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# Group Actions and Divisors on Tropical Curves

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# Group Actions and Divisors on Tropical Curves

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May, 2011

**HARVEY MUDD**  
COLLEGE

Department of Mathematics

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# Abstract

Tropical geometry is algebraic geometry over the tropical semiring, or min-plus algebra. In this thesis, I discuss the basic geometry of plane tropical curves. By introducing the notion of abstract tropical curves, I am able to pass to a more abstract metric-topological setting. In this setting, I discuss divisors on tropical curves. I begin a study of  $G$ -invariant divisors and divisor classes.



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# Chapter 1

## Tropical Geometry

This chapter is a short introduction to tropical algebraic geometry. There are several ways to approach tropical geometry. Here, it is defined as algebraic geometry over the tropical semiring. While this approach may seem somewhat unmotivated, it is the most direct way to begin a study of the subject. For alternative approaches to tropical geometry that illustrate how it arises naturally from classical algebraic geometry, see Gathmann (2006).

Here, we define a plane tropical curve to be the corner locus of a tropical polynomial in two variables. We then transition to a more abstract construction of tropical curves, which will be in the in subsequent chapters.

### 1.1 The Tropical Semiring

The *tropical semiring* is the set  $\mathbb{T} = \mathbb{R} \cup \{\infty\}$  under addition  $\oplus$  and multiplication  $\odot$ , defined as

$$x \oplus y := \min(x, y) \quad \text{and} \quad x \odot y := x + y.$$

Because of this definition,  $\mathbb{T}$  is sometimes referred to as the *min-plus algebra* (some authors prefer to use the max-plus algebra instead; the differences are superficial). Additive and multiplicative identities in  $\mathbb{T}$  are given by infinity and zero, respectively, as

$$x \oplus \infty = \min(x, \infty) = x \quad \text{and} \quad x \odot 0 = x + 0 = x$$

for all  $x \in \mathbb{T}$ . Multiplication distributes over addition, because for all  $x, y, z \in \mathbb{T}$ ,

$$x \odot (y \oplus z) = x + \min(y, z) = \min(x + y, x + z) = (x \odot y) \oplus (x \odot z).$$

It is not difficult to verify that the tropical semiring satisfies the remaining commutative ring axioms, except for the existence of additive inverses, because the equation

$$x \oplus y = \infty$$

has only  $x = y = \infty$  as a solution. Hence, the additive structure of the tropical semiring is a monoid, a semigroup with identity.

The  $\odot$  sign is often omitted when variables are being multiplied. For example, in writing tropical polynomials,  $x \odot y$  is represented simply as  $xy$ . Repeated multiplication is denoted by exponentiation; for example,  $x \odot x \odot x = x^{\odot 3} = x^3$ . Note that repeated tropical addition is *not* equivalent to tropical multiplication; for instance,  $x \oplus x = \min(x, x) = x$ , whereas  $2 \odot x = 2 + x$ . Hence,  $\alpha \odot x$  denotes that the tropical “coefficient” of  $x$  is  $\alpha$ , and  $\alpha x$  denotes  $x$  multiplied by  $\alpha$  in the classical sense.

We will take the topology on  $\mathbb{T}$  to be the topology generated by all open sets of  $\mathbb{R}$  and all sets  $(a, \infty] := (a, \infty) \cup \{\infty\}$  for  $a \in \mathbb{R}$ . That is, every open subset of  $\mathbb{T}$  is a union of a collection of these sets. For more discussion of topology, see Section 1.4.

## 1.2 Tropical Polynomials

A *tropical monomial* is any tropical product of variables  $x_1, x_2, \dots, x_n$ , with repetition allowed,

$$m(x_1, \dots, x_n) = x_1^{i_1} \odot x_2^{i_2} \odot \dots \odot x_n^{i_n} = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n},$$

where each  $i_k \in \mathbb{Z}$ . We will often only consider the case where the exponents  $i_k$  are nonnegative. Writing  $m$  in terms of classical arithmetic, we have

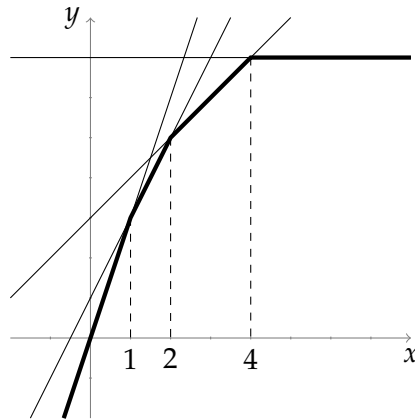
$$m(x_1, \dots, x_n) = i_1 x_1 + i_2 x_2 + \dots + i_n x_n.$$

Thus,  $m: \mathbb{T}^n \rightarrow \mathbb{T}$  is a linear function with integer coefficients that passes through the origin. Moreover, every such linear function may be written as a tropical monomial: If  $f(x_1, \dots, x_n) = \sum_{k=1}^n s_k x_k$ , with each  $s_k \in \mathbb{Z}$ , is such a function, then  $f$  is the tropical monomial

$$f(x_1, \dots, x_n) = x_1^{s_1} \odot \dots \odot x_n^{s_n}.$$

A *tropical polynomial*  $p: \mathbb{T}^n \rightarrow \mathbb{T}$  is a finite linear combination of tropical monomials,

$$p(x_1, \dots, x_n) = \bigoplus_{j=1}^N c_j \odot m_j(x_1, \dots, x_n),$$



**Figure 1.1** The tropical polynomial  $p(x) = x^3 \oplus 1 \odot x^2 \oplus 3 \odot x \oplus 7$  (thick line) as the minimum of a finite set of linear functions, with its roots.

where  $c_j \in \mathbb{R}$  and  $N$  is a positive integer. The reason for the restriction  $c_j \in \mathbb{R}$  (as opposed to  $c_j \in \mathbb{T}$ ) is to exclude the constant function  $p = \infty$ , and a term with an infinite coefficient is irrelevant to any other polynomial. Hence, nothing is lost by considering a tropical polynomial as a function  $p: \mathbb{R}^n \rightarrow \mathbb{R}$ . Writing  $p$  classically, we see that  $p$  is the minimum of a finite number of linear functions:

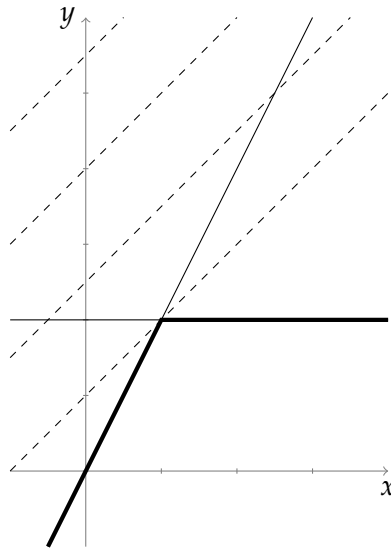
$$p(x_1, \dots, x_n) = \min(c_1 + i_1 x_1 + \dots + i_n x_n, \dots, c_N + i_{N_1} x_1 + \dots + i_{N_n} x_n).$$

The tropical polynomials in  $n$  variables are precisely the continuous, piecewise linear, concave functions on  $\mathbb{R}^n$  with integer coefficients, where the number of pieces is finite. The points where a tropical polynomial fails to be linear are its *roots*. The Fundamental Theorem of Algebra holds for tropical polynomials; that is, every (nonconstant) tropical polynomial in one variable can be written uniquely as a tropical product of linear factors. However, distinct polynomials may represent the same function. We illustrate these facts with the following examples.

**Example 1.1.** Consider the polynomial

$$p(x) = x^3 \oplus 1 \odot x^2 \oplus 3 \odot x \oplus 7.$$

To graph  $p$ , we plot the lines  $y = 3x$ ,  $y = 2x + 1$ ,  $y = x + 3$ , and  $y = 7$ . The value of  $p$  at  $x$  is the minimum of the  $y$ -values of the four lines at  $x$ , as illustrated in Figure 1.1.



**Figure 1.2**  $x^2 \oplus 2 = x^2 \oplus r \odot x \oplus 2$  for any  $r \geq 1$ .

It is easily seen that the roots of  $p$  are 1, 2, and 4. These points are analogous to the roots (zeros) of a classical polynomial, as we see by factoring  $p$ :

$$\begin{aligned}
 p(x) &= x^3 \oplus 1 \odot x^2 \oplus 3 \odot x \oplus 7 = \min(3x, 2x + 1, x + 3, 7) \\
 &= \min(x, 1) + \min(x, 2) + \min(x, 4) \\
 &= (x \oplus 1) \odot (x \oplus 2) \odot (x \oplus 4). \quad (1.1)
 \end{aligned}$$

This factorization may be verified by expanding Equation 1.1 using the distributive law.

**Example 1.2.** To see that distinct polynomials can represent the same function, observe that

$$x^2 \oplus 2 = \min(2x, 2) = \min(2x, x + r, 2) = x^2 \oplus r \odot x \oplus 2$$

for any  $r \geq 1$  (see Figure 1.2). The most judicious choice is  $r = 1$ , because we may factor

$$x^2 \oplus 1 \odot x \oplus 2 = (x \oplus 1)^2.$$

This factorization is easily seen if we write the polynomial classically, as  $\min(2x, 2) = 2 \min(x, 1)$ .

A good introduction to tropical polynomials, including the material covered here, is given by Speyer and Sturmfels (2009).

### 1.3 Plane Tropical Curves

Let  $p: \mathbb{R}^n \rightarrow \mathbb{R}$  be a tropical polynomial. The *hypersurface*  $V(p)$  is the set of all roots of  $p$ . Because  $V(p)$  is the set of points in  $\mathbb{R}^n$  where  $p$  fails to be linear, we also call  $V(p)$  the *corner locus* of  $p$ . Because  $p$  is the minimum of a finite collection of linear functions,  $x \in \mathbb{R}^n$  is a root of  $p$  if and only if this minimum is attained at least twice at  $x$ . In this section, we are concerned with the case  $n = 2$ . If

$$p(x, y) = \bigoplus_{(i,j)} c_{ij} \odot x^i y^j,$$

is a tropical polynomial in two variables, then we call  $V(p)$  a *plane tropical curve*. The *degree* of  $V(p)$  is the degree of the polynomial  $p$ .

**Example 1.3.** A general tropical *line* is defined by

$$p(x, y) = a \odot x \oplus b \odot y \oplus c,$$

for  $a, b, c \in \mathbb{R}$ . The line  $V(p)$  is the set of all points  $(x, y) \in \mathbb{R}^2$  where  $\min(x + a, y + b, c)$  is obtained at least twice. That is,  $V(p)$  consists of all points  $(x, y)$  satisfying one of the following conditions:

- $x + a = y + b < c$ ; that is,  $y = x + (a - b)$  for  $x < c - a$
- $x + a = c < y + b$ ; that is,  $x = c - a$  for  $y > c - b$
- $y + b = c < x + a$ ; that is,  $y = c - b$  for  $x > c - a$
- $x + a = y + b = c$ ; that is,  $(x, y) = (c - a, c - b)$ .

Thus, we see that a general tropical line, illustrated in Figure 1.3, consists of three half rays emerging from the point  $(c - a, c - b)$  in the directions  $(1, 0)$ ,  $(0, 1)$ , and  $(-1, -1)$ . Summing these direction vectors, we have

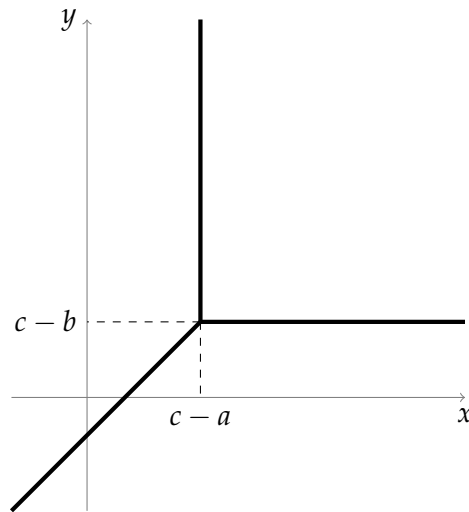
$$(1, 0) + (0, 1) + (-1, -1) = (0, 0).$$

We shall see that all plane tropical curves satisfy relationships of this form.

**Example 1.4.** As in the classical case, a tropical *conic* is defined by a degree 2 polynomial, of the form

$$p(x, y) = a \odot x^2 \oplus b \odot xy \oplus c \odot y^2 \oplus d \odot y \oplus e \odot x \oplus f.$$





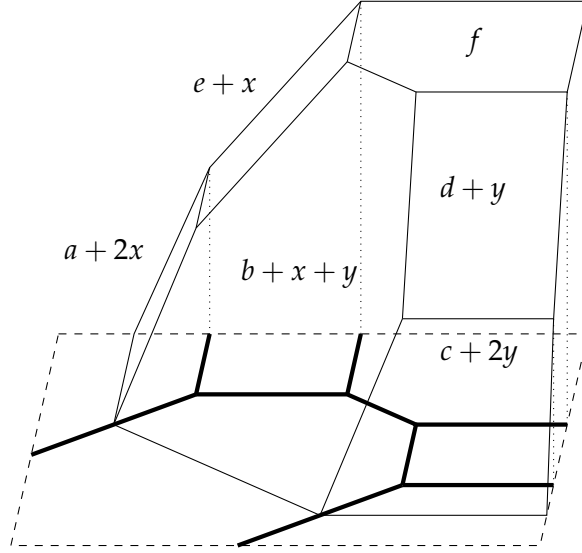
**Figure 1.3** A tropical line.

