Cesaro Limits of Analytically Perturbed Stochastic Matrices

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Overview

- Motivating example
- Definitions
- Main problem
- Eigenvalues
- Results

Motivating example

The peculiar case of Roland the hot dog street vendor

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$$p_{n+1}(1) = (0.5 + \varepsilon)p_n(1) + (0.5 - 2\varepsilon)p_n(2)$$
$$p_{n+1}(2) = (0.5 - \varepsilon)p_n(1) + (0.5 + 2\varepsilon)p_n(2)$$

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or . . .

$$\begin{bmatrix} p_{n+1}(1) & p_{n+1}(2) \end{bmatrix} = \begin{bmatrix} p_n(1) & p_n(2) \end{bmatrix} \begin{bmatrix} 0.5 + \varepsilon & 0.5 - \varepsilon \\ 0.5 - 2\varepsilon & 0.5 + 2\varepsilon \end{bmatrix}$$

Motivating example (cont.)

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From the previous recursive relationship,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \begin{bmatrix} p_k(1) & p_k(2) \end{bmatrix} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5+\varepsilon & 0.5-\varepsilon \\ 0.5-2\varepsilon & 0.5+2\varepsilon \end{bmatrix}^k$$

Motivating example (cont.)

$$P^* = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \begin{bmatrix} 0.5+\varepsilon & 0.5-\varepsilon \\ 0.5-2\varepsilon & 0.5+2\varepsilon \end{bmatrix}^k$$

$$= \frac{1}{1-3\varepsilon} \begin{bmatrix} 0.5-2\varepsilon & 0.5-\varepsilon \\ 0.5-2\varepsilon & 0.5-\varepsilon \end{bmatrix}$$

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Roland’s long-term average daily earnings are thus

$$\frac{0.5-2\varepsilon}{1-3\varepsilon} \cdot 90 + \frac{0.5-\varepsilon}{1-3\varepsilon} \cdot 100 = 95 + \frac{5\varepsilon}{1-3\varepsilon}$$

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What would happen if we let $\varepsilon \downarrow 0$ and $N \to \infty$ simultaneously?

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An *analytic perturbation* of a matrix $T_0 \in \mathbb{C}^{n\times n}$ is a power series

$$T(\varepsilon) = T_0 + A(\varepsilon) = T_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots$$

in which the “coefficients” A_1, A_2, $\ldots$ are in $\mathbb{C}^{n\times n}$ as well.

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We are interested in the hybrid Cesaro limit

$$\lim_{\varepsilon \downarrow 0} \frac{1}{N(\varepsilon)} \sum_{k=1}^{N(\varepsilon)} P^k(\varepsilon),$$

where $N(\varepsilon) \uparrow \infty$ as $\varepsilon \downarrow 0$.

Foundation for My Thesis

In 2002, Filar, Krieger, and Syed characterized the hybrid Cesaro limit when P_0 has no eigenvalues λ satisfying $|\lambda| = 1$ except for $\lambda = 1$.

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- Each eigenvalue λ of P_0 has a separate contribution to the limit.
- If $|\lambda| < 1$, this contribution is always equal to 0.
- If $\lambda = 1$, the contribution depends on the rate at which $N(\varepsilon) \uparrow \infty$.

Dependence of Limit on $N(\varepsilon)$

If

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$$\lim_{\varepsilon \downarrow 0} \frac{1}{N(\varepsilon)} \sum_{k=1}^{N(\varepsilon)} P^k(\varepsilon) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \frac{1-e^{2L}}{2L} \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

Perturbed eigenvalues

If $T(\varepsilon) = T_0 + A(\varepsilon)$ and λ is an eigenvalue of T_0, then $T(\varepsilon)$ has a collection of eigenvalues $\lambda_1(\varepsilon)$, $\lambda_2(\varepsilon)$, $\ldots$, $\lambda_s(\varepsilon)$ that converge to λ as $\varepsilon \to 0$.

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Each $\lambda_j(\varepsilon)$ has a *Puiseux series*

$$\lambda_j(\varepsilon) = \lambda + c_{1,j}\varepsilon^{1/p_j} + c_{2,j}\varepsilon^{2/p_j} + \cdots$$

for some positive integer p_j and complex numbers $c_{1,j}$, $c_{2,j}$, ...

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λ is *completely reducible* for $T(\varepsilon)$ if 0 is reducible for $T_0(\varepsilon) = T(\varepsilon) - \lambda I$ and, inductively, 0 is reducible for $T_i(\varepsilon)$, where $T_i(\varepsilon)$ is obtained by reducing $T_{i-1}(\varepsilon)$ for 0.

More on eigenvalues

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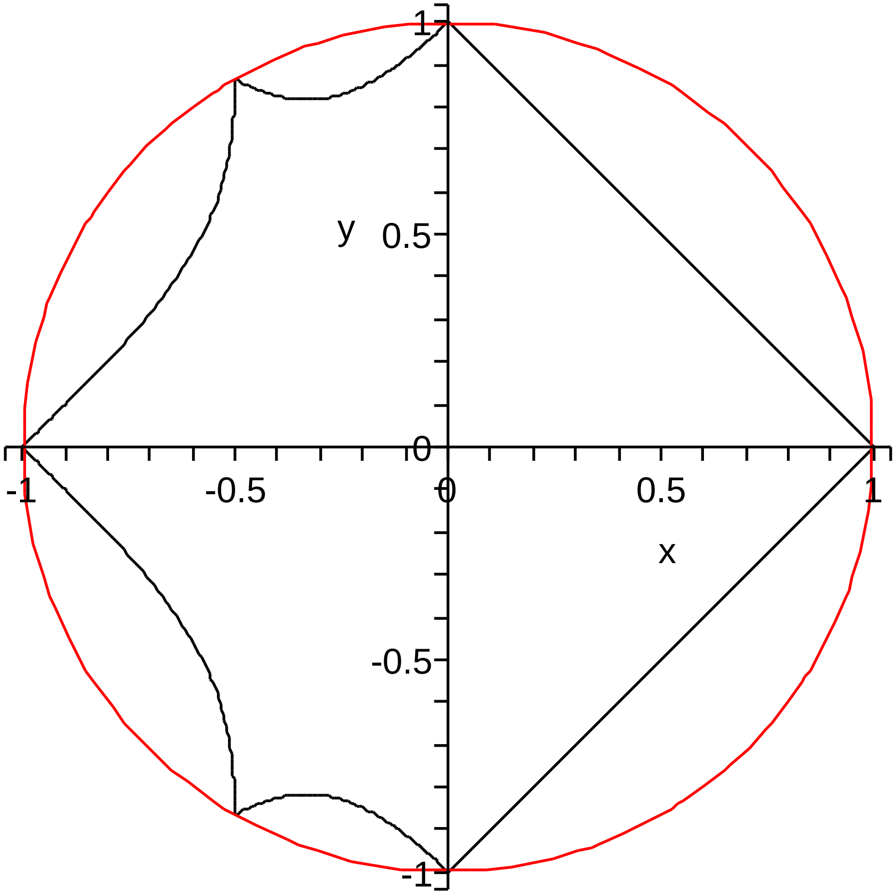
- All kth roots of unity, where $k \leq n$
- Curvilinear arcs connecting consecutive roots of unity
- Each arc implicitly parametrized in t by one of the following equations:

$$z^q(z^p - t)^r = (1-t)^r$$

$$(z^b - t)^d = (1-t)^d z^q$$

Region for $n = 4$



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Results

- Curvilinear boundary arcs make nonzero angles with lines tangent to the unit circle
- If $\lambda(\varepsilon)$ is a λ-group eigenvalue of $P(\varepsilon)$ where $|\lambda| = 1$, then the direction of approach of $\lambda(\varepsilon)$ to λ has a nonzero radial component.
- If $\lambda \neq 1$ is a completely reducible unit-circle eigenvalue of $P(\varepsilon)$, its contribution to the hybrid Cesaro limit is 0.