlyvee \vec{b}$.

Example 4.5

$$\begin{pmatrix}1\\3\end{pmatrix} \curlywedge \begin{pmatrix}2\\4\end{pmatrix} = (1 \wedge 2) \vee (3 \wedge 4) = 1 \vee 3 = 3$$

and

$$\begin{pmatrix}1\\3\end{pmatrix} \curlyvee \begin{pmatrix}2\\4\end{pmatrix} = (1 \vee 2) \wedge (3 \vee 4) = 2 \wedge 4 = 2,$$

so

$$\begin{pmatrix}1\\3\end{pmatrix} \curlywedge \begin{pmatrix}2\\4\end{pmatrix} > \begin{pmatrix}1\\3\end{pmatrix} \curlyvee \begin{pmatrix}2\\4\end{pmatrix}.$$

4.2 Subtropical Lines (Euclidean Analogy)

In developing our subtropical geometry, we might next attempt to develop the notion of a line. Since the easiest way to start is by analogy to known arithmetics, we begin by considering analogues to standard Euclidean geometry. Analytically, we are accustomed to seeing lines expressed in standard form: $ax + by + c = 0$. With this expression we may modify a, b, c appropriately to describe any line in the plane. However, the presence of additive inverses is what allows for this canonical representation and allows us to equate the line $5x + 3y + 9 = 0$ with the line $10x + 5y + 4 = 5x + 2y - 5$. Due to a lack of subtropical additive inverses, we might then guess that the form for a line would be equations of the form

$$(a \vee x) \wedge (b \vee y) \wedge c = (d \vee x) \wedge (e \vee y) \wedge f.$$

We shall now explore this system thoroughly. We begin by examining the left side.

To induce a symmetry for the sake of generality, we will instead consider

$$(a \vee x) \wedge (b \vee y) \wedge (c \vee z) \quad \text{for } a \leq b \leq c.$$

Since the names of the variables are arbitrary, we will homogenize the system this way and hold whichever variable corresponds to the "constant term" at a constant value of $-\infty$. For example, the expression $(5 \vee u) \wedge (3 \vee v) \wedge 4$ corresponds to $(3 \vee x) \wedge (4 \vee y) \wedge (5 \vee z)$ with $3 \leq 4 \leq 5$ and $y = -\infty$.

In considering $L = L(x, y, z) = (a \vee x) \wedge (b \vee y) \wedge (c \vee z)$ for $a \leq b \leq c$, we will start by examining cases.

- If $x \leq a$, then we have $L = a$, since $(a \vee x) = a$ and each of $(b \vee y)$ and $(c \vee z)$ is at least a.
- If $a \leq x \leq b$, then we have $L = x$, since $(a \vee x) = x$ and each of $(b \vee y)$ and $(c \vee z)$ is at least x.
- If $y \leq b \leq x$, then we have $L = b$, since $(b \vee y) = b \leq x = (a \vee x)$ and $(c \vee z)$ is at least $c \geq b$.
- If $b \leq x \leq y \leq c$, then we have $L = x$, since $(a \vee x) = x$ and each of $(b \vee y)$ and $(c \vee z)$ is at least y, which is greater than or equal to x.
- If $b \leq y \leq x \leq c$, then we have $L = y$, since $(b \vee y) = y \leq x = (a \vee x)$ and $(c \vee z)$ is at least c, which is greater than or equal to y.
- If $b \leq x \leq c \leq y$, then we have $L = x$, since $(a \vee x) = x$ and each of $(b \vee y)$ and $(c \vee z)$ is at least c, which is greater than or equal to x.
- If $b \leq y \leq c \leq x$, then we have $L = y$, since $(b \vee y) = y$ and each of $(a \vee x)$ and $(c \vee z)$ is at least c, which is greater than or equal to y.

- If $z \leq c \leq x \leq y$, then $L = c$.
- If $z \leq c \leq y \leq x$, then $L = c$.
- If $c \leq z \leq x \leq y$, then $L = z$.
- If $c \leq z \leq y \leq x$, then $L = z$.
- If $c \leq x \leq z \leq y$, then $L = x$.
- If $c \leq x \leq y \leq z$, then $L = x$.
- If $c \leq y \leq z \leq x$, then $L = y$.
- If $c \leq y \leq x \leq z$, then $L = y$.

So, in summary:

$$L(x,y,z) = \begin{cases} a & \text{if} \quad x \leq a \\ b & \text{if} \quad y \leq b \leq x \\ c & \text{if} \quad z \leq c \leq x \leq y, \quad z \leq c \leq y \leq x \\ x & \text{if} \quad a \leq x \leq b, \quad b \leq x \leq y \leq c, \quad b \leq x \leq c \leq y, \\ & \quad\quad c \leq x \leq y \leq z, \quad c \leq x \leq z \leq y \\ y & \text{if} \quad b \leq y \leq x \leq c, \quad b \leq y \leq c \leq x, \\ & \quad\quad c \leq y \leq z \leq x, \quad c \leq y \leq x \leq z \\ z & \text{if} \quad c \leq z \leq x \leq y, \quad c \leq z \leq y \leq x \end{cases}$$

The right hand side of the equation is handled similarly, except we that we now no longer have the ability to treat the variables as arbitrary, since the left hand side forces properties onto the right hand side. The solution to this system of equations would then be all values $(x,y) \in \overline{\mathbb{R}}^2$ such that equality holds. We shall see with the next examples, however, that this creates a very unintuitive image of a line.

Example 4.6 $(2 \vee x) \wedge (3 \vee y) \wedge 4 = 3.$

We see that a necessary condition for equality would be either $x = 3$ a minimum, $y = 3$ a minimum, or 3 (the coefficient of y) a minimum. By our formula for $L(x,y,z)$ we see that this happens when $x = 3$ and y is any value or when $x \geq 4$ and $y \leq 3$. We draw observe this “line” below, in Figure 4.1.

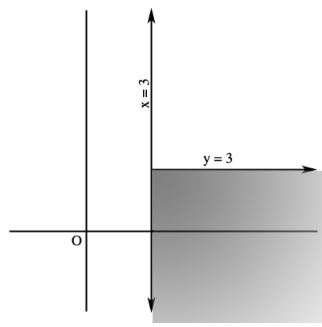

Figure 4.1: $(2 \vee x) \wedge (3 \vee y) \wedge 4 = 3$.

This "line" is actually a sheet in the plane. The next example is a little more satisfying, but simplistic in form.

Example 4.7 $x \wedge y = 1$.

The solutions to this equation are $x = 1$ and $y \geq 1$ or $y = 1$ and $x \geq 1$. It appears in Figure 4.2.

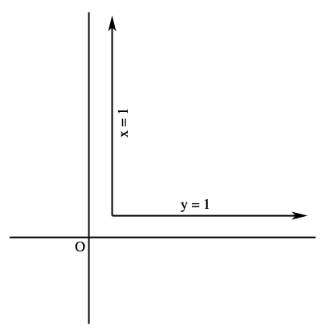

Figure 4.2: $x \wedge y = 1$.

But next, an even more degenerate "line."

Example 4.8 $x \wedge y = x \wedge 2$.

The solutions to this equation are $x \leq y$ and $x \leq 2$ or $y = 2$ and $x \geq 2$. This region can be seen in Figure 4.3.

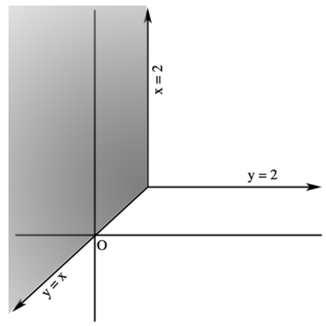

Figure 4.3: $x \wedge y = x \wedge 2$.

The “line” in Figure 4.3 is a sheet in the plane attached to a ray. While the line in Figure 4.2 nearly satisfies our geometric intuitions for what a line might be in terms of dimension, intersections, and so forth, the general form for a line as suggested by analogy to analytic geometry in a traditional algebraic setting yields far more exceptions to this intuition than it does satisfactory “lines.”

4.3 Subtropical Lines (Tropical Analogy)

As we have grown accustomed to, the next step would then be to attempt to formulate a notion of a line by analogy to tropical geometry. In tropical geometry, a line is defined in terms of ideals of tropical polynomials. Since the subtropical polynomials we explored in an earlier section never proved fruitful for most meaningful criteria of analysis, defining a subtropical line analogously to a tropical line will not yet have much rationale other than wishful thinking. In that spirit, we consider the solutions to

$$(a \vee x) \wedge (b \vee y) \wedge c, \text{ with } a, b, c \in \overline{\mathbb{R}},$$

where the minimum of $(a \vee x), (b \vee y), c$ is achieved at least twice. We note that one of three conditions must hold for this to happen:

- $(a \vee x) = (b \vee y) \leq c$,
- $(a \vee x) = c \leq (b \vee y)$, or
- $(b \vee y) = c \leq (a \vee x)$.

We begin by assuming $c = \infty$ so that we need only examine the first case. Consider the equation $a \vee x = b \vee y$, we observe that the solutions are as follows.

- If $a < b$, then we have $x = b$ and $y \leq b$ or $y \geq b$ and $y = x$.
- If $a = b$, we have $x = y \geq a$ or $x \leq a$ and $y \leq a$.
- If $a > b$, then we have $y = a$ and $x \leq a$ or $x \geq a$ and $y = x$.

We graph these cases below.

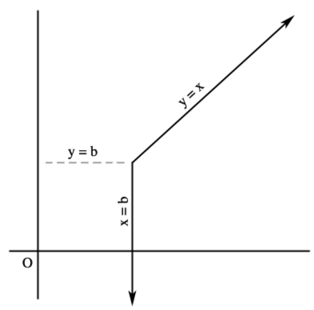

Figure 4.4: $(a \vee x) \wedge (b \vee y)$ for $a < b$.

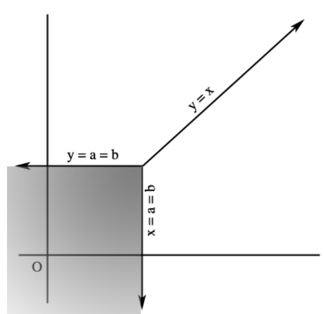

Figure 4.5: $(a \vee x) \wedge (b \vee y)$ for $a = b$.

With these lines, the only degenerate case which involves a sheet of the plane is when $a = b$. For the other lines we have the nice properties that the lines are in some sense "one-dimensional" and intersect each other never, once, or infinitely many times.

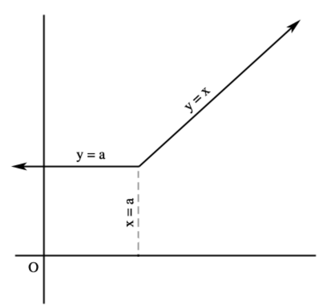

Figure 4.6: $(a \vee x) \wedge (b \vee y)$ for $a > b$.

Chapter 5

Future Work

While the notions of subtropical polynomial and linear space are not yet fully fleshed out, it is hoped that one or both will be found, leading to a richer understanding of the other and other possible subtropical objects. Our primary goal for the immediate future is the development of the following:

- Extension of the inner product and inner sum to matrices.
- Subtropical convexity.
- Subtropical metrics and arc-lengths.

Above all, once these objects and structures have been successfully defined and detailed, it is our hope to give them mathematical interpretations such that they might become useful objects in the study of other mathematical or scientific fields.

Bibliography

[1] M. Develin, F. Santos, and B. Sturmfels, *On the rank of a tropical matrix*, In "Discrete and Computational Geometry" (E. Goodman, J. Pach and E. Welzl, eds), MSRI Publications, Cambridge Univ. Press, 2005. ISBN-10: 0521848628.

[2] Zur Izhakian, *Tropical Arithmetic & Algebra of Tropical Matrices*.

[3] ______, *Tropical Varieties, Ideals and An Algebraic Nullstellensatz*.

[4] Ki Hang Kim and Fred W. Roush, *Factorization of polynomials in one variable over the tropical semiring*.

[5] Jürgen Richter-Gebert, Bernd Sturmfels, and Thorsten Theobald, *First steps in tropical geometry*.

[6] Eugenii Shustin and Zur Izhakian, *A tropical Nullstellensatz*.

[7] David Speyer and Bernd Sturmfels, *The Tropical Grassmannian*.

[8] ______, *Tropical Mathematics*.

[9] David E Speyer, *Tropical Linear Spaces*.