The conic  $V(p)$  is the corner locus of the minimum of six planes in 3-space. The positioning of these six planes in 3-space is determined by the coefficients  $a, \dots, f$  in the polynomial  $p$ . In contrast to the line in Example 1.3, where the coefficients of the polynomial only affect the location of the curve in the plane, the shape of  $V(p)$  depends heavily on the coefficients. One possibility is illustrated in Figure 1.4. In fact, this is one of four possible shapes for a nondegenerate tropical conic. All four types are illustrated in Gathmann (2006), and nondegeneracy is defined at the end of this section.

In general, a plane tropical curve is a metric graph in the plane with a finite number of vertices and edges, where some of the edges are infinite. More specifically, we have the following equivalent definition, which is purely geometric, that appears in Gathmann (2006):

**Definition 1.1** (Gathmann, Definition C). A plane tropical curve  $\Gamma$  of degree  $d$  is a weighted graph  $\Gamma$  in  $\mathbb{R}^2$  such that

1. Every edge of  $\Gamma$  is a line segment with rational slope;
2.  $\Gamma$  has  $d$  ends each in the directions  $(1, 0)$ ,  $(0, 1)$ , and  $(-1, -1)$ , where an end of weight  $w$  counts  $w$  times; and
3. At every vertex  $v$  of  $\Gamma$ , the *balancing condition* holds: the weighted sum of the primitive integral vectors of the edges around  $v$  is zero.



**Figure 1.4** A quadratic polynomial  $p$  is the minimum of six linear functions. The corner locus  $V(p)$  defines a tropical conic in the plane.

The balancing condition and weighting work as follows. Consider a tropical plane curve  $\Gamma$  locally around a vertex  $v$ . Because we may change coordinates so that  $v = (0, 0)$ , we can, without loss of generality, take  $v$  to be the origin in  $\mathbb{R}^2$ . Then  $\Gamma$  is locally the corner locus of a tropical polynomial where every term has a coefficient of zero:

$$p(x, y) = \bigoplus_{i=1}^n x^{a_i} y^{b_i} = \min_{1 \leq i \leq n} (a_i x + b_i y), \quad a_i, b_i \in \mathbb{Z}.$$

Note that in such a polynomial every linear term achieves a common minimum value of 0 at the origin. Consider the points  $q_i = (-a_i, -b_i) \in \mathbb{R}^2$ , and let  $\Delta$  denote the convex hull of these points.

First, if  $q_i = (-a_i, -b_i)$  is not a vertex of  $\Delta$ , then it is impossible for  $a_i x + b_i y$  to be strictly smaller than all the other  $a_j x + b_j y$  for any  $(x, y) \in \mathbb{R}^2$  (this situation is analogous to the situation in Example 1.2, in which the term  $x + r$  was irrelevant to the minimum). This fact can be proved by observing that if  $q_i$  is not a vertex of  $\Delta$ , then there exist vertices  $q_j, q_k$  of  $\Delta$  such that  $a_j \leq a_i \leq a_k$ . Moreover,  $q_j$  and  $q_k$  can be chosen so that either  $b_j \leq b_i \leq b_k$  or  $b_k \leq b_i \leq b_j$ . Considering cases proves the claim.

Now, if

$$p(x, y) = a_i x + b_i y = a_j x + b_j y \quad (1.2)$$

for two adjacent vertices  $q_i, q_j$  of  $\Delta$ , then the point  $(x, y)$  is in  $\Gamma$ . Moreover, points  $(x, y) \in \mathbb{R}^2$  satisfying Equation 1.2 are (locally) exactly the points of  $\Gamma$ , because if Equation 1.2 holds for nonadjacent  $q_i, q_j$ , then it must hold for all vertices  $q_k$  in between. Rearranging Equation 1.2, we get

$$y = -\left(\frac{a_j - a_i}{b_j - b_i}\right)x.$$

The slope of the line connecting  $q_i$  and  $q_j$  is  $\frac{b_j - b_i}{a_j - a_i}$ . Thus,  $\Gamma$  is locally the union of all half-rays emanating from  $v$  (the origin) and pointing in the directions of the outward normal vectors of the edges of  $\Delta$ . In other words, the graph of  $\Gamma$  is (locally) *dual* to  $\Delta$ . We have also demonstrated property 1 of Definition 1.1: each edge of  $\Gamma$  has rational slope.

If  $q_1, \dots, q_n$  are the vertices of  $\Delta$ , listed in counterclockwise order, then an outward normal vector of the edge joining  $q_i$  and  $q_{i+1}$  (where  $q_{n+1}$  is taken to be  $q_1$ ) is  $n_i := (b_i - b_{i+1}, a_{i+1} - a_i)$ . For each  $i$ , write  $n_i = w_i u_i$ , where  $u_i$  is the primitive integral vector in the direction of  $v_i$  and  $w$  is a positive integer. The number  $w_i$  is called the *weight* of the corresponding edge in  $\Gamma$ ; using these weights, we consider  $\Gamma$  to be a weighted graph. Summing all of the  $n_i$ , we have

$$\sum_{i=1}^n n_i = \sum_{i=1}^n w_i u_i = 0;$$

that is, the weighted sum of the primitive integral vectors around a vertex  $v$  of  $\Gamma$  is 0, which is precisely the balancing condition of part 3 of Definition 1.1. We saw a simple case of the balancing condition in Example 1.3.

**Example 1.5.** Consider the polynomial

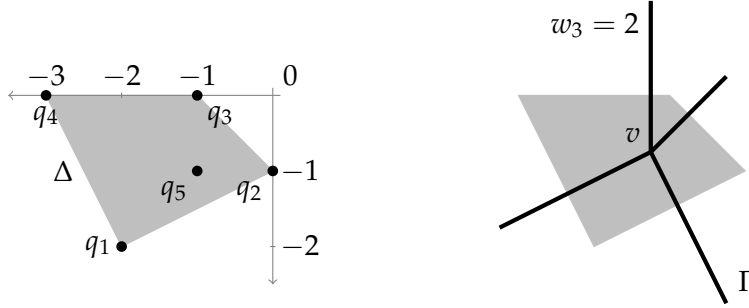
$$\begin{aligned} p(x, y) &= 3 \odot x^4 \oplus 17 \odot y^4 \oplus 5 \odot x^2 y^2 \\ &\quad \oplus 1 \odot x^3 \oplus 6 \odot xy \oplus 5 \odot x \oplus 8 \odot y \oplus 10 \\ &= \min(4x + 3, 4y + 17, 5 + 2x + 2y, \\ &\quad 1 + 3x, 6 + x + y, 5 + x, 8 + y, 10). \end{aligned}$$

The plane tropical curve  $\Gamma = V(p)$  has a vertex at  $v = (2, -1)$ , because

$$5 + 2x + 2y = 1 + 3x = 6 + x + y = 5 + x = 8 + y = 7$$

is the minimum there. Shifting coordinates so that  $v$  is the origin, we have

$$p(x, y) = 11 \odot x^4 \oplus 13 \odot y^4 \oplus 7 \odot x^2 y^2 \oplus 7 \odot x^3 \oplus 7 \odot xy \oplus 7 \odot x \oplus 7 \odot y \oplus 10,$$



**Figure 1.5** Plot of the convex polygon  $\Delta$  and local sketch of the graph  $\Gamma$ , defined by the polynomial in Example 1.5.

which has the same corner locus as

$$4 \odot x^4 \oplus 6 \odot y^4 \oplus x^2 y^2 \oplus x^3 \oplus xy \oplus x \oplus y \oplus 3$$

(changing the coefficients only shifts the graph of  $p$  in the  $z$ -direction in  $\mathbb{R}^3$ ). The  $x^4$ ,  $y^4$ , and constant terms have nonzero coefficients, and therefore they are irrelevant near the origin. So, locally,

$$p(x, y) = x^2 y^2 \oplus x^3 \oplus xy \oplus x \oplus y.$$

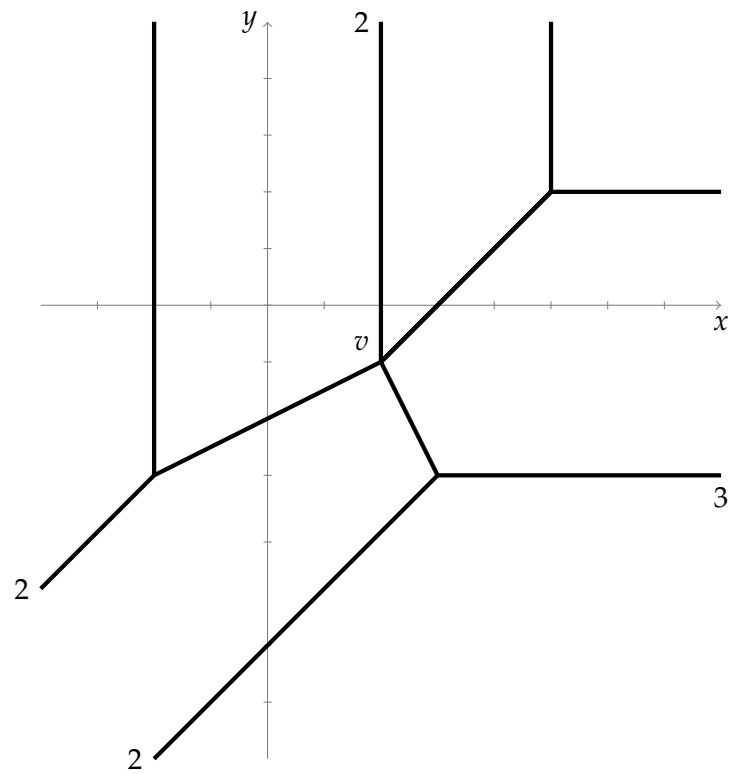
The vertices of  $\Delta$  are given by  $q_1 = (-2, -2)$ ,  $q_2 = (0, -1)$ ,  $q_3 = (-1, 0)$ , and  $q_4 = (-3, 0)$ . Observe that  $q_5 = (-1, -1)$  is not a vertex of  $\Delta$ , and, indeed, the  $xy$  term is irrelevant to  $p$ : If  $x + y \geq 0$ , then  $x + y \geq x$  or  $x + y \geq y$ ; if  $x + y < 0$ , then  $x + y > 2x + 2y$ ; in all cases,  $x + y$  cannot uniquely obtain the minimum. The normal vectors  $n_i$ , which point in the directions that the edges of  $\Gamma$  emanate from  $v$ , are  $n_1 = (1, -2)$ ,  $n_2 = (1, 1)$ ,  $n_3 = (0, 2)$ , and  $n_4 = (-2, -1)$ . Observe that  $n_3 = (0, 2) = 2 \cdot (0, 1)$ , and so the vertical edge of  $\Gamma$  at  $v$  has weight  $w_3 = 2$ . The weighted sum of the primitive integral vectors is then

$$(1, 1) + 2 \cdot (0, 1) + (-2, -1) + (1, -2) = (0, 0),$$

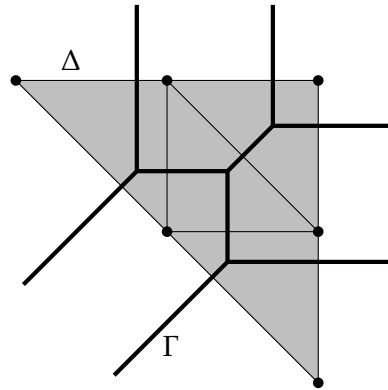
illustrating the balancing condition. The graphs of  $\Delta$  and  $\Gamma$  (locally around  $v$ ) are shown in Figure 1.5.

By repeating this process for the other vertices (located at  $(-2, -3)$ ,  $(3, -3)$ , and  $(5, 2)$ ), we can obtain the entire graph  $\Gamma$ , as shown in Figure 1.6. Note also that part 2 of Definition 1.1 is satisfied.

The graphing process illustrated in Example 1.5, which locally considers each vertex of a plane tropical curve, may be performed “globally” by



**Figure 1.6** Plot of entire curve  $\Gamma$  from Example 1.5. Edge weights not equal to 1 are labeled.



**Figure 1.7** A nondegenerate tropical conic  $\Gamma$  superimposed on its Newton subdivision  $\Delta$ .

considering all vertices simultaneously. If  $\Gamma$  is a plane tropical curve of degree  $d$ , then the result is a partition  $\Delta$  of a  $d \times d$  triangle. This partition is called the *Newton subdivision* corresponding to  $\Gamma$ . We say that  $\Gamma$  is *smooth*, or *nondegenerate*, if its Newton subdivision consists of  $d^2$  triangles, each of area  $\frac{1}{2}$  (the curve in Example 1.5 is not smooth). Figure 1.7 shows a smooth tropical conic and its Newton subdivision. The Newton subdivision provides a convenient way to classify types of tropical curves. Moreover, the Newton subdivision can be a useful tool when working with tropical curves. For instance, the tropical analog of Bézout's theorem (two distinct plane curves, of degrees  $d_1$  and  $d_2$ , intersect in  $d_1 d_2$  points when counted with multiplicity) is easily understood through Newton subdivisions. For more details on Newton subdivisions, including their use in Bézout's theorem, consult Gathmann (2006).

## 1.4 Topology

This section is devoted to stating several basic definitions from topology, which will be useful in the following section. This material is found in Willard (1970).

The first definition is familiar from a first course in analysis.

**Definition 1.2** (Willard 2.1). A *metric space* is a set  $M$  together with a function  $d: M \times M \rightarrow \mathbb{R}$ , called the *metric* on  $M$ , satisfying

1.  $d(x, y) \geq 0$ , with equality if and only if  $x = y$ ;

2.  $d(x, y) = d(y, x)$ ; and
3.  $d(x, z) \leq d(x, y) + d(y, z)$

for any  $x, y, z \in M$ .

**Definition 1.3** (Willard 3.1). A *topology* on a set  $X$  is a collection  $\tau$  of subsets of  $X$ , such that

1. Any union of sets in  $\tau$  is in  $\tau$ ;
2. Any finite intersection of sets in  $\tau$  is in  $\tau$ ; and
3. The empty set  $\emptyset$  and  $X$  are in  $\tau$ .

