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Teaching Differential Equations with Graphics and without Linear Algebra

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Abstract: We present our approach to teaching the Method of Eigenvectors to solve linear systems of ODEs without assuming a prerequisite course in Linear Algebra. Rather we depend heavily on a graphical approach to systems in two dimensions to motivate the eigenvalue equation.

1 The Role of a First ODEs Course

At least as far back as the seminal College Mathematics Journal Special Issue on Differential Equations [7], there has been a clear direction in the teaching of ordinary differential equations toward the use of modern technology and graphical techniques. Slope field and phase plane diagrams are frequently standard parts of the curriculum. In addition, a modern first course in ordinary differential equations (henceforth, simply "ODEs") often has a focus on modeling and applications. Such a course is positioned, typically, at a crossroads in a student's mathematical career. It serves as a continuation of the calculus sequence, an introduction to applied mathematics, and also as a technical requirement for many majors other than math, our *partner disciplines*. Today, these partner disciplines are no longer limited to just the traditional physics and engineering majors.

Recently we had to question the necessity of Linear Algebra as a prerequisite for ODEs. A new major in Biophysics was introduced at the Claremont Colleges. The major required Differential Equations, but not Linear Algebra. We asked ourselves: could we reasonably teach this cohort of biophysics students (along with the rest of the class) without assuming linear algebra and without watering down the content of ODEs? And could we do this without simply rushing through a course in Linear Algebra inserted in the middle of our course?

The greatest difficulty would be the significant and *central* unit on linear systems. So our question was, primarily, whether we could give reasonable instruction on the Method of Eigenvectors and Eigenvalues for solving linear systems to a group of students who had not had a course in Linear Algebra. We found that graphics, fully integrated in our

ODEs course, allowed for a *better* development and explanation of this Method, rather than Linear Algebra. In fact, we would encourage math programs to consider placing ODEs before Linear Algebra. A more applied and visual ODE course can work better as a transition from calculus to the theoretical abstractions in linear algebra and as a necessary course for our partner majors.

We want to emphasize that we are *not* claiming to teach ODEs without a Linear Algebra prerequisite by squeezing a course in Linear Algebra into our ODE class. We want to stick to simple versions of linear algebra concepts that are familiar from high school algebra and geometry, or easily understood in the plane. For example, "two solutions are *linearly dependent* if they are (constant) multiples of one another" does not require a sophisticated machinery about linear dependence relations.

In this paper, we will

• outline what we are assuming our students already know before they come into our class,

• outline the (relevant) material they have seen in our course up to our unit on linear systems,

• demonstrate our lesson through some excerpts,

• address possible concerns regarding the limitations of our presentation, particularly

(a) dealing with systems with dimension higher than 2, and

(b) dealing with deficient eigenspaces and generalized eigenvectors,

• discuss how our lesson fits in with the remained of the course.

2 What They Know Before the Class Starts

It is useful for us to spell out exactly what previous mathematical skills we are expecting from our students in this class. We assume they know calculus through at least integral calculus (covered in the AP Calculus BC course, for example). Our students have had a multivariable and vector-valued calculus class. (Let us refer to such a course, covering material at least through Green's Theorem, as *MVCalc*.) In some math programs, however, Linear Algebra is a prerequisite to MVCalc, so we try to be specific about what MVCalc material is to be assumed.

We will assume that they have seen vectors in the plane, perhaps in an MVCalc course or in a general physics course. In our experience, this is a mild assumption. Often even precalculus texts will cover vectors (although maybe not the courses themselves). We will assume that the incoming students in this class have seen 3×3 determinants. If they have taken cross products in MVCalc or in physics, then they have seen 3×3 determinants. And if they can take a 3×3 determinant, we may assume they are familiar with (and can calculate) 2×2 determinants.

We only assume that our students can solve 2 (linear) equations in 2 unknowns, something they should know from high school algebra. They can learn to solve (perhaps with assistance) 3 equations in 3 unknowns.

If a student has seen matrix-vector notation, enough so that the equality

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

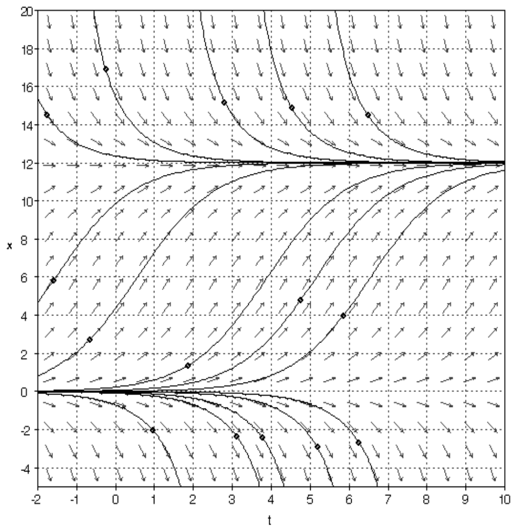

Figure 1: Slope field for a logistic model

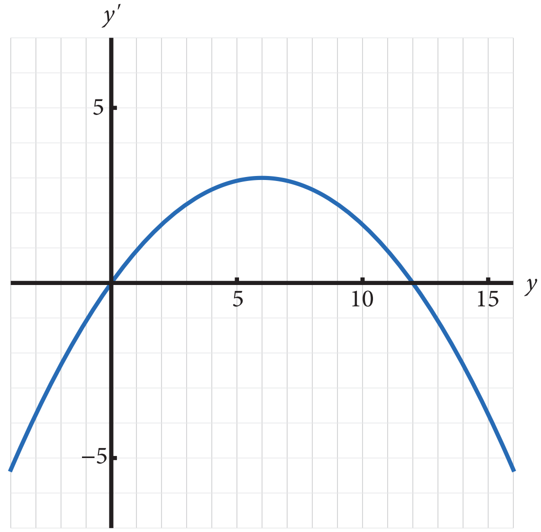

Figure 2: Phase plane for logistic model

makes sense, that is helpful. But we (re)introduce the equation, as we will demonstrate in Example 4.1, as a notational formality.

The only additional topics from MVCalc that the students need are partial derivatives (for existence and uniqueness theorems and for computing the Jacobian when linearizing non-linear systems), and perhaps Green's Theorem if the class covers a proof of Bendixson's Negative Criterion (for example see Theorem 9.2.4 in [2]).

In the next section we will highlight in detail the (relevant) topics that the students have seen in our course before the presentation of the Method of Eigenvectors.

3 The Start of Our Course

Like most ODEs, we start with basic first-order techniques. We have taken a modeling approach, so, for example, the Method of Integrating Factor has been presented as a solution to one-compartment *tank* ([2], page 43) or *mixing* ([1], pg. 49) problems. Two-compartment tank problems have briefly been discussed as a way to introduce the students to systems of equations. In particular, the students can model and solve *linear cascades*. We have also discussed the simpler idea of an *uncoupled system*.

Damped and undamped spring motion problems have been used to motivate solutions to general second order linear ODEs with constant coefficients. Students are familiar, in this context, with characteristic polynomials and solutions involving distinct real roots, complex conjugate roots, and double roots. (Such solutions were presented as an (educated) guess-and-check, essentially. The CODEE Journal paper [4] has a nice method for addressing the repeated roots case.)

Population models have been used to introduce analytic vs. graphical solutions. The Logistic growth models have allowed us to introduce slope fields as well as the phase plane. For example, Figures 1 and 2 show these graphs for $y' = y(1 - y/12)$.

Students can create the phase plane curve and sketch qualitative behavior of solutions.

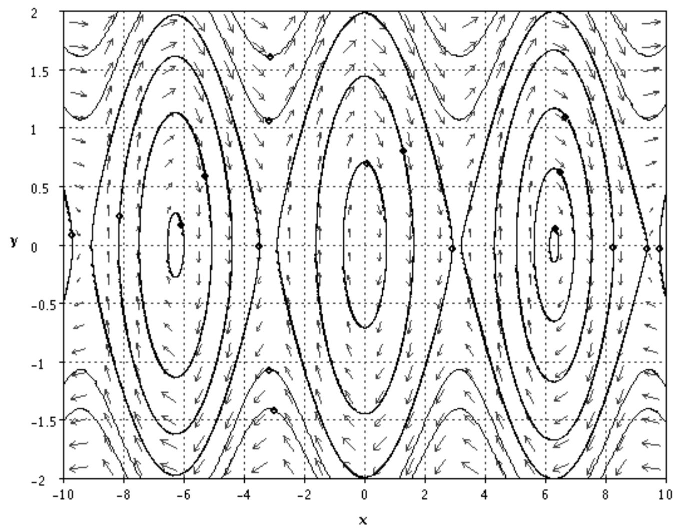

Figure 3: Frictionless Pendulum: $x' = y,\ y' = -\sin x$

We choose to use the software programs *pplane* and *dfield* [5] for the creation of slope fields and families of solution curves.

As a more advanced example, we have discussed the motion of a pendulum, with and without friction. We did not derive the model ourselves, but "left that for the physicists." We have used the model as a way to tie together several ideas. First, we furthered our discussion (started with spring motion) of friction/damping terms in models. Next, we continued our exploration (started with logistic population growth) of linearization of non-linear models. Most importantly, we used this model as an opportunity for students to practice "reading" a phase-plane diagram. Again, the students have used *pplane* to create slope fields and example orbits, as shown in Figure 3.

Finally, we have briefly presented slope fields for predator-prey models, such as

$$\begin{aligned} x' &= xy/12 - x \\ y' &= y - xy/6 \end{aligned}$$

and discussed the use of *nullclines* to sketch orbits in the phase plane and to predict long term behavior. (See Figure 4.) Students are able to make predictions about the shape of orbits before experimenting with *pplane*.

It is at this point that we move on to one of the core units in ODEs: linear systems. Here we opt to not depend on any understanding of linear algebra beyond high school algebra. Rather, we motivate and explore eigenvectors graphically, building upon previous types of problems just described.

4 Solving Linear Systems

We motivate the concept of eigenvalues and eigenvectors by considering systems of differential equations through a series of examples. We begin by getting students comfortable with matrix-vector notation for linear systems using familiar problems from earlier in the course.

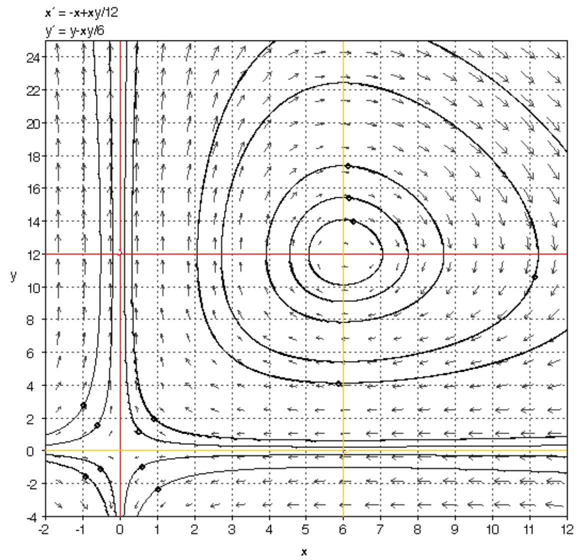

Figure 4: Some orbits in the predator prey phase plane

Example 4.1 (Systems and matrix notation). We recall the second order linear equation with constant coefficients: $y'' + py' + qy = 0$. We can convert this second order equation to a *first* order *system* with the introduction of a second variable. Let $y'(t) = v(t)$ and $y''(t) = v'(t)$. Then we get the *system* of first order equations

$$\begin{cases} y' = v \\ v' = -qy - pv. \end{cases} \qquad \text{or} \qquad \begin{cases} y' = 0y + 1v \\ v' = -qy - pv. \end{cases}$$

We can rewrite this system using matrix-vector notation as follows. Let

$$\vec{x} = \begin{pmatrix} y \\ v \end{pmatrix}, \qquad \text{so that} \qquad \vec{x}\,' = \begin{pmatrix} y' \\ v' \end{pmatrix}$$

and let the matrix $A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$. Then the system can be written

$$\begin{pmatrix} y' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} \qquad \text{or simply} \qquad \vec{x}\,' = A\vec{x}$$

if we allow that $\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix}$ is short-hand notation for $\begin{pmatrix} