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Howard Dwyer

William Green

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Integrating Factors and Repeated Roots of the Characteristic Equation

Howard I. Dwyer
Monmouth College

William R. Green
Eastern Illinois University

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Abstract: Most texts on elementary differential equations solve homogeneous constant coefficient linear equations by introducing the characteristic equation; once the roots of the characteristic equation are known the solutions to the differential equation follow immediately, unless there is a repeated root. In this paper we show how an integrating factor can be used to find all of the solutions in the case of a repeated root without depending on an assumption about the form that these solutions will take. We also show how an integrating factor can be used to explain the “extra” power of t which appears in the trial form of the solution when using the method of undetermined coefficients on a nonhomogeneous equation in the case where the right hand side is a polynomial multiple of the corresponding homogeneous solution.

1 Motivation and Intuition

Constant coefficient, linear differential equations are well-studied in introductory differential equation classes. The standard method is to use an ansatz to transform the differential equation into a polynomial algebraic equation, which is easily solved. The standard approach works well when the algebraic equation doesn't have repeated roots. We offer an alternative approach to explain the form of solutions obtained from repeated roots. We further show that our approach, which is based on simple first order methods, applies equally well to homogeneous and nonhomogeneous equations with repeated roots.

Consider the first order, constant coefficient linear differential equation,

$$y'(t) + r_1 y(t) = g(t). \tag{1.1}$$

Here r_1 is a constant and $g(t)$ is an arbitrary function. Every equation of this form is solved by use of the integrating factor $\mu(t) = e^{r_1 t}$. Multiplying both sides of the equation

by $\mu(t)$, we have

$$\frac{d}{dt}(e^{r_1 t} y(t)) = e^{r_1 t}(y'(t) + r_1 y(t)) = e^{r_1 t} g(t).$$

Integrating both sides, we have

$$e^{r_1 t} y(t) = \int e^{r_1 t} g(t) dt.$$

This integrating factor technique allows us to solve the equation whether (1.1) is homogeneous, $g(t) = 0$, or non-homogeneous, $g(t) \neq 0$.

Let us now consider a homogeneous second order equation of similar form,

$$y''(t) + a_1 y'(t) + a_2 y(t) = 0. \quad (1.2)$$

The standard approach for this type of equation is to posit a solution of the form e^{rt} and study the resulting algebraic characteristic equation

$$r^2 + a_1 r + a_2 = 0.$$

If there are two distinct real solutions r_1 and r_2 , we obtain the linearly independent solutions $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$. When there is a repeated root, this approach yields only one solution $y(t) = e^{r_1 t}$. One can use a reduction of order type approach to solve for the second solution as in [4, p. 119]. One can also see [5] which develops a reduction of order technique that applies for equations of any order $n \geq 2$. We propose a slightly different approach. If $r = r_1$ is a repeated root to (1.2), then we can rewrite the equation as

$$y''(t) - 2r_1 y'(t) + r_1^2 y(t) = 0$$

We now multiply the equation by the integrating factor $\mu(t) = e^{-r_1 t}$, then a simple calculation shows that the left hand side of the equation collapses to a second derivative,

$$\frac{d^2}{dt^2}(e^{-r_1 t} y) = 0.$$

So that

$$e^{-r_1 t} y(t) = c_1 t + c_2.$$

This leads to the solutions $y_1(t) = e^{r_1 t}$ and $y_2(t) = t e^{r_1 t}$.

At this point consider the third order equation with a triple root $r = r_1$, which takes the form

$$y^{(3)}(t) - 3r_1 y''(t) + 3r_1^2 y'(t) - r_1^3 y(t) = 0. \quad (1.3)$$

At this point, most texts take the approach of suggesting multiplying by t again to obtain a third solution without any intuition or additional motivation. We suggest the following

motivation, multiply by the integrating factor $\mu(t) = e^{-r_1 t}$, this will collapse the lefthand side of (1.3) to a third derivative, (see (2.4) below)

$$\frac{d^3}{dt^3}(e^{-r_1 t} y(t)) = 0.$$

Thus the solution must be of the form

$$y(t) = c_1 t^2 e^{r_1 t} + c_2 t e^{r_1 t} + c_3 e^{r_1 t}$$

which yields the three linearly independent solutions $y_1(t) = t^2 e^{r_1 t}$, $y_2(t) = t e^{r_1 t}$ and $y_3(t) = e^{r_1 t}$. At this point we note that as in the first order equation (1.1) this integrating factor approach applies to equations of the form (1.2) and (1.3) even when the equations are not homogeneous.

The rest of the article is outlined as follows. In Section 2 we develop the machinery necessary to show that the integrating factor approach works for any n^{th} order equation with an n -fold repeated root. In Section 3 we discuss the nonhomogeneous n^{th} order equation with an n -fold repeated root, and finally in Sections 4 and 5 we show that the integrating factor approach works for multiple roots and certain types of variable coefficient equations, though these cases become a bit unwieldy.

2 General n^{th} order case

Most textbooks first handle distinct real roots and later come back to discuss when $r = r_1$ is a repeated solution to (2.3). Consider the following equation,

$$y^{(n)} + \binom{n}{1}(-r_1)y^{(n-1)} + \dots + \binom{n}{n-1}(-r_1)^{n-1}y' + (-r_1)^n y = 0. \quad (2.1)$$

This differential equation has characteristic equation

$$r^n + \binom{n}{1}(-r_1)r^{n-1} + \dots + \binom{n}{n-1}(-r_1)^{n-1}r + (-r_1)^n = (r - r_1)^n = 0.$$

It is often shown, with varying levels of explanation and motivation, that the repeated roots give rise to the solutions $y_i(t) = t^{i-1} e^{r_1 t}$ for $1 \leq i \leq n$. We show that this can be seen via first order differential equation solution techniques. In particular, we show that these solutions can easily be found through the use of the integrating factor $\mu(t) = e^{-r_1 t}$.

Consider the n^{th} order constant coefficient homogeneous linear differential equation

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + a_2 y^{(n-2)}(t) + \dots + a_{n-1} y'(t) + a_n y(t) = 0. \quad (2.2)$$

Here a_i are constants. We most often study the case when $n = 2$ due to Newton's laws of motion and with ease one can generalize the theory of second order equations to higher order. Under these conditions, one often makes the ansatz $y = e^{rt}$ which transforms (2.2) into an algebraic equation, which is more easily solved. In particular with the above ansatz, we have the characteristic equation

$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \dots + a_{n-1} r + a_n = 0. \quad (2.3)$$

Lemma 2.1. *The following derivative identity holds.*

$$\frac{d^n}{dt^n} \left(e^{-r_1 t} y \right) = e^{-r_1 t} \left[y^{(n)} + \binom{n}{1} (-r_1) y^{(n-1)} + \cdots + \binom{n}{n-1} (-r_1)^{n-1} y' + (-r_1)^n y \right]. \quad (2.4)$$

Proof. We prove this inductively. Take the base case of $n = 1$,

$$\frac{d}{dt} \left(e^{-r_1 t} y \right) = e^{-r_1 t} y' - r_1 e^{-r_1 t} y.$$

Assume that (2.4) holds for n derivatives, we show that it must hold for $n + 1$ derivatives as well. Now,

$$\begin{aligned} \frac{d^{n+1}}{dt^{n+1}} \left(e^{-r_1 t} y \right) &= \frac{d}{dt} \left\{ e^{-r_1 t} \left[y^{(n)} + \binom{n}{1} (-r_1) y^{(n-1)} + \cdots + (-r_1)^n y \right] \right\} \\ &= \frac{d}{dt} \left\{ e^{-r_1 t} \sum_{j=0}^n \binom{n}{j} (-r_1)^j y^{(n-j)} \right\} = e^{-r_1 t} \sum_{j=0}^{n+1} b_j y^{(n+1-j)} \end{aligned}$$

If we examine the coefficients on the $n - j^{\text{th}}$ derivative, b_{j+1} , we have contributions from when the derivative acts on the exponential or the $y^{(n+1-j)}$, so

$$b_{j+1} = (-r)^{j+1} \left(\binom{n}{j} + \binom{n}{j+1} \right) = (-r_1)^{j+1} \binom{n+1}{j+1}$$

by Pascal's rule. Thus,

$$\frac{d^{n+1}}{dt^{n+1}} \left(e^{-r_1 t} y \right) = e^{-r_1 t} \sum_{j=0}^{n+1} \binom{n+1}{j} (-r_1)^j y^{(n+1-j)}$$

as desired. □

Returning to the differential equation with an n -fold repeated root, we now multiply (2.1) by the integrating factor $\mu(t) = e^{-r_1 t}$ to see

$$e^{-r_1 t} \left[y^{(n)} + \binom{n}{1} (-r_1) y^{(n-1)} + \cdots + \binom{n}{n-1} (-r_1)^{n-1} y' + (-r_1)^n y \right] = 0.$$

Applying (2.4), we have

$$\frac{d^n}{dt^n} \left(e^{-r_1 t} y \right) = 0.$$

As the only functions whose n^{th} derivative is exactly zero are polynomials of order strictly less than n , integrating both sides of the equation n times yields

$$e^{-r_1 t} y(t) = c_{n-1} t^{n-1} + c_{n-2} t^{n-2} + \cdots + c_1 t + c_0,$$

equivalently, the solution to the differential equation is

$$y(t) = e^{r_1 t}(c_{n-1}t^{n-1} + c_{n-2}t^{n-2} + \cdots + c_1t + c_0).$$

While this approach does not apply directly to (2.2) when there are distinct roots, it does provide an alternative to the standard methods; factoring differential operators (see for example [3, p. 304]), using partial derivatives (see for example [2, p. 232]), or the “method of the lucky guess” of [1, p. 329]. The case of a second order differential equation using first a substitution and then an integrating factor was explored in [4, p. 119]. Our approach is logically equivalent to the factoring of operators, we can rewrite (2.4) as a sum of terms of the form $(D + r_1)^n[e^{-r_1 t}y] = e^{-r_1 t}D^n y$, though it loses the familiar first-order feel and the ability to tackle the nonhomogeneous applications we discuss below.

3 Nonhomogeneous Equations

We also offer an application of the integrating factor technique outlined above to non-homogeneous equations. Consider when the nonhomogeneity of the n^{th} order equation, (2.1) is a polynomial multiplying the exponential solution $e^{r_1 t}$.

$$y^{(n)} + \binom{n}{1}(-r_1)y^{(n-1)} + \cdots + \binom{n}{n-1}(-r_1)^{n-1}y' + (-r_1)^n y = P(t)e^{r_1 t}. \quad (3.1)$$

Here $P(t)$ is a polynomial. Multiplying (3.1) by the integrating factor $\mu(t) = e^{-r_1 t}$ we have

$$e^{-r_1 t} \left\{ y^{(n)} + \binom{n}{1}(-r_1)y^{(n-1)} + \cdots + \binom{n}{n-1}(-r_1)^{n-1}y' + (-r_1)^n y \right\} = P(t).$$

Again applying (2.4), this is equivalent to

$$\frac{d^n}{dt^n} \left(e^{-r_1 t} y \right) = P(t).$$

This can be easily integrated n times to obtain both the particular and the homogeneous solution. The particular solution arises from integrating $P(t)$ n times and the constants of integration yield the homogeneous solution.

Example 3.1. Consider the following nonhomogeneous problem whose characteristic equation has the triple root $r_1 = 2$.

$$y^{(3)} - 6y'' + 12y' - 8y = t^9 e^{2t}$$

Multiplying by the integrating factor $\mu(t) = e^{-2t}$, and using (2.4), we have

$$\frac{d^3}{dt^3} \left(e^{-2t} y \right) = t^9.$$

Integrating three times, we have

$$e^{-2t} y = \frac{1}{(12)(11)(10)} t^{12} + c_2 t^2 + c_1 t + c_0.$$

So that the solution is

$$y(t) = e^{2t} \left(\frac{1}{1320} t^{12} + c_2 t^2 + c_1 t + c_0 \right).$$

This approach provides a more natural explanation than the standard approach of multiplying by t until it works [2, p. 181], [3, p. 331]. We were able to find this solution without solving a system of equations as necessitated in using the method of undetermined coefficients. The standard approach of undetermined coefficients first forces us to guess the correct form of the solution and then to solve a resulting system of ten equations and ten unknowns. Our approach avoids the linear algebra and requires only integration. We note that this approach works with any non-homogeneity $g(t)$, we chose $g(t) = e^{r_1 t} P(t)$ since integrating a polynomial requires no special techniques and to offer more motivation for why multiplication by t yields the correct form for the solution.

4 Unequal Repeated Roots

While the above approach does not apply exactly to the case when there are distinct repeated roots to the characteristic equation, an approach that relies on integrating factors does still work. We first give the following example.