A set  $E \in \tau$  is called an *open set*. We call  $(X, \tau)$ , or just  $X$  if  $\tau$  is clear, a *topological space*

**Example 1.6.** Any metric space has a natural topological structure, called the *metric topology*. If  $M$  is a metric space with metric  $d$ , then  $E \subseteq M$  is open if for each  $x \in E$  there exists  $\epsilon > 0$  such that

$$\{y \in M \mid d(x, y) < \epsilon\} \subset E.$$

Two very important topological notions are compactness and connectedness. First, recall that an *open cover* of a topological space  $X$  is a collection  $\{U_\alpha \mid \alpha \in A\}$  of open subsets of  $X$  whose union is all of  $X$  (here  $A$  is some, possibly infinite, indexing set). A *subcover* is a subcollection of  $\{U_\alpha \mid \alpha \in A\}$  that is also a cover.

**Definition 1.4** (Willard 17.1). The topological space  $X$  is *compact* if each open cover of  $X$  has a finite subcover.

**Definition 1.5** (Willard 26.1). The topological space  $X$  is *disconnected* if there are disjoint nonempty open subsets  $H$  and  $K$  of  $X$  such that  $X = H \cup K$ . If no such disconnection exists, then  $X$  is *connected*.

Finally, we need the following two notions of equivalence.

**Definition 1.6** (Willard 7.8). If  $X$  and  $Y$  are topological spaces, a function  $f: X \rightarrow Y$  is a *homeomorphism* if  $f$  is bijective and continuous and  $f^{-1}$  is also continuous. In this case, we say  $X$  and  $Y$  are *homeomorphic*.

**Definition 1.7** (Willard 24.3). If  $M$  and  $N$  are metric spaces with metrics  $d_M$  and  $d_N$ , respectively, then a function  $f: M \rightarrow N$  is an *isometry* if  $f$  is bijective and  $d_N(f(x), f(y)) = d_M(x, y)$  for all  $x, y \in M$ . In this case, we say  $M$  and  $N$  are *isometric*.

## 1.5 Abstract Tropical Curves

It is sometimes useful to consider a tropical curve as independent of a defining equation. In this section, we will describe one way of doing so, which was used by Joyner and colleagues (2010) to prove Theorem 2.2.

For positive  $n \in \mathbb{Z}$  and positive  $r \in \mathbb{R}$ , define the *star-shaped set*

$$S(n, r) = \{z \in \mathbb{C} \mid x = te^{\frac{2\pi ik}{n}} \text{ for some } t \in [0, r), k \in \mathbb{Z}\}.$$

For fixed  $k$ , the set

$$S_k(n, r) = \{z \in \mathbb{C} \mid x = te^{\frac{2\pi ik}{n}} \text{ for some } t \in [0, r)\}$$

is an *arm* of  $S(n, r)$ , and the point  $z = 0$  is the *center* of  $S(n, r)$ . Each arm  $S_k(n, r)$  is given the metric induced from the Euclidean metric on  $\mathbb{C}$ , and the entire star-shaped set is given the path metric (i.e., the distance between two points on different arms is given by the sum of their distances from the center). We take the metric topology on  $S(n, r)$ .

The goal is now to define (abstract) tropical curves to locally “look” like star-shaped sets at every point. First, we make an intermediate definition.

**Definition 1.8** (Joyner, Ksir, Melles; Definition 2). Let  $\Gamma$  be a compact connected topological space. We say that  $\Gamma$  is a *metric topological graph* if

1. Each point  $P \in \Gamma$  has a neighborhood homeomorphic to a star-shaped set  $S(n_P, r_P)$ , where  $P$  is mapped to the center of the star-shaped set under the homeomorphism; and
2. If  $P \in \Gamma$  has  $n_P \neq 1$  (where  $n_P$  is the positive integer determined by the homeomorphism above), then  $P$  has a neighborhood isometric to  $S(n_P, r_P)$ .

The number  $n_P$  is called the *valence* of  $P$ .

If  $\Gamma$  is a metric topological graph not homeomorphic to a circle, define  $V = \{P \in \Gamma \mid n_P \neq 2\}$ . It is a consequence of the compactness of  $\Gamma$  that  $V$  is finite. Then  $E = X \setminus V$  is homeomorphic to a finite disjoint union of open intervals. If we consider  $E$  instead as a finite set of disjoint open intervals, then  $G = (V, E)$  is a finite graph, called the *minimal graph* of  $\Gamma$ . If an edge in  $G$  is adjacent to a 1-valent vertex, we call that edge a *leaf*; otherwise, it is an *interior edge*.



**Definition 1.9** (Joyner, Ksir, Melles; Definition 4). Let  $\Gamma$  be a metric topological graph with minimal graph  $G$ . If each interior edge of  $G$  has finite length and each leaf of  $G$  has infinite length, and each edge of  $G$  has an associated positive integer *multiplicity*, then we say that  $\Gamma$  is an *abstract tropical curve*. In this case, the 1-valent vertices of  $\Gamma$  are called *infinite points*, and all other points are called *finite points*.

Note that any tropical plane curve as defined by Definition 1.1 can be thought of as a special case of Definition 1.9. However, there are advantages to using this newer definition. When we are studying intrinsic properties of tropical curves, the defining polynomial is not of interest, and using this definition of an abstract tropical curve is much less cumbersome than considering a generic polynomial  $p$  (which, for example, may be of arbitrarily large degree). Additionally, the metric and topological structure inherited from star-shaped sets provides a much more concrete framework with which to analyze abstract tropical curves than is available for the curves of Definition 1.1. Finally, abstract tropical curves are independent of the ambient space. As with real curves, tropical curves need not lie in the plane. Thus, we may think of Definition 1.9 as a means for analyzing curves embedded in higher-dimensional space, and so this definition is less restrictive than the original one.

## Chapter 2

# Divisors on Tropical Curves

This chapter is an introduction to divisors on tropical curves. This material applies to metric topological graphs as well, and so it is recommended to read this section with the abstract tropical curves of Definition 1.9 in mind. Of course, everything in this chapter it applies to tropical plane curves. We introduce the notion of equivalence of divisors, state the tropical Riemann–Roch theorem, and address a recent result on  $G$ -invariant divisors on tropical curves.

### 2.1 Rational Functions

A *rational function* defined on an (abstract) tropical curve  $\Gamma$  is a continuous, real-valued function on  $\Gamma^\circ = \Gamma \setminus \{P \in \Gamma \mid \text{valence}(P) = 1\}$  ( $\Gamma$  without its “infinite points”), which is piecewise linear with only finitely many singular points. A singular point of a rational function is a point at which the slope is not defined. Compare this definition of a rational function with the definition of a tropical polynomial in Section 1.2. A rational function can be thought of as a tropical polynomial defined on  $\Gamma$ . The term “rational” is arguably more suitable if we only allow tropical polynomials to contain terms with nonnegative exponents, in which case the analogy to the classical case is apparent. Denote the set of rational functions on  $\Gamma$  by  $M(\Gamma)$ , and notice that  $M(\Gamma)$  is a group under addition (i.e., tropical multiplication).

Let  $f \in M(\Gamma)$  and let  $P \in \Gamma^\circ$ . By definition,  $\Gamma$  is locally homeomorphic to a star-shaped set centered at  $P$ . On each arm of the star-shaped set, there is a neighborhood of  $P$  on which  $f$  has constant slope (because  $f$  is singular at finitely many points). The *order* of  $f$  at  $P$ , denoted  $\text{ord}_P(f)$ , is the weighted sum of these slopes, where each slope is given the weight of the

corresponding edge in  $\Gamma$ . If instead  $P$  is a 1-valent point (i.e., an “infinite point” at the end of a leaf), then  $f$  is constant close to  $P$  (on an interval isometric to  $(a, \infty)$  for some  $a$ ). The slope of  $f$  in this region, multiplied by the appropriate weight, is taken to be  $\text{ord}_P(f)$ .

Note that if  $f$  is linear at  $P$ , then  $\text{ord}_P(f) = 0$ , and so the order of  $f$  may only be nonzero at the finitely many singular points of  $f$  and the finitely many 1-valent points of  $\Gamma$ .

## 2.2 Divisors

A divisor  $D$  on a tropical curve  $\Gamma$  is a finite formal linear combination of points on  $\Gamma$ ,

$$D = \sum_{P \in \Gamma} \lambda_P P,$$

where each coefficient  $\lambda_P$  is an integer, all but finitely many of which are zero. The set of all divisors on  $\Gamma$ , denoted  $\text{Div } \Gamma$ , is an abelian group under addition. The *degree* of  $D \in \text{Div } \Gamma$  is  $\deg D = \sum_{P \in \Gamma} \lambda_P$ . A divisor is *effective* if  $\lambda_P \geq 0$  for all  $P \in \Gamma$ . Denote the set of all effective divisors on  $\Gamma$  by  $\text{Div}_+ \Gamma$ , and denote the set of effective divisors of degree  $s$  by  $\text{Div}_+^s \Gamma$ .

Each rational function defined on  $\Gamma$  determines a divisor  $(f)$ . A divisor  $D$  is *principal* if

$$D = (f) := \sum_{P \in \Gamma} \text{ord}_P(f) P,$$

for some  $f \in M(\Gamma)$ . Note that  $\deg(f) = 0$ , because for each line segment on which  $f$  is linear, the slopes of  $f$  emanating from the two endpoints are negatives of each other and cancel in the sum  $\sum_{P \in \Gamma} \text{ord}_P(f)$ . The set of all principal divisors forms a subgroup, denoted  $\text{Prin } \Gamma$  of  $\text{Div } \Gamma$ .

## 2.3 Equivalence of Divisors

For divisors  $D$  and  $D'$  on a tropical curve  $\Gamma$ , we say that  $D$  is *linearly equivalent*, or simply *equivalent*, to  $D'$  if  $D - D'$  is principal; that is, if there exists  $f \in M(\Gamma)$  such that

$$D = D' + (f).$$

This notion of linear equivalence is in fact an equivalence relation on  $\text{Div } \Gamma$ . The equivalence class of  $D$  is denoted  $[D]$  and called the *linear system associated to  $D$*  or the *divisor class of  $D$* . The set of effective divisors in the linear

system  $[D]$  is denoted  $[D]_+$ . The set of all linear systems is a group, called the *Picard group*, which is the quotient

$$\text{Pic } \Gamma = \text{Div } \Gamma / \text{Prin } \Gamma.$$

For a divisor  $D$  on  $\Gamma$ , define the *rank* of  $D$ ,  $r(D)$ , to be the maximal integer  $s \geq 0$  such that  $[D - E]_+ \neq \emptyset$  for all  $E \in \text{Div}_+^s \Gamma$ . If no such  $s$  exists (i.e.,  $[D]_+ = \emptyset$ ), then set  $r(D) = -1$ . Define the *canonical divisor* on  $\Gamma$  to be the divisor  $K = \sum_{P \in \Gamma} (\text{valence}(P) - 2)P$ .

The Riemann–Roch theorem provides a relationship between divisor properties, and a purely topological quantity, the genus of the curve. Recall that the genus  $g$  of a finite graph  $G = (V, E)$  is given by the first Betti number (the number of “holes” in  $G$ ). It is calculated by  $g = |E| - |V| + 1$ . The genus of the (abstract) tropical curve  $\Gamma$  is defined to be the genus of the minimal graph of  $\Gamma$ . The tropical analogue of the Riemann–Roch theorem was proved by Gathmann and Kerber (2008)

**Theorem 2.1** (Riemann–Roch) (Gathmann and Kerber, Corollary 3.8). *Let  $\Gamma$  be a tropical curve of genus  $g$  and  $K$  the canonical divisor on  $\Gamma$ . Then for all  $D \in \text{Div } \Gamma$ , we have*

$$r(D) - r(K - D) = \deg D + 1 - g.$$

## 2.4 G-Invariance

An *automorphism* of the tropical curve  $\Gamma$  is a map  $\sigma: \Gamma \rightarrow \Gamma$  that is a homeomorphism, is an isometry on  $\Gamma^\circ$ , and that preserves edge multiplicities. The group of automorphisms of  $\Gamma$ ,  $\text{Aut } \Gamma$ , is a group under composition.

An automorphism  $\sigma \in \text{Aut } \Gamma$  acts naturally on a divisor  $D$  by extending its action linearly:

$$\sigma(D) = \sum_{P \in \Gamma} \lambda_P \sigma(P).$$

The divisor class  $[D]$  is also acted on by  $\sigma$ , with

$$\sigma([D]) = \{\sigma(D') \mid D' \in [D]\}.$$

Automorphisms of  $\Gamma$  also act naturally on  $M(\Gamma)$ . Specifically, for  $\sigma \in \text{Aut } \Gamma$  and rational function  $f \in M(\Gamma)$ ,  $\sigma f$  is the rational function given by

$$(\sigma f)(P) = f(\sigma^{-1}(P))$$

for each point  $P \in \Gamma$ . This action is compatible with the action of  $\text{Aut } \Gamma$  on  $\text{Div } \Gamma$ , in that  $\sigma((f)) = (\sigma f)$  for all principal divisors  $(f)$ . It follows, then, that  $\sigma([D]) = [\sigma(D)]$ . That is, automorphisms of  $\Gamma$  preserve equivalence of divisors.

If  $G \leq \text{Aut } \Gamma$ , then we say a divisor  $D$  is  $G$ -invariant if  $\sigma(D) = D$  for all  $g \in G$ , and, similarly,  $[D]$  is  $G$ -invariant if  $\sigma([D]) = [D]$  for all  $g \in G$ . The  $G$ -invariant divisors form a subgroup, denoted  $\text{Div}^G \Gamma$ , of  $\text{Div } \Gamma$ . Similarly, the  $G$ -invariant linear systems form a subgroup  $\text{Pic}^G \Gamma$  of  $\text{Pic } \Gamma$ .

Because an automorphism  $\sigma$  of  $\Gamma$  acts on  $\text{Pic } \Gamma$  by  $\sigma([D]) = [\sigma(D)]$ , it follows that the linear system  $[D]$  is  $G$ -invariant if it contains a  $G$ -invariant representative. The following theorem, proved recently by Joyner and colleagues (2010), states that the converse is true. The proof uses techniques in group cohomology.

**Theorem 2.2** (Joyner, Ksir, Melles; Theorem 3). *Let  $\Gamma$  be an abstract tropical curve and let  $G$  be a finite subgroup of the automorphism group  $\text{Aut } \Gamma$  of  $\Gamma$ . Then the map*

$$\text{Div}^G \Gamma \rightarrow \text{Pic}^G \Gamma,$$

*where a divisor  $D$  is mapped to its divisor class  $[D]$ , is surjective. That is, every  $G$ -invariant divisor class contains a  $G$ -invariant divisor.*

## Chapter 3

# Relating Group Actions to Quotient Spaces

Let  $G$  be a finite group acting on the (abstract) tropical curve  $\Gamma$ . This chapter begins with an introduction to the topological notion of a quotient space. A natural relationship is then illustrated between the group of  $G$ -invariant divisors on  $\Gamma$ , and the group of divisors on the quotient  $\Gamma/G$ . In general, a similar relationship does not hold between the Picard groups of  $\Gamma$  and  $\Gamma/G$ . However, we will see that partial analogues exist.

### 3.1 Quotient Spaces

The material in this section is taken from Willard (1970).

**Definition 3.1** (Willard 9.1). If  $X$  is a topological space,  $Y$  is a set and  $q: X \rightarrow Y$  is a surjective mapping, then the collection  $\tau_q$  of subsets of  $Y$  defined by

$$\tau_q = \{E \subset Y \mid q^{-1}(E) \text{ is open in } X\}$$

is a topology on  $Y$ , called the *quotient topology* induced on  $Y$  by  $q$ . When  $Y$  is given some such quotient topology, it is called a *quotient space* of  $X$ , and the inducing map  $q$  is called a *quotient map*.

The quotient topology may not always be the most practical topology to work with. If the set  $Y$  in Definition 3.1 is already a topological space, it would be useful to know when the existing topology is identical to the quotient topology. The condition is relatively simple, and is given after the following definition.

**Definition 3.2** (Willard 8.5). Let  $X$  and  $Y$  be topological spaces. A mapping  $q: X \rightarrow Y$  is called *open* (*closed*) if for each open (closed) set  $A$  in  $X$ ,  $q(A)$  is an open (closed) set in  $Y$ .

**Theorem 3.1** (Willard 9.2). Let  $X$  and  $Y$  be topological spaces, with  $q: X \rightarrow Y$  a surjective mapping. Then the topology on  $Y$  is the quotient topology  $\tau_q$  if and only if  $q$  is continuous and either open or closed.

### 3.2 Quotients of Abstract Tropical Curves

Let  $\Gamma$  be an abstract tropical curve, and let  $G$  be a subgroup of  $\text{Aut } \Gamma$ . Define  $\Gamma/G$  to be the set of orbits of  $\Gamma$  under the action of  $G$ . Define a mapping  $q: \Gamma \rightarrow \Gamma/G$  by

$$q: P \mapsto \tilde{P} := \{P' \in \Gamma \mid \sigma(P) = P' \text{ for some } \sigma \in G\};$$

that is,  $q$  maps each point  $P$  to its orbit under  $G$ . Because the  $G$ -orbits partition  $\Gamma$ ,  $q$  is well-defined and surjective. Hence,  $\Gamma/G$  is a topological space with the quotient topology induced by  $q$ . Informally, we may think of  $\Gamma/G$  as the space obtained from  $\Gamma$  by “gluing” together points in the same  $G$ -orbit.

**Example 3.1.** Let  $\Gamma$  be an abstract tropical curve homeomorphic to a circle. Then  $G = \text{Aut } \Gamma$  contains an infinite number of rotations. In particular, given any two distinct points on  $\Gamma$ , there exists an automorphism that maps one to the other. Thus,  $\Gamma/G$  consists of a single orbit, the orbit containing all points in  $\Gamma$ .

**Example 3.2.** Let  $\Gamma$  be an abstract tropical curve such that the minimal graph of  $\Gamma$  has only one edge. That is,  $\Gamma$  has two 1-valent points, both infinite points, and all of the remaining points in  $\Gamma$  have valence 2, and so  $\Gamma$  is homeomorphic to an infinite line. A translation of the line, say by a unit length, generates an infinite group  $G \leq \text{Aut } \Gamma$ . Given a closed interval  $I$  in  $\Gamma$  of unit length, the points in  $I$  are all in distinct  $G$ -orbits, except for the two endpoints, which are in the same orbit. Moreover, all of the  $G$ -orbits are represented, because every point in  $\Gamma$  is some integer multiple of a unit length away from a point in  $I$ . Thus, we can identify  $\Gamma/G$  with an interval of unit length. However, in  $\Gamma/G$  the endpoints of this interval are the same point, and  $\Gamma/G$  is homeomorphic to a circle.

The phenomena illustrated in Example 3.1 and Example 3.2 are somewhat unusual. For the remainder of this chapter, we will avoid such behavior by taking  $G$  to be a finite subgroup of  $\text{Aut } \Gamma$ . In fact, in all cases

apart from those in Example 3.1 and Example 3.2,  $G$  is forced to be finite, as illustrated by the next theorem. The proof follows from the fact that if  $\Gamma$  is not homeomorphic to a circle, then any automorphism of  $\Gamma$  is a graph automorphism on the minimal graph of  $\Gamma$ , taking vertices to vertices and edges to edges.

**Theorem 3.2** (Joyner, Ksir, Melles; Theorem 1). *If an abstract tropical curve  $\Gamma$  has a minimal graph with only one edge, or is homeomorphic to a circle, then the automorphism group  $\text{Aut } \Gamma$  of  $\Gamma$  is an infinite group. Otherwise,  $\text{Aut } \Gamma$  is finite, and moreover if  $\ell$  is the number of leaves of the minimal graph of  $\Gamma$  and  $i$  is the number of inner edges, then  $\text{Aut } \Gamma$  is a subgroup of the product of symmetric groups  $S_\ell \times S_{2i}$ .*

We now show that  $\Gamma/G$  retains most of the structure of an abstract tropical curve.

**Lemma 3.1.** *The quotient  $\Gamma/G$  of an abstract tropical curve is a compact space.*

*Proof.* Let  $\{U_\alpha \mid \alpha \in A\}$  be an open cover of  $\Gamma/G$ . Then, by definition of the quotient topology,  $q^{-1}(U_\alpha)$  is an open set in  $\Gamma$  for each  $\alpha \in A$ , where  $q: \Gamma \rightarrow \Gamma/G$  is the quotient map defined above. If  $P \in \Gamma$ , then  $P$  is in some  $G$ -orbit  $\tilde{P} \in \Gamma/G$ . Because  $\{U_\alpha \mid \alpha \in A\}$  is a cover, there is an  $\alpha \in A$  such that  $\tilde{P} \in U_\alpha$ . The preimage of the orbit  $\tilde{P}$  under  $q$  is the set of points in  $\Gamma$  that are in the orbit, and so  $P \in q^{-1}(\tilde{P}) \subseteq q^{-1}(U_\alpha)$ . Thus,  $\{q^{-1}(U_\alpha) \mid \alpha \in A\}$  is an open cover of  $\Gamma$ , and because  $\Gamma$  is defined to be compact, it has a finite subcover, say  $\{q^{-1}(U_{\alpha_i})\}_{i=1}^n$ .

I claim that  $\{U_{\alpha_i}\}_{i=1}^n$  is a cover of  $\Gamma/G$ . Let  $\tilde{P}$  be an orbit in  $\Gamma/G$  and suppose  $\tilde{P} \notin U_{\alpha_i}$ . If another orbit  $\tilde{P}'$  is in  $U_{\alpha_i}$ , then  $q^{-1}(\tilde{P})$  and  $q^{-1}(\tilde{P}')$  are distinct  $G$ -orbits in  $\Gamma$ . Therefore,  $q^{-1}(\tilde{P}) \not\subseteq q^{-1}(U_{\alpha_i})$ . It follows that if  $\tilde{P} \notin \cup_{i=1}^n U_{\alpha_i}$ , then  $q^{-1}(\tilde{P}) \not\subseteq \cup_{i=1}^n q^{-1}(U_{\alpha_i}) = \Gamma$ , which is impossible. It must be, then, that  $\{U_{\alpha_i}\}_{i=1}^n$  is a finite subcover of  $\{U_\alpha \mid \alpha \in A\}$ , and so  $\Gamma/G$  is compact.  $\square$

**Lemma 3.2.** *The quotient  $\Gamma/G$  of an abstract tropical curve is a connected space.*

*Proof.* Suppose, to the contrary, that  $\Gamma/G$  is disconnected. Then there exist disjoint nonempty open subsets  $H, K \subseteq \Gamma/G$  such that  $\Gamma/G = H \cup K$ . By definition of the quotient topology,  $q^{-1}(H)$  and  $q^{-1}(K)$  are open subsets of  $\Gamma$ . The sets  $q^{-1}(H)$  and  $q^{-1}(K)$  are nonempty, because  $H$  and  $K$  are nonempty and each orbit in  $\Gamma/G$  has a nonempty preimage, namely the set of points in the orbit. Moreover, if a point  $P \in \Gamma$  is in both  $q^{-1}(H)$  and  $q^{-1}(K)$ , then this implies that the orbit of  $P$ —a single point in  $\Gamma/G$ —is in



both  $H$  and  $K$ . But  $H$  and  $K$  are disjoint, and so such a  $P$  cannot exist. That is,  $q^{-1}(H) \cap q^{-1}(K) = \emptyset$ .

Finally, given any  $P \in \Gamma$ , we know that  $P$  is in some  $G$ -orbit  $\tilde{P}$ . Now,  $\tilde{P}$  must be in either  $H$  or  $K$ , and so  $P$  is in  $q^{-1}(H)$  or  $q^{-1}(K)$ . This implies that  $\Gamma = q^{-1}(H) \cup q^{-1}(K)$ ; that is,  $\Gamma$  is disconnected. But this contradicts the fact that, by definition, an abstract tropical curve is a connected space. It must be, then, that  $\Gamma/G$  is connected.  $\square$

**Remark 3.1.** Lemma 3.1 and Lemma 3.2 are specific versions of more general statements about quotients of a compact connected space  $X$ . This can be seen by using the *decomposition space* interpretation of a quotient  $X/Y$  (see Willard (1970), 9.5–9.11). In our case, the decomposition of  $\Gamma$  is given by the  $G$ -orbits.

The metric structure of  $\Gamma$  gives rise to a natural metric structure on  $\Gamma/G$ .

**Proposition 3.1.** *Let  $d$  be the metric on the abstract tropical curve  $\Gamma$ . Then the quotient  $\Gamma/G$  is a metric space with metric  $\tilde{d}$ , where for orbits  $\tilde{P}_1, \tilde{P}_2 \in \Gamma/G$ ,*

$$\tilde{d}(\tilde{P}_1, \tilde{P}_2) = \min_{\substack{P \in \tilde{P}_1 \\ P' \in \tilde{P}_2}} d(P, P').$$

(Note: Here we are using that  $G$  is finite, and so each  $G$ -orbit is finite. If we were to allow  $G$  to be infinite, then we would need to redefine  $\tilde{d}$  as an infimum.)

*Proof.* Because  $d$  is a metric on  $\Gamma$ ,  $d(P, P') \geq 0$ , with equality if and only if  $P = P'$ . Hence,  $\tilde{d}(\tilde{P}_1, \tilde{P}_2) \geq 0$ , with equality if and only if  $\tilde{P}_1$  and  $\tilde{P}_2$  share a point. But the  $G$ -orbits partition  $\Gamma$ , and so  $\tilde{d}(\tilde{P}_1, \tilde{P}_2) = 0$  if and only if  $\tilde{P}_1 = \tilde{P}_2$ . Also, because  $d$  is a metric,  $d(P, P') = d(P', P)$ , and it follows that  $\tilde{d}(\tilde{P}_1, \tilde{P}_2) = \tilde{d}(\tilde{P}_2, \tilde{P}_1)$ .

The only thing left to show is that  $\tilde{d}$  satisfies the triangle inequality. This follows from the fact that  $d$  satisfies the triangle inequality. We have

$$\tilde{d}(\tilde{P}_1, \tilde{P}_2) = \min_{\substack{P \in \tilde{P}_1 \\ P' \in \tilde{P}_2}} d(P, P') \leq \min_{\substack{P \in \tilde{P}_1 \\ P'' \in \tilde{P}_2}} (d(P, P'') + d(P'', P'))$$

for any  $P'' \in \Gamma$ . Thus, for any choice of  $\tilde{P}_3 \in \Gamma/G$ ,

$$\begin{aligned} \tilde{d}(\tilde{P}_1, \tilde{P}_2) &\leq \min_{\substack{P \in \tilde{P}_1 \\ P'' \in \tilde{P}_2 \\ P'' \in \tilde{P}_3}} (d(P, P'') + d(P'', P')) \leq \min_{\substack{P \in \tilde{P}_1 \\ P'' \in \tilde{P}_3}} d(P, P'') + \min_{\substack{P'' \in \tilde{P}_2 \\ P'' \in \tilde{P}_3}} d(P'', P') \\ &= \tilde{d}(\tilde{P}_1, \tilde{P}_3) + \tilde{d}(\tilde{P}_3, \tilde{P}_2), \end{aligned}$$

and so  $\tilde{d}$  is a metric on  $\Gamma/G$ .  $\square$

Fortunately, the metric  $\tilde{d}$  on  $\Gamma/G$  is compatible with the quotient topology.

**Proposition 3.2.** *Let  $q: \Gamma \rightarrow \Gamma/G$  be the map that sends a point to its orbit, and let  $\tilde{d}$  be the metric on  $\Gamma/G$  given in Proposition 3.1. The metric topology on  $\Gamma/G$  coincides with the quotient topology  $\tau_q$ .*

*Proof.* By Theorem 3.1, it will suffice to show that  $q$  is continuous and either open or closed (in the metric topology). Continuity is easy. Let  $\epsilon > 0$  and choose  $P_1, P_2 \in \Gamma$  such that  $d(P_1, P_2) < \epsilon$ . Then, if  $\tilde{P}_1 = q(P_1)$  and  $\tilde{P}_2 = q(P_2)$ , we have  $\tilde{d}(\tilde{P}_1, \tilde{P}_2) = \min_{P \in \tilde{P}_1, P' \in \tilde{P}_2} d(P, P') \leq d(P_1, P_2) < \epsilon$ , and  $q$  is continuous.

It is somewhat more difficult to show that  $q$  is open. Recall that a *base* for a topology is a subcollection of the topology (i.e., a collection of open sets) such that every open set can be written as the union of sets in the base. For a metric space, the collection of all open disks about points in the space is a base for the metric topology. Consult Willard (1970), Section 5 for a more detailed discussion.

Because  $q(U_1 \cup U_2) = q(U_1) \cup q(U_2)$  for all open sets  $U_1, U_2 \subseteq \Gamma$ , we need only consider how  $q$  behaves on a base for  $\Gamma$ . We use the base given by all open disks; that is, sets of the form

$$U_{P,r} = \{Q \in \Gamma \mid d(P, Q) < r\}$$

for  $P \in \Gamma, r > 0$ . Fix such a  $P$  and  $r$ , and let  $\tilde{P} = q(P)$  denote the orbit of  $P$  as a point in  $\Gamma/G$ . We have

$$q(U_{P,r}) = \{\tilde{Q} \in \Gamma/G \mid d(P, Q') < r \text{ for some } Q' \text{ in the orbit } \tilde{Q}\}.$$

To show that  $q(U_{P,r})$  is open in  $\Gamma/G$ , we need to show it contains an open ball around each of its points. Let  $\tilde{Q} \in q(U_{P,r})$  and set

$$B_{\tilde{Q}} = \{\tilde{R} \in \Gamma/G \mid \tilde{d}(\tilde{Q}, \tilde{R}) < r - \tilde{d}(\tilde{P}, \tilde{Q})\},$$

the open ball of radius  $r - \tilde{d}(\tilde{P}, \tilde{Q})$  about  $\tilde{Q}$  in  $\Gamma/G$  (note that, for any point  $Q$  in the orbit  $\tilde{Q}$ ,  $\tilde{d}(\tilde{P}, \tilde{Q}) \leq d(P, Q) < r$ , and so  $B_{\tilde{Q}}$  does indeed have positive radius).

Now, let  $\tilde{R} \in B_{\tilde{Q}}$ . We wish to show that  $\tilde{R} \in q(U_{P,r})$ . By definition,

$$\tilde{d}(\tilde{Q}, \tilde{R}) < r - \tilde{d}(\tilde{P}, \tilde{Q}) \leq r - d(P, Q) \tag{3.1}$$

for any point  $Q$  in the orbit  $\tilde{Q}$ . By definition of the metric  $\tilde{d}$ , there exist points  $Q \in \tilde{Q}$  and  $R \in \tilde{R}$  such that  $\tilde{d}(\tilde{Q}, \tilde{R}) = d(Q, R)$ . Without loss of

generality, we may take this  $Q$  to be the same as the  $Q$  on the right hand side of Equation 3.1 (this is because  $d(\sigma(Q), \sigma(R)) = d(Q, R)$  for any  $\sigma \in G$ ). Therefore, for these  $Q$  and  $R$ , we have

$$d(Q, R) < r - d(P, Q).$$

By the triangle inequality,

$$d(P, R) \leq d(P, Q) + d(Q, R) < d(P, Q) + (r - d(P, Q)) = r.$$

Thus, we have shown that  $\tilde{R}$  is an orbit in  $\Gamma/G$  with representative  $R$  such that  $d(P, R) < r$ . That is,  $\tilde{R} \in q(U_{P,r})$ , implying that  $B_{\tilde{Q}} \subseteq q(U_{P,r})$  and  $q(U_{P,r})$  is open. It follows that  $q$  is an open map, and that the metric topology on  $\Gamma/G$  coincides with  $\tau_q$ .  $\square$

Finally, we have the following theorem on the structure of the quotient  $\Gamma/G$ .

**Theorem 3.3.** *Let  $\Gamma$  be an abstract tropical curve and let  $G \leq \text{Aut } \Gamma$  be finite. Then  $\Gamma/G$  is a metric topological graph.*

*Proof.* By Lemma 3.1 and Lemma 3.2,  $\Gamma/G$  is a compact connected topological space.

Let  $\tilde{P} \in \Gamma/G$ . Now,  $q^{-1}(\tilde{P}) \subseteq \Gamma$  is the set of points in the orbit  $\tilde{P}$ . Each of these points  $P$  has a neighborhood  $N_P$  homeomorphic to some star-shaped set. Call  $\mathcal{S}_{\tilde{P}}$  the set of all of these neighborhoods, and view  $\mathcal{S}_{\tilde{P}}$  as a subset of  $\Gamma$ . Because all of the neighborhoods  $N_P$  are homeomorphic by automorphisms in  $G$ , each neighborhood  $N_P$  in  $\mathcal{S}_{\tilde{P}}$  is homeomorphic to a star-shaped set with the same number,  $n_P$ , of arms. Moreover, the arm length  $r_P$  can be chosen so that it is also the same for each  $N_P$ , and so that the neighborhoods  $N_P$  are disjoint. That is, each connected component of  $\mathcal{S}_{\tilde{P}}$  is homeomorphic to the star-shaped set  $S(n_P, r_P)$ , and the centers of these star-shaped sets correspond to the points in the orbit  $\tilde{P}$ . Now, any automorphism  $\sigma \in G$  maps each center of one of these star-shaped sets to another center. Because  $\sigma$  is an isometry, it must map an arm of a connected component of  $\mathcal{S}_{\tilde{P}}$  (by which I mean a segment of  $\Gamma$  corresponding to an arm of the appropriate star-shaped set) to another arm of a connected component  $\mathcal{S}_{\tilde{P}}$ . In other words, every point in  $\mathcal{S}_{\tilde{P}}$  is mapped into the set  $\{\tilde{Q} \in \Gamma/G \mid \tilde{d}(\tilde{P}, \tilde{Q}) < r_P\}$  under the quotient map  $q$ . If  $M$  is a neighborhood of  $\tilde{P}$  of radius  $r$ , then by taking  $r$  small enough we can ensure that  $q$  restricted to  $\mathcal{S}_{\tilde{P}}$  is surjective and that  $\tilde{P}$  is the only point in  $M$  that is (possibly) of valence greater than 2.

Now, examine  $M$ . It consists of all points whose distance from  $\tilde{P}$  is less than or equal to some  $r$  ( $0 < r \leq r_P$ ). The action of  $G$  partitions the arms of the connected components of  $S_{\tilde{P}}$  into orbits. Suppose there  $m$  orbits. Then  $m \leq n_P$ , because each automorphism in  $G$  is a permutation of the arms. Given a fixed positive  $\rho < r$ , there exist exactly  $m$  distinct points in  $\Gamma/G$  of distance  $\rho$  away from  $\tilde{P}$ . Namely, these are the orbits of the points on the star-shaped sets of distance  $\rho$  away from the centers. That is to say, there are  $m$  segments in  $M$  emanating from  $\tilde{P}$ . Thus, there is a natural bijection between  $M$  and the star-shaped set  $S(m, r)$ . Because we have made the arm length of the star-shaped set equal to the radius of  $M$ , this map is an isometry if  $n_P \neq 1$  (in which case  $\tilde{P}$  may be a point at infinity).  $\square$

**Example 3.3.** It is not the case, in general, that  $\Gamma/G$  is an abstract tropical curve. As an easy counterexample, consider the case where  $\Gamma$  is a tropical line (see Example 1.3). Any of the three rays emanating from the base vertex may be interchanged, and so  $\text{Aut } \Gamma$  is isomorphic to the symmetric group  $S_3$ . Under the full action of  $S_3$ , however, all three rays are identified, and so  $\Gamma/G$  consists of a single ray. That is  $\Gamma/G$  has a 1-valent point that is not an infinite point. We may, however, allow an edge of a quotient  $\Gamma/G$  to continue to have a positive integer multiplicity.

### 3.3 The Group of $G$ -Invariant Divisors

Let  $D$  be a divisor on  $\Gamma$ . If

$$D = \sum_{P \in \Gamma} \lambda_P P,$$

then define

$$D/G = \sum_{\tilde{P} \in \Gamma/G} \left( \sum_{P \in \tilde{P}} \lambda_P \right) \tilde{P} \in \text{Div } \Gamma/G.$$

The map  $D \mapsto D/G$  may be thought of a linear extension of the quotient map  $q: \Gamma \rightarrow \Gamma/G$  to a map  $\varphi: \text{Div } \Gamma \rightarrow \text{Div } \Gamma/G$ . If  $D$  is  $G$ -invariant, then  $D = \sigma(D) = \sum_{P \in \Gamma} \lambda_P \sigma(P)$  for all  $\sigma \in G$ . In order for this to hold, it must be that  $\lambda_P = \lambda_{\sigma(P)}$  for all  $g \in G$ ; that is,  $D$  is constant on  $G$ -orbits. Hence, we may rewrite  $D$ , as

$$D = \sum_{\tilde{P} \in \Gamma/G} \lambda_{\tilde{P}} \sum_{P \in \tilde{P}} P. \tag{3.2}$$

In this case, we have

$$D/G = \sum_{\tilde{P} \in \Gamma/G} |\tilde{P}| \lambda_{\tilde{P}} \tilde{P}, \tag{3.3}$$

where  $|\tilde{P}|$  denotes the number of points of  $\Gamma$  in the orbit  $\tilde{P}$ .

The following theorem gives a description of the group of divisors on  $\Gamma/G$ .

**Theorem 3.4.** *Let  $\Gamma$  be an abstract tropical curve and let  $G$  be a finite subgroup of  $\text{Aut } \Gamma$ . The map  $\varphi: D \rightarrow D/G$  is a surjective homomorphism, and  $\text{Div } \Gamma/G$  is isomorphic to a quotient of  $\text{Div } \Gamma$ . This quotient is proper if  $G$  is nontrivial.*

*Proof.* Let  $D_1 = \sum_{P \in \Gamma} \lambda_P P$ ,  $D_2 = \sum_{P \in \Gamma} \mu_P P$  be in  $\text{Div } \Gamma$ . Then

$$\begin{aligned} (D_1 + D_2)/G &= \sum_{\tilde{P} \in \Gamma/G} \left( \sum_{P \in \tilde{P}} \lambda_P + \mu_P \right) \tilde{P} \\ &= \sum_{\tilde{P} \in \Gamma/G} \left( \sum_{P \in \tilde{P}} \lambda_P \right) \tilde{P} + \sum_{\tilde{P} \in \Gamma/G} \left( \sum_{P \in \tilde{P}} \mu_P \right) \tilde{P} \\ &= D_1/G + D_2/G. \end{aligned}$$

Therefore, the map  $\varphi$  is a homomorphism. Let

$$\tilde{D} = \sum_{\tilde{P} \in \Gamma/G} \lambda_{\tilde{P}} \tilde{P}$$

be a divisor in  $\text{Div } \Gamma/G$ , and let  $\mathcal{S} \subseteq \Gamma$  be a complete set of  $G$ -orbit representatives (so that for each orbit  $\tilde{P} \in \Gamma/G$ , there is exactly one  $S_{\tilde{P}} \in \mathcal{S}$  such that  $S_{\tilde{P}} \in \tilde{P}$ ). Consider

$$D = \sum_{\tilde{P} \in \Gamma/G} \lambda_{\tilde{P}} S_{\tilde{P}} \in \text{Div } \Gamma.$$

Then  $D/G = \tilde{D}$ , and  $\varphi$  is surjective onto  $\text{Div } \Gamma/G$ , establishing the result.

Moreover, the kernel of  $\varphi$  is the subgroup

$$K = \left\{ D = \sum_{P \in \Gamma} \lambda_P P \mid \sum_{P \in \tilde{P}} \lambda_P = 0 \text{ for each orbit } \tilde{P} \in \Gamma/G \right\} \leq \text{Div}^0 \Gamma. \quad (3.4)$$

If  $G$  is nontrivial, then so is  $K$ , and the quotient is proper.  $\square$

**Corollary 3.1.** *Let  $\Gamma$  be an abstract tropical curve and let  $G$  be a finite subgroup of  $\text{Aut } \Gamma$ . Then  $\text{Div}^G \Gamma$  is isomorphic to a subgroup of  $\text{Div } \Gamma/G$ .*

*Proof.* Restricting  $\varphi$  to  $\text{Div}^G \Gamma$ , we lose surjectivity, but we gain injectivity. Indeed, if

$$D/G = \sum_{\tilde{P} \in \Gamma/G} |\tilde{P}| \lambda_{\tilde{P}} \tilde{P} = 0,$$

then  $\lambda_{\tilde{P}} = 0$  for each orbit  $\tilde{P}$  of  $G$  on  $\Gamma$ . That is,  $D$  takes on the value 0 at every point of every orbit of  $G$  on  $\Gamma$ . The union of all  $G$ -orbits is  $\Gamma$  (all points are in some orbit), and so  $D$  must be the divisor that is everywhere zero.  $\square$

The next result is obtained by replacing the map  $\varphi$  with a similar one. If  $D \in \text{Div}^G \Gamma$ , then we can write  $D$  as in Equation 3.2. Consider the mapping

$$\psi: D = \sum_{\tilde{P} \in \Gamma/G} \lambda_{\tilde{P}} \sum_{P \in \tilde{P}} P \mapsto \sum_{\tilde{P} \in \Gamma/G} \lambda_{\tilde{P}} \tilde{P} \quad (3.5)$$

from  $\text{Div}^G \Gamma$  to  $\text{Div} \Gamma/G$ .

**Theorem 3.5.** *Let  $\Gamma$  be an abstract tropical curve and let  $G$  be a finite subgroup of  $\text{Aut} \Gamma$ . The map  $\psi$  is an isomorphism and  $\text{Div}^G \Gamma \cong \text{Div} \Gamma/G$ .*

*Proof.* Arguments analogous to those in Theorem 3.4 and Corollary 3.1, immediately give us that the map is an injective homomorphism. Surjectivity follows from Equation 3.5, because the coefficients  $\lambda_{\tilde{P}}$  may take on any integer values.  $\square$

We have now established that

$$\text{Div}^G \Gamma \cong \text{Div} \Gamma/G \cong (\text{Div} \Gamma)/K, \quad (3.6)$$

where  $K$  is as in Equation 3.4. By Corollary 3.1,  $\text{Div}^G \Gamma$  is isomorphic to a (proper, if  $G$  is nontrivial) subgroup of  $\text{Div} \Gamma/G$ . Hence, by the above isomorphisms,  $\text{Div} \Gamma/G$  is isomorphic to a proper subgroup of itself. (There are, in fact, many subgroups of  $\text{Div} \Gamma/G$  isomorphic to the entire group. This is similar to the way that the additive group  $\mathbb{Z}$  contains several isomorphic copies of itself, namely  $n\mathbb{Z}$  for any integer  $n$ .) Also of note is that we have shown, courtesy of Theorem 3.4 and Theorem 3.5, that  $\text{Div}^G \Gamma$ , which is a subgroup of  $\text{Div} \Gamma$ , is in fact isomorphic to a (proper, if  $G$  is nontrivial) quotient of  $\text{Div} \Gamma$ . We can see this concretely. The composite map  $\psi^{-1}\varphi: \text{Div} \Gamma \rightarrow \text{Div} \Gamma$ , where

$$\psi^{-1}\varphi: \sum_{P \in \Gamma/G} \lambda_P P \mapsto \sum_{\tilde{P} \in \Gamma/G} \left( \sum_{P \in \tilde{P}} \lambda_P \right) \sum_{P \in \tilde{P}} P,$$

is surjective onto  $\text{Div}^G \Gamma$  and has kernel  $K$ .

### 3.4 The Group of $G$ -Invariant Divisor Classes

The next logical step is to attempt to show that a result analogous to Equation 3.6 holds for the Picard groups. The maps  $\varphi$  and  $\psi$  from Section 3.3 induce maps between Picard groups. This section examines those maps, and demonstrates that they lead to partial analogues of Theorem 3.4 and Theorem 3.5. In particular, we shall see that it is critically important to understand how the maps  $\varphi$  and  $\psi$  behave on principal divisors.

Define a map  $\bar{\varphi}: \text{Pic } \Gamma \rightarrow \text{Pic } \Gamma/G$  by

$$\bar{\varphi}: [D] \mapsto [D/G].$$

Define another map  $\bar{\psi}: \text{Pic}^G \Gamma \rightarrow \text{Pic } \Gamma/G$  by

$$\bar{\psi}: [D] \mapsto [\psi(D)],$$

where only  $G$ -invariant representatives are used (so that  $\psi(D)$  is defined). By Theorem 2.2, every linear system in  $\text{Pic}^G \Gamma$  has a  $G$ -invariant representative, and so  $\bar{\psi}$  is defined on all of  $\text{Pic}^G \Gamma$ .

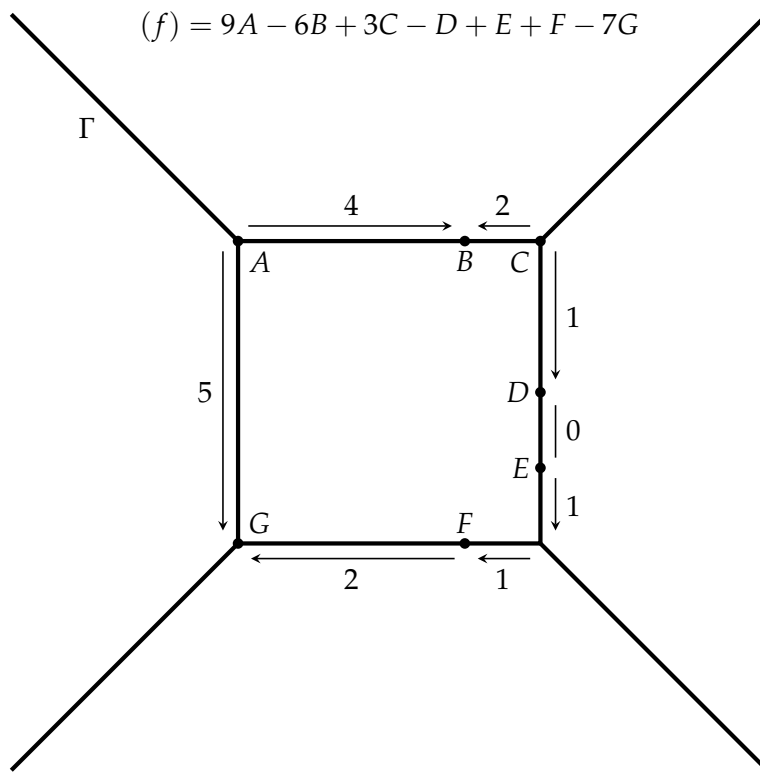
Now, because  $\varphi$  and  $\psi$  are homomorphisms and the Picard group is a quotient group, it is easy to see that  $\bar{\varphi}$  and  $\bar{\psi}$  will be homomorphisms if they are well-defined. These maps will be well-defined if  $\varphi$  and  $\psi$  map principal divisors to principal divisors, because then the images of equivalent divisors will remain equivalent. Note that rational functions and principal divisors are defined on metric topological graphs in exactly the same way as they were for abstract tropical curves in Section 2.1 and Section 2.2

We first search for an analogue to Theorem 3.4. We wish to determine if, for each principal divisor  $(f)$  on  $\Gamma$ , there exists a rational function on  $\Gamma/G$  with divisor equal to  $\varphi((f)) = (f)/G$ . There is no obvious way to find such a rational function on  $\Gamma/G$ , so we consider some examples as motivation.

**Example 3.4.** Consider the abstract tropical curve illustrated in Figure 3.1. The divisor

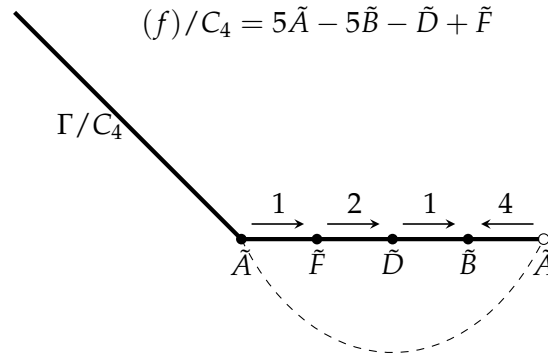
$$9A - 6B + 3C - D + E + F - 7G$$

is principal. Indeed, the arrows drawn in Figure 3.1 demonstrate a rational function that realizes this divisor: The net change of around the cycle is zero, and so any function  $f$  with these slopes, and which is constant on the infinite arms, is continuous and therefore principal. Let  $\Gamma$  be acted upon by  $C_4$ , the cyclic group of order 4 generated by a rotation of  $\Gamma$  by  $\pi/2$ .



**Figure 3.1** An abstract tropical curve  $\Gamma$  and principal divisor  $(f)$ . The interior (finite) edges of  $\Gamma$  have unit length, and vertices  $B$ ,  $E$ , and  $F$  are each of distance  $\frac{1}{4}$  away from the nearest 3-valent point. The labeled arrows represent the slopes of the rational function  $f$ .





**Figure 3.2** The quotient of the curve from Figure 3.1 under the action of  $C_4$ . The divisor  $(f)/C_4$  is seen to be principal.

This action partitions the support of  $(f)$  into four orbits:  $\tilde{A} = \{A, C, G\}$ ,  $\tilde{B} = \{B, E\}$ ,  $\tilde{D} = \{D\}$ , and  $\tilde{F} = \{F\}$ . Hence,

$$(f)/C_4 = 5\tilde{A} - 5\tilde{B} - \tilde{D} + \tilde{F} \in \text{Div } \Gamma/C_4.$$

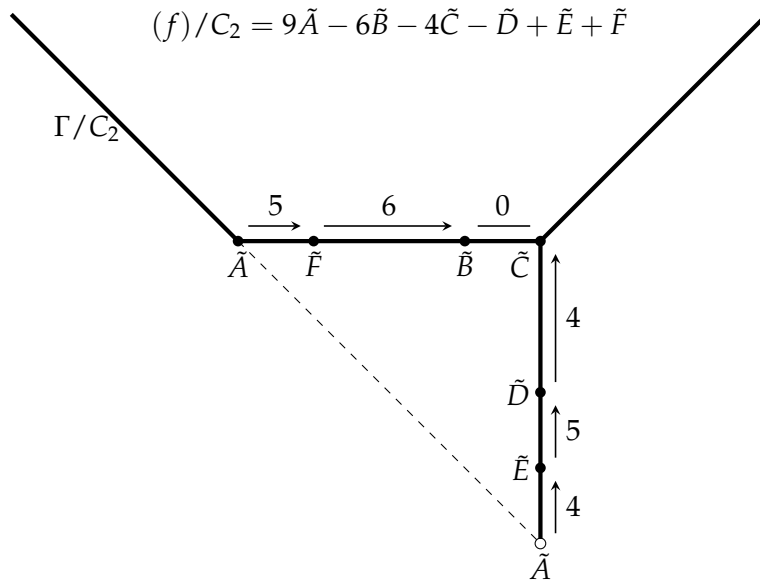
We wish to show that  $(f)/C_4$  is in fact principal on  $\Gamma/C_4$ . In order to do so, we must find a rational function on  $\Gamma/C_4$  that realizes this divisor. This is easily done, as illustrated in Figure 3.2. Note that the quotient graph remains of genus 1, and so continuity must be verified.

It is worth noting that the rational function on  $\Gamma/C_4$  that realizes the divisor  $(f)/C_4$  has slopes given by “superimposing” the slopes of  $f$  on  $\Gamma$ . For instance, the slope between  $\tilde{A}$  and  $\tilde{F}$  is given by the sum of all of the slopes of  $f$  on the corresponding segments in  $\Gamma$ :  $4 + 1 + 1 - 5 = 1$ . The other slopes in Figure 3.2 are obtained in a similar manner.

**Example 3.5.** Again, consider the abstract tropical curve  $\Gamma$  and principal divisor  $(f)$  in Figure 3.1. This time, let  $\Gamma$  be acted upon by  $C_2$ , the cyclic group of order 2 generated by a rotation of  $\Gamma$  by  $\pi$ . (We could consider the cyclic group of order 2 generated by a reflection, but because the resulting quotient would be of genus zero, this case is less interesting.) This action partitions the support of  $(f)$  into six orbits:  $\tilde{A} = \{A\}$ ,  $\tilde{B} = \{B\}$ ,  $\tilde{C} = \{C, G\}$ ,  $\tilde{D} = \{D\}$ ,  $\tilde{E} = \{E\}$ , and  $\tilde{F} = \{F\}$ . Therefore,

$$(f)/C_2 = 9\tilde{A} - 6\tilde{B} - 4\tilde{C} - \tilde{D} + \tilde{E} + \tilde{F}.$$

Again, we can construct a rational function on  $\Gamma/C_2$  that realizes this divisor. The result is shown in Figure 3.3.



**Figure 3.3** The quotient of the curve from Figure 3.1 under the action of  $C_2$ . The divisor  $(f)/C_2$  is seen to be principal.

Once again, the rational function that realizes  $(f)/C_2$  is obtained by “superimposing” the slopes of  $f$  over all orbits.

It appears, based on these examples, that this method of “superimposing” slopes will always work. However, the next example shows that this is not the case.

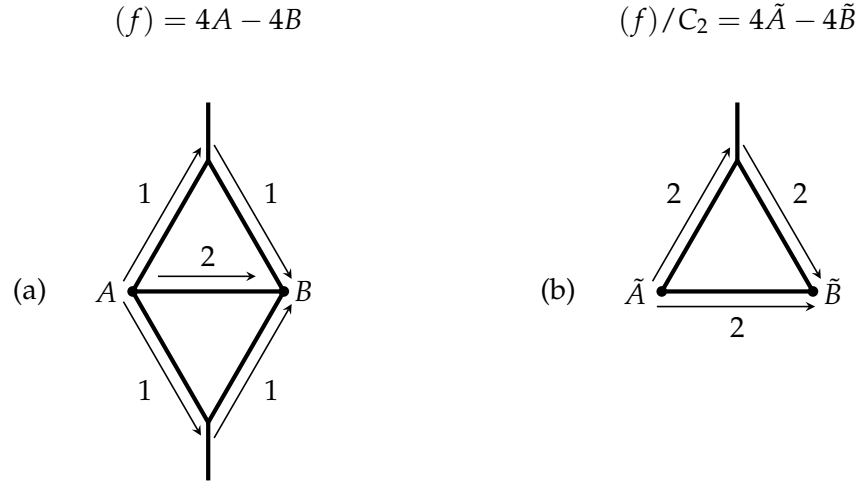
**Example 3.6.** Let  $\Gamma$  be the abstract tropical curve shown in Figure 3.4a. The divisor

$$(f) = 4A - 4B$$

is principal, as illustrated by the slopes in the figure. Let  $C_2$  act on  $\Gamma$  by reflection about its horizontal line of symmetry. The quotient  $\Gamma/C_2$  is shown in Figure 3.4b. Under this action, both  $A$  and  $B$  are fixed points, and so they are their own orbits,  $\tilde{A}$  and  $\tilde{B}$ , respectively. Hence, the image of  $(f)$  under  $\varphi$  is

$$(f)/C_2 = 4\tilde{A} - 4\tilde{B}.$$

Figure 3.4b also shows the slopes on  $\Gamma/C_2$  that result from the same “superimposing” trick used in Example 3.4 and Example 3.5. However, in this case, it is easily seen that these slopes do *not* describe a rational function on



**Figure 3.4** (a) An abstract tropical curve  $\Gamma$  and principal divisor  $(f)$ . All inner edges of  $\Gamma$  have unit length. (b) Under the action of  $C_2$  on  $\Gamma$  by reflection, the divisor  $(f)/C_2$  is not principal.

$\Gamma/C_2$ , because there is a net change around the cycle. Moreover, it is impossible to create a rational function  $g$  on  $\Gamma/C_2$  that only has singularities at  $\tilde{A}$  and  $\tilde{B}$  such that  $\text{ord}_A(g) = 4$  and  $\text{ord}_B(g) = -4$ .

Therefore, in this case, the map  $\bar{\varphi}$  is not a homomorphism, because it is not well-defined:  $[4A] = [4B]$ , but  $\bar{\varphi}([4A]) \neq \bar{\varphi}([4B])$ .

As Example 3.6 demonstrated, the “superimposing” technique can fail when the genus of  $\Gamma$  is greater than one. In fact, if we restrict the genus to be at most one, then we can guarantee the superimposing method works. First, we need the following lemma.

**Lemma 3.3.** *Let  $\Gamma$  be an abstract curve and  $G \leq \text{Aut } \Gamma$  be finite. Suppose that  $\Gamma/G$  has nonzero genus. Then if  $\tilde{P}$  lies on a cycle in  $\Gamma/G$ , all of the points  $P \in \Gamma$  in the orbit  $\tilde{P}$  lie on a cycle in  $\Gamma$ . Moreover, the genus of  $\Gamma/G$  is at most the genus of  $\Gamma$ .*

*Proof.* Consider a cycle in  $\Gamma/G$ . Let  $\tilde{P}_1$  and  $\tilde{P}_2$  be points on the cycle of maximal distance apart. That is, there exist two paths on the cycle of equal distance, say  $D$ , from  $\tilde{P}_1$  to  $\tilde{P}_2$ . Let  $P_1 \in \Gamma$  be some point in the cycle  $\tilde{P}_1$ . Then there must exist two distinct paths of length  $D$  in  $\Gamma/G$  from  $P_1$  to points in the orbit  $\tilde{P}_2$ , say  $P_2$  and  $P'_2$ . Now, if  $P_2 = P'_2$ , then these points lie on a cycle. Otherwise, there is another path on the cycle of length  $D$  from  $P_2$  to a point  $P'_1$  in the orbit  $\tilde{P}_1$ , and similarly, there is a path of length  $D$  on

the cycle from  $P'_2$  to a point  $P''_1 \in \tilde{P}_1$ . If  $P'_1 = P''_1$ , then these points lie on a cycle in  $\Gamma$ . Otherwise, we may repeat this process. However, because  $G$  is finite, the orbits of  $G$  on  $\Gamma$  contain finitely many points, and therefore this process must terminate. Thus, all of the points in  $\Gamma$  corresponding to the orbits  $P'_1$  and  $P'_2$  lie on a cycle in  $\Gamma$ .

We have seen that for each cycle in  $\Gamma/G$ , there exists at least one cycle in  $\Gamma$ . Moreover, if the process described above is applied to distinct cycles in  $\Gamma/G$ , then the result will be distinct cycles in  $\Gamma$ . The reason for this is if  $\tilde{Q} \in \Gamma/G$  is on a cycle other than the one above (containing  $\tilde{P}_1$  and  $\tilde{P}_2$ ), then  $\tilde{Q}$  can be chosen so that it is a distance greater than  $D$  away from one of the points on the original cycle; without loss of generality,  $\tilde{d}(\tilde{P}_1, \tilde{Q}) > D$ . But then, by definition of the metric  $\tilde{d}$ , the points in  $\Gamma$  in the orbit  $\tilde{Q}$  must all be a distance greater than  $D$  from the points  $P_1, P'_1, P''_1, \dots$ . That is, each point in  $\Gamma$  corresponding to  $\tilde{Q}$  lies on a cycle distinct from the one constructed above. Consequently, the genus of  $\Gamma/G$  can be no greater than the genus of  $\Gamma$ .  $\square$

**Proposition 3.3.** *Let  $\Gamma$  be an abstract tropical curve of genus  $g(\Gamma) \leq 1$ . If  $(f) \in \text{Prin } \Gamma$ , then  $\varphi((f))/G \in \text{Prin } \Gamma/G$ .*

*Proof.* We define a function  $\hat{f}$  on  $\Gamma/G$  by

$$\hat{f}(\tilde{P}) = \sum_{P \in \tilde{P}} f(P)$$

for each orbit  $\tilde{P} \in \Gamma/G$ , where the sum is taken over all points of  $\Gamma$  in the orbit.

Every  $P \in \tilde{P}$  has a neighborhood homeomorphic to the same star-shaped set,  $S(n, r)$ . Let  $\mathcal{S}_{\tilde{P}}$  be the collection of these star-shaped sets. Because  $f$  has finitely many singular points,  $r$  can be chosen small enough so that  $f$  is linear on each arm of each star-shaped sets in  $\mathcal{S}_{\tilde{P}}$  (by which I mean  $f$  is linear on the corresponding intervals in  $\Gamma$ ). By Theorem 3.3, there is a neighborhood of  $\tilde{P}$  in  $\Gamma/G$  that is homeomorphic to a star-shaped set  $T = S(m, r)$ , where  $m \leq n$ . Suppose that  $X \in \Gamma$  is a point (i.e., corresponds under the homeomorphism to a point) in some star-shaped set  $S \in \mathcal{S}_{\tilde{P}}$ . Then any automorphism  $\sigma \in G$  maps  $X$  to a point on another star-shaped set  $S' \in \mathcal{S}_{\tilde{P}}$ , and because  $\sigma$  preserves distances, it must map the entire arm in  $S$  that contains  $X$  to a single arm of  $S'$ . Thus, each arm of  $T$  corresponds to some number of arms (an “orbit” of arms under  $G$ ) on the star-shaped sets in  $\mathcal{S}_{\tilde{P}}$ . The value of  $\hat{f}$  on an arm of  $T$  is obtained by summing the values of  $f$  on the corresponding arms. But the sum of linear functions is linear,

and thus  $\hat{f}$  is linear on each arm of  $T$ . Because this holds for all  $\tilde{P} \in \Gamma/G$ , this shows that  $\hat{f}$  is piecewise-linear on  $\Gamma/G$ .

The slope of  $\hat{f}$  on a segment of  $\Gamma/G$  is obtained by summing the corresponding slopes of  $f$ , and so  $\hat{f}$  has integer slopes. Moreover, by the construction above, the sum of the outgoing slopes of  $\hat{f}$  at  $\tilde{P}$  is

$$\sum_{P \in \tilde{P}} \text{ord}_P(f). \quad (3.7)$$

Because  $\text{ord}_P(f)$  is zero except for at finitely many points (the singular points of  $f$ ), the quantity in Equation 3.7 is also zero except for at finitely many orbits  $\tilde{P}$ . Thus,  $\hat{f}$  is piecewise-linear with integer slopes and finitely many pieces.

In order to show that  $\hat{f}$  is a rational function, then, we would need to show that it is continuous. However, as we have constructed it,  $\hat{f}$  is *not* necessarily continuous. We will, then, construct a continuous function  $\tilde{f}$  defined on  $\Gamma/G$  that everywhere has the same slope as  $\hat{f}$ . What this amounts to, in essence, is adding a different constant to the value of  $\hat{f}$  on each of its pieces so that the result is continuous. If the genus of  $\Gamma/G$  is zero, then this is trivial. However, otherwise we must check that the sum of the slopes multiplied by the segment length around any cycle in  $\Gamma/G$  is zero.

Suppose, then, that  $\Gamma/G$  has nonzero genus. By Lemma 3.3,  $\Gamma/G$  must be of genus one. Moreover, each  $\tilde{P}$  in the unique cycle in  $\Gamma/G$  corresponds to an orbit of points in  $\Gamma$ , all of which lie on the unique cycle in  $\Gamma$ . The cycle in  $\Gamma/G$  corresponds to several disjoint line segments in  $\Gamma$ , each of which lies on the cycle in  $\Gamma$  and whose union is the entire cycle.

Divide the cycle in  $\Gamma/G$  into segments on which  $\hat{f}$  has constant slope. Say the  $i$ th segment has length  $\ell_i$  and the slope of  $\hat{f}$  there is  $m_i$ . We may write  $m_i$  as the sum  $m_i = \sum_j m_{ij}$ , where  $m_{ij}$  is the slope of  $f$  on the  $j$ th segment of  $\Gamma$  (which lies on the cycle) in the corresponding orbit. Then

$$\sum_i \ell_i m_i = \sum_i \ell_i \left( \sum_j m_{ij} \right) = \sum_{i,j} \ell_i m_{ij}.$$

Now, this sum is the net change of  $f$  over the cycle in  $\Gamma$ , and because  $f$  is a continuous function, this quantity is zero. Therefore, it is possible to adjust  $\hat{f}$  so that we get a continuous function  $\tilde{f}$  on  $\Gamma/G$  with the same slopes as  $\hat{f}$ .

Finally, by Equation 3.7, we have that

$$\text{ord}_P(\tilde{f}) = \sum_{P \in \tilde{P}} \text{ord}_P(f).$$

It follows that  $(\tilde{f}) = (f)/G$ ; that is,  $(f)/G$  is a principal divisor on  $\Gamma/G$ .  $\square$

By Proposition 3.3 and the surjectivity of  $\varphi$ , we have the following result, which provides an analogue to Theorem 3.4 in low genus.

**Theorem 3.6.** *Let  $\Gamma$  be an abstract tropical curve of genus  $g(\Gamma) \leq 1$ , and let  $G$  be a finite subgroup of  $\text{Aut } \Gamma$ . Then  $\bar{\varphi}: \text{Pic } \Gamma \rightarrow \text{Pic } \Gamma / G$  is a surjective homomorphism and  $\text{Pic } \Gamma / G$  is isomorphic to a quotient of  $\text{Pic } \Gamma$ . We have*

$$\text{Pic } \Gamma / G \cong \text{Pic } \Gamma$$

*if and only if  $\bar{\varphi}$  is injective; that is, if and only if  $\varphi(D) \in \text{Prin } \Gamma / G$  implies  $D \in \text{Prin } \Gamma$  for every divisor  $D \in \text{Div } \Gamma$ .*

We now turn our attention to finding an analogue to finding an analogue to Theorem 3.5. This is difficult, because  $\bar{\psi}$  does not, in general, respect principal divisors, and I have so far been unable to determine a rule for when it does. However, we can work around this by applying Theorem 2.2. Because each equivalence class in  $\text{Pic}^G \Gamma$  has at least one  $G$ -invariant representative, we can consider  $\text{Pic}^G \Gamma$  as equivalence classes of  $G$ -invariant divisors. Because the difference of two  $G$ -invariant divisors is  $G$ -invariant, we have that

$$\text{Pic}^G \Gamma \cong (\text{Div}^G \Gamma) / (\text{Prin}^G \Gamma).$$

Thus, instead of using the (possibly ill-defined) new map  $\bar{\psi}$ , we can instead simply use our old map  $\psi$ .

**Theorem 3.7.** *Let  $\Gamma$  be an abstract tropical curve and let  $G$  be a finite subgroup of  $\text{Aut } \Gamma$ . Then*

$$\text{Pic}^G \Gamma \cong (\text{Div } \Gamma / G) / N,$$

*where  $N = \psi(\text{Prin}^G \Gamma) \leq \text{Div } \Gamma / G$ .*

*Proof.* This follows from Theorem 3.5 and the fact that

$$H / K \cong \varphi(H) / \varphi(K)$$

for an injective homomorphism  $\varphi$  defined on a group  $H$  with normal subgroup  $K$ . □

While I have not found a general description of the quotient groups involved in Theorem 3.6 and Theorem 3.7, there are situations in which they are easily described. The next theorem illustrates a particularly nice case, in which  $\bar{\varphi}$  is injective and  $\psi(\text{Prin}^G \Gamma) = \text{Prin } \Gamma / G$ .

**Theorem 3.8.** *If  $\Gamma$  is an abstract tropical curve of genus zero and  $G \leq \text{Aut } \Gamma$  is finite, then*

$$\text{Pic}^G \Gamma \cong \text{Pic } \Gamma / G \cong \text{Pic } \Gamma \cong \mathbb{Z}.$$

*Proof.* Fix a base point  $O \in \Gamma$ . Consider any point  $P \in \Gamma$ . Then  $P$  is equivalent to  $O$ , because a rational function that has slope 1 on the path from  $O$  to  $P$  and is constant everywhere else will have  $P - O$  as its principal divisor. Therefore, if  $D \in \text{Div } \Gamma_n$ , then  $D$  is equivalent to  $(\deg D)O$ . Thus, the linear systems in  $\text{Pic } \Gamma$  are precisely the collections of all divisors of a given degree. That is, the map  $[D] \mapsto \deg D$  is a bijection between  $\text{Pic } \Gamma$  and the integers  $\mathbb{Z}$ . Moreover, this is an isomorphism, as  $\deg(D_1 + D_2) = \deg D_1 + \deg D_2$  for any divisors  $D_1, D_2$ .

Because  $\deg D = \deg \sigma(D)$  for any divisor  $D \in \text{Div } \Gamma_n$  and automorphism  $\sigma \in G$ , we have that  $\sigma([D]) = [D]$  for all  $[D] \in \text{Pic } \Gamma$ . Thus,  $\text{Pic}^G \Gamma = \text{Pic } \Gamma$ .

By Lemma 3.3, if  $\Gamma/G$  has nonzero genus, then  $\Gamma$  must have nonzero genus as well. Therefore, because  $\Gamma$  has genus zero, so does  $\Gamma/G$ . Thus, the same analysis applies to  $\Gamma/G$ : If  $O/G = \bar{\varphi}(O)$ , then a divisor  $\tilde{D} \in \text{Div } \Gamma/G$  is equivalent to  $(\deg \tilde{D})(O/G)$ , and therefore  $\text{Pic } \Gamma/G \cong \mathbb{Z}$ .

In summary, we have

$$\text{Pic}^G \Gamma = \text{Pic } \Gamma \cong \mathbb{Z} \cong \text{Pic } \Gamma/G$$

when  $\Gamma$  has genus zero. □

## Chapter 4

# Conclusion and Future Work

We have seen that there is a nice relationship between the group of  $G$ -invariant divisors on a tropical curve  $\Gamma$  and the group of divisors on the quotient  $\Gamma/G$ . Namely,

$$\mathrm{Div}^G \Gamma \cong \mathrm{Div} \Gamma / G \cong (\mathrm{Div} \Gamma) / K,$$

where  $K$  is a subgroup of  $\mathrm{Div} \Gamma$ . We have also seen that, under certain conditions, the group of linear systems of divisors on  $\Gamma$  can satisfy similar relationships. These results establish connections between groups that are potentially very large, and knowledge of these connections may lead to further understanding of the groups involved. The work presented here has been conducted in a very concrete framework. Going forward, it is likely that this study would benefit from also working in a more abstract setting, where there is more of an established theory to work with. For example, Joyner and colleagues (2010) successfully used group cohomology to study  $G$ -invariant divisors on abstract tropical curves.





# Appendix A

## Chip-Firing

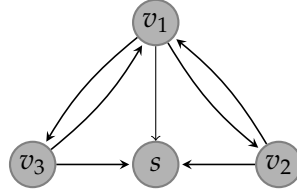
This appendix describes the chip-firing game, also called the abelian sand-pile model. In order to introduce the model, several definitions and assertions are needed. To avoid simply transcribing well-known proofs, we choose to omit proofs altogether in favor of examples. The interested reader is referred to Holroyd et al. (2008). While chip-firing did not play a role in the investigations in Chapter 3, it is possible that these ideas can be applied to study group actions on abstract tropical curves. For instance, an abstract tropical curve  $\Gamma$  can be made into a directed graph by replacing the edges of the minimal graph with a pair of opposite-pointing directed edges. More points from  $\Gamma$  can be added to this graph as vertices, if desired. An effective divisor on  $\Gamma$  can then be interpreted as a chip configuration.

### A.1 Basic Definitions

The chip-firing game is played on a finite directed graph (digraph)  $G = (V, E)$ , with self-loops allowed. The *out-degree*  $d_v$  of a vertex  $v \in V$  is the number of directed edges emanating from  $v$ . A vertex  $s$  is a *sink* if it has out-degree zero, and a *global sink* if it is a sink and there is a path from every other vertex to  $s$ . If a global sink exists, it is the unique sink in the digraph.

Note: To connect the material in this appendix to tropical curves, observe that we may construct a (bi)directed graph from any (abstract) tropical curve  $\Gamma$ . To do so, let  $G$  be the minimal graph of  $\Gamma$ , as defined in Section 1.5, and replace each edge of  $G$  with a pair of opposite-pointing directed edges.

Label the vertices of  $G$  as  $v_1, \dots, v_n$ . The *adjacency matrix*  $A = (a_{ij})$  of



**Figure A.1** A digraph with global sink  $s$ .

$G$  is an  $n \times n$  matrix where  $a_{ij}$  is the number of edges from  $v_i$  to  $v_j$ . The *Laplacian* of  $G$  is the matrix  $\Delta = D - A$ , where  $D$  is the diagonal matrix with  $i$ th entry  $d_i$ . The  $ij$ th entry of  $\Delta$  is then

$$\Delta_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j, \\ d_i - a_{ii} & \text{if } i = j. \end{cases}$$

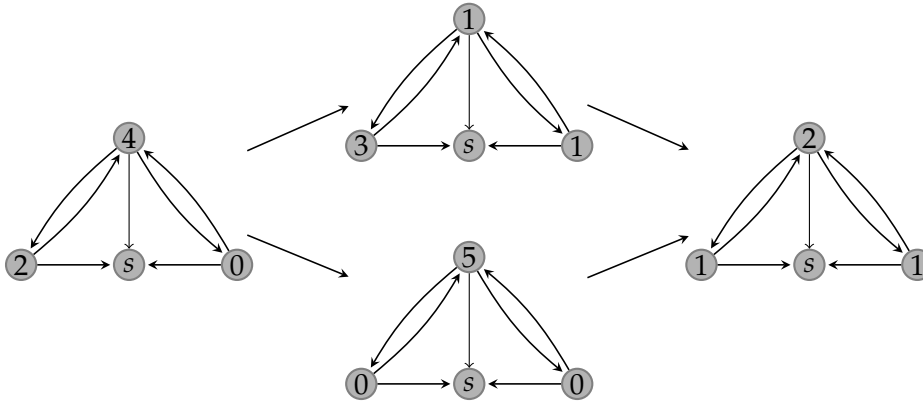
Because the sum across the  $i$ th row of  $A$  is  $d_i$ , the rows of  $\Delta$  sum to zero. If  $v_i$  is a sink, then the  $i$ th row of  $\Delta$  consists only of zeros. For our examples, we will be using the directed graph on four vertices with a global sink shown in Figure A.1. For this graph, the Laplacian is

$$\Delta = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A *chip configuration* or *sandpile*  $\sigma$  on  $G$  is an assignment of a nonnegative integer  $\sigma(v)$  to each nonsink vertex  $v$  of  $G$ . If  $G$  has  $k$  nonsink vertices, then  $\sigma$  can be written as a vector in  $\mathbb{Z}^k$ . The chip configuration  $\sigma$  is *stable* if  $\sigma(v) < d_v$  for all nonsink vertices  $v$ . If  $\sigma(v) \geq d_v$  for some nonsink  $v$ , then we say that the vertex  $v$  is *active*. An active vertex  $v$  may *fire*, which consists of removing  $d_v$  chips from  $v$  and placing one of them at each of the vertices adjacent to  $v$ . Denoting the resulting chip configuration by  $\sigma'$ , we have  $\sigma'(w) = \sigma(w) + a_{vw}$  for nonsink  $w \neq v$ , and  $\sigma'(v) = \sigma(v) - d_v + a_{vv}$ . The chip configuration  $\sigma'$  is called a *successor* of  $\sigma$ .

One interesting fact is that the outcome of the chip-firing game is independent of the order in which vertices are fired. This is illustrated in Figure A.2. More precisely,

**Lemma A.1.** *Let  $G$  be a digraph, and let  $\sigma_0, \sigma_1, \dots, \sigma_n$  be a sequence of chip configurations on  $G$ , each of which is a successor of the previous one. Let  $\sigma'_0, \sigma'_1, \dots, \sigma'_m$  be another such sequence with the same initial configuration, that is,  $\sigma'_0 = \sigma_0$ .*



**Figure A.2** The order of firing is irrelevant to the final outcome.

1. If  $\sigma_n$  is stable, then  $m \leq n$ , and, moreover, no vertex fires more times in  $\sigma'_0, \sigma'_1, \dots, \sigma'_m$  than in  $\sigma_0, \sigma_1, \dots, \sigma_n$ .
2. If  $\sigma_n$  and  $\sigma'_m$  are both stable, then  $m = n$ ,  $\sigma_n = \sigma'_m$ , and each vertex fires the same number of times in both histories.

It is a consequence of Lemma A.1 that, starting from a given configuration  $\sigma$ , there is at most one stable chip configuration that can be reached. If such a stable configuration exists, we call it the *stabilization* of  $\sigma$  and write  $\sigma^\circ$ . We have the following useful result.

**Lemma A.2.** *If the digraph  $G$  has a global sink, then every chip configuration on  $G$  stabilizes.*

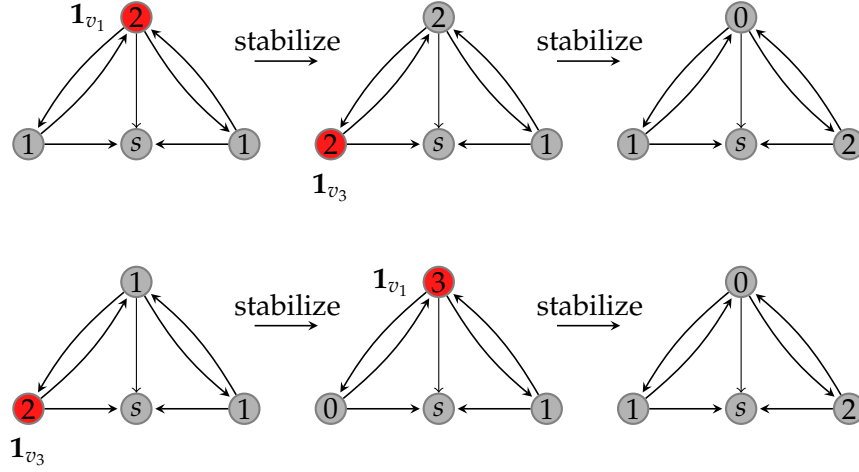
Thus, every chip configuration we place on the digraph in Figure A.1 will stabilize.

## A.2 The Sandpile Group

We now begin to investigate the algebraic structure of sandpiles. To start with, we define the *chip-addition operator*  $E_v$ , a map which is applied to chip configurations, which adds a single chip at the vertex  $v$ . That is,

$$E_v \sigma = (\sigma + \mathbf{1}_v)^\circ,$$

where  $\mathbf{1}_v$  is the chip configuration that consists of a single chip at  $v$ . The next result is called the *abelian property* (hence the name of the model).



**Figure A.3** The abelian property. A red vertex indicates that a chip has just been added there.

**Lemma A.3.** *On a digraph with a global sink, the chip-addition operators commute.*

Lemma A.3 is illustrated in Figure A.3. It also has the consequence that, rather than adding chips through chip-addition operators, we may add all of the chips at once, and then stabilize once to achieve the same result.

If  $G$  is a digraph with a global sink, then we may consider chip configurations on  $G$  as vectors in  $\mathbb{Z}^{n-1}$ . Remove the row and column corresponding to the sink from the Laplacian  $\Delta$  of  $G$ . The resulting  $(n-1) \times (n-1)$  matrix  $\Delta'$  is called the *reduced Laplacian*. Observe that, starting from chip configuration  $\sigma$ , firing the nonsink vertex  $v$  results in the configuration  $\sigma - \Delta'_v$ , where  $\Delta'_v$  is the row of the reduced Laplacian corresponding to  $v$ . We wish to identify the configurations before and after firing as equivalent. Hence, we consider the chip configurations as living in the quotient

$$\mathbb{Z}^{n-1} / \mathbb{Z}\Delta',$$

where  $\mathbb{Z}\Delta'$  is the integer row-span of  $\Delta'$ . We see, then, that the chip configurations on a digraph have a quotient group structure.

**Definition A.1.** Let  $G$  be a digraph with  $n$  vertices and a global sink. The *sandpile group* of  $G$  is the quotient

$$\mathcal{S}(G) = \mathbb{Z}^{n-1} / \mathbb{Z}\Delta',$$

where  $\Delta'$  is the reduced Laplacian of  $G$ .

**Example A.1.** For the digraph  $G$  in Figure A.1, the reduced Laplacian is

$$\Delta' = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

It can be seen that

$$\mathcal{S}(G) = \mathbb{Z}^{n-1} / \mathbb{Z}\Delta' \cong \mathbb{Z}/8\mathbb{Z},$$

and  $\mathcal{S}(G)$  is generated by the coset containing  $(0, 0, 1)$ .

**Lemma A.4.** *The order of  $\mathcal{S}(G)$  is the determinant of the reduced Laplacian  $\Delta'$ .*

Indeed, for our example,  $\det \Delta' = 8$ .

There is the issue as to which sandpiles to choose as the representative elements in  $\mathcal{S}(G)$ . Fortunately, there is a natural choice. A chip configuration  $\sigma$  is *accessible* if from any other chip configuration it is possible to obtain  $\sigma$  by a combination of adding chips and selectively firing active vertices. A chip configuration is *recurrent* if it is both stable and accessible. We have the following result:

**Theorem A.1.** *Let  $G$  be a digraph with a global sink. Every equivalence class of  $\mathbb{Z}^{n-1}$  modulo  $\Delta'$  contains exactly one recurrent chip configuration of  $G$ .*

It follows that we may alternatively view  $\mathcal{S}(G)$  as the set of all recurrent chip configurations on  $G$  under the group operation

$$(\sigma, \sigma') \mapsto (\sigma + \sigma')^\circ.$$

This operation works because the sum of two accessible configurations is accessible, and stabilizing does not alter the equivalence class of a configuration.



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