Cesaro Limits of Analytically Perturbed Stochastic Matrices

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Markov Chains

The theory of discrete-time Markov chains on a finite state space has a wide array of applications in modeling everything from credit ratings to population genetics.

A particular Markov chain can be represented by a stochastic matrix containing the transition probabilities, as follows:

$$P = \begin{bmatrix} 0.5 & 0.1 & 0.4 \\ 0.2 & 0.6 & 0.2 \\ 1 & 0 & 0 \end{bmatrix}.$$

Information about the long-term behavior of such a system (that is, the relative frequency it inhabits each state) is encoded in the *Cesaro limit*

$$P^* = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P^k.$$

Imperfect Information and a New Type of Limit

In actual situations, the precise transition probabilities for a system may not be known exactly (for example if they are estimated from data). In this case we may have a sort of transition matrix

$$P(\varepsilon) = P_0 + A(\varepsilon)$$
$$= P_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots$$

where P_0 is the actual transition matrix for the system and $A(\varepsilon)$ is an error term that depends upon the parameter ε. We require that $P(\varepsilon)$ be stochastic for all sufficiently small positive ε. For such *analytically perturbed* stochastic matrices, we can investigate a hybrid Cesaro limit similar to that above:

$$\lim_{\varepsilon \downarrow 0} \frac{1}{N(\varepsilon)} \sum_{k=1}^{N(\varepsilon)} P^k(\varepsilon).$$

Here $N(\varepsilon)$ increases to ∞ as ε decreases to 0. In particular, we are concerned with cases where the unperturbed stochastic matrix P_0 has eigenvalues on the unit circle in the complex plane other than 1 (in such cases, the associated system exhibits periodic behavior).

Previous Results

In 2002, Filar, Krieger, and Syed characterized this hybrid Cesaro limit when P_0 has no eigenvalues on the unit circle other than 1. In particular, they showed that the limit always exists, but that it can depend on the rate at which $N(\varepsilon) \uparrow \infty$. For example, letting

$$P(\varepsilon) = \begin{bmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{bmatrix},$$

then three cases arise:

- if $N(\varepsilon)\,\varepsilon \to \infty$ as $\varepsilon \downarrow 0$, then the hybrid Cesaro limit is

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix};$$

- if $N(\varepsilon)\,\varepsilon \to 0$ as $\varepsilon \downarrow 0$, the hybrid Cesaro limit is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

- and if $N(\varepsilon)\,\varepsilon \to L$ as $\varepsilon \downarrow 0$, where $0 < L < \infty$, the limit is

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \frac{1-e^{-2L}}{2L} \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

Eigenvalues of Perturbed Matrices

For any analytically perturbed stochastic matrix $P(\varepsilon)$, its eigenvalues vary continuously with ε and hence approach the eigenvalues of P_0 as $\varepsilon \to 0$ (in fact, this is true for all analytically perturbed matrices, not just stochastic ones). For instance, in the example above the eigenvalues of $P(\varepsilon)$ are

$$\lambda_1(\varepsilon) = 1, \qquad \lambda_2(\varepsilon) = 1 - 2\varepsilon,$$

both of which approach 1, the eigenvalue of P_0, the identity matrix, as $\varepsilon \to 0$.

The eigenvalues of a stochastic matrix, as it turns out, always lie within the closed unit disc in the complex plane. Hence, for an analytically perturbed stochastic matrix, the perturbed eigenvalues will trace out curves within the unit disc as $\varepsilon \downarrow 0$.

The results obtained by Filar, Krieger, and Syed relied on showing that for perturbed eigenvalues approaching 1, these curves are bounded away from the unit circle in a sense; specifically, these curves cannot be tangent to the unit circle at 1.

unit circle

The green line is a possible eigenvalue approach curve; the blue curve is not.
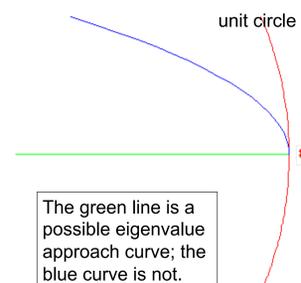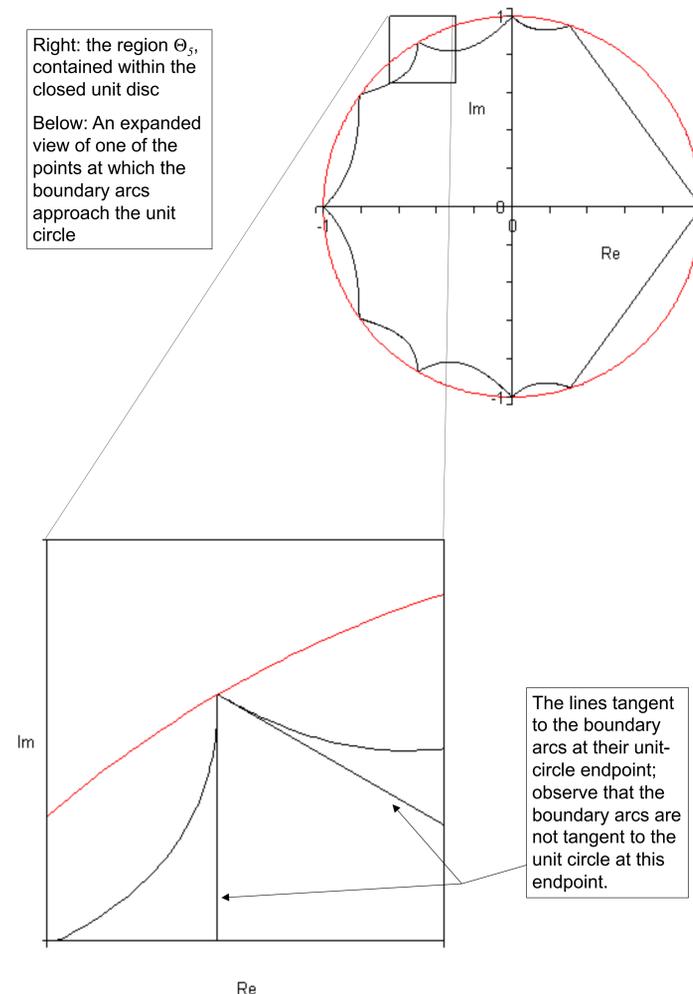
More on Eigenvalues

In 1951, Karpelevic obtained a general characterization of the set of complex numbers that are the eigenvalue of some $n \times n$ stochastic matrix for each individual n (this set or region is denoted by Θ_n).

The boundary of Θ_n consists of all the kth roots of unity, $k \leq n$, together with curved arcs connecting these roots of unity in circular order. Each curved arc is implicitly parametrized by an equation of one of the two following forms:

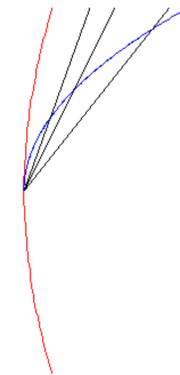$$z^q(z^p - t)^r = (1-t)^r$$
$$(z^b - t)^d = (1-t)^d z^q$$

Here the parameter t is allowed to run over the interval [0, 1], and b, d, p, q, and r are nonnegative integers that depend on the endpoints of the arc.

Right: the region Θ_5, contained within the closed unit disc

Below: An expanded view of one of the points at which the boundary arcs approach the unit circle

The lines tangent to the boundary arcs at their unit-circle endpoint; observe that the boundary arcs are not tangent to the unit circle at this endpoint.

New Results

As the diagrams for Θ_5 suggest, none of the boundary arcs are tangent to the unit circle at their respective endpoints. Using the equations for the arcs and the implicit function theorem, I have been able to show that this is the case for every Θ_n.

This implies that for an analytically perturbed stochastic matrix $P(\varepsilon)$, any perturbed eigenvalue that approaches a value on the unit circle as $\varepsilon \downarrow 0$ is bounded away from the unit circle in the same way that perturbed eigenvalues approaching 1 were previously know to be; that is, the eigenvalue curves cannot be tangent to the unit circle at their limiting points, the unperturbed eigenvalues.

If an eigenvalue curve were tangent to the unit circle, as in the blue curve at the right, points on it would fall outside of the relevant Θ_n, even if the boundary arcs are very close to being tangent to the unit circle (as illustrated by the black lines).

Under an additional assumption that a specific type of reduction process can be performed on $P(\varepsilon)$, this bounding of the perturbed eigenvalues allows the methods used by Filar, Krieger, and Syed to be adapted and applied to the new situation. In this case, however, the presence of eigenvalues on the unit circle other than 1 adds nothing new, in a sense, to the value of the hybrid Cesaro limit.

References

[1] Jerzy Filar, Henry A. Krieger, and Zamir Syed. Cesaro limits of analytically perturbed stochastic matrices. *Linear Algebra and Its Applications,* 353: 227-243, 2002.

[2] Henryk Minc. *Nonnegative Matrices.* John Wiley & Sons, 1988.