Example 4.1. Consider the following differential equation whose characteristic equation has a triple root $r_1 = 4$ and double root $r_2 = -2$,

$$y^{(5)} - 8y^{(4)} + 4y^{(3)} + 80y'' - 64y' + 256y = 0.$$

We first multiply by the integrating factor $\mu_1(t) = e^{-4t}$,

$$e^{-4t}[y^{(5)} - 8y^{(4)} + 4y^{(3)} + 80y'' - 64y' + 256y] = 0,$$

using (2.4), we have the equivalent equation

$$\frac{d^5}{dt^5} [e^{-4t}y] + 12 \frac{d^4}{dt^4} [e^{-4t}y] + 36 \frac{d^3}{dt^3} [e^{-4t}y] = 0.$$

Now, let $u = \frac{d^3}{dt^3} [e^{-4t}y]$, the equation in terms of u becomes

$$u'' + 12u' + 36u = 0$$

Now we use the integrating factor $\mu_2(t) = e^{6t}$, notice that the exponent is the difference between the two roots to the original equation, again using (2.4) we have

$$\frac{d^2}{dt^2} [e^{6t}u] = 0$$

So that, upon integrating twice, we have

$$e^{6t}u = A_1t + A_2.$$

Thus, we have

$$\frac{d^3}{dt^3} [e^{-4t}y] = e^{-6t}(A_1t + A_2)$$

Using integration by parts on the right hand side, and noting that A_1, A_2 are arbitrary constants, we have

$$e^{-4t}y = e^{-6t}(B_1t + B_2) + B_3t^2B_4t + B_5.$$

So that, the solution is

$$y(t) = e^{-2t}(B_1t + B_2) + e^{4t}(B_3t^2 + B_4t + B_5)$$

with B_i arbitrary constants, as desired.

We outline the approach for an equation of the form in (2.2) whose characteristic equation is of the form

$$(r - r_1)^k(r - r_2)^\ell = 0,$$

where without loss of generality, we take $k \geq \ell$. We use two integrating factors, first we multiply the original equation, (2.2), by the integrating factor $\mu_1(t) = e^{-r_1t}$ and use (2.4) to expand the differential equation in terms of $\frac{d^j}{dt^j}[e^{-r_1t}y]$. This leads to an ℓ^{th} order equation in $u = \frac{d^k}{dt^k}[e^{-r_1t}y]$. The integrating factor $\mu_2(t) = e^{(r_1-r_2)t}$ reduces the equation to

$$\frac{d^\ell}{dt^\ell} \left[e^{(r_1-r_2)t}u \right] = 0.$$

Upon integrating ℓ times, we have

$$e^{(r_1-r_2)t}u = P_{\ell-1}(t).$$

Here $P_j(t)$ indicates an arbitrary polynomial of order j . Noting, by integration by parts, integrating a polynomial of degree j multiplied by an exponential, yields a polynomial of degree j multiplied by the same exponential. We have

$$\frac{d^k}{dt^k} \left[e^{-r_1t}y \right] = u = e^{(r_2-r_1)t}P_{\ell-1}(t).$$

Now, integrating k times, we have

$$e^{-r_1t}y = e^{(r_2-r_1)t}P_{\ell-1}(t) + P_{k-1}(t).$$

Thus, the solution is

$$y(t) = e^{r_2t}P_{\ell-1}(t) + e^{r_1t}P_{k-1}(t)$$

as desired.

5 Non-Constant Coefficient Equations

We briefly discuss how the integrating factor approach can apply to non-constant coefficient equations. We offer the following example.

Example 5.1. Consider the variable coefficient second order equation

$$y''(t) + 2 \cos(t)y'(t) + (\cos^2 t - \sin t)y(t) = 0.$$

At first glance, this equation is not easily solved. We consider the following integrating factor $\mu(t) = e^{\sin(t)}$ and the equation

$$\frac{d^2}{dt^2} \left(e^{\sin t} y(t) \right) = y''(t) + 2 \cos(t)y'(t) + (\cos^2 t - \sin t)y(t).$$

Thus, multiplying both sides of the equation will collapse the left hand side to a second derivative.

$$\frac{d^2}{dt^2} \left(e^{\sin t} y(t) \right) = 0$$

Integrating twice and solving for $y(t)$ yields the solutions $y_1(t) = e^{-\sin t}$ and $y_2(t) = te^{-\sin t}$.

For a second order equation of the form

$$y''(t) + f'(t)y'(t) + (f''(t) + [f'(t)]^2)y(t) = g(t)$$

we can use the integrating factor $\mu(t) = e^{f(t)}$ to collapse to the equation

$$\frac{d^2}{dt^2} \left(e^{f(t)} y(t) \right) = e^{f(t)} g(t),$$

at which point we can integrate twice to find the solution. If we have a homogeneous equation, $g(t) = 0$, we see solutions of the form $y_1(t) = e^{-f(t)}$ and $y_2(t) = te^{-f(t)}$. As this is possible only for a very special class of equations, we do not discuss equations of order higher than two.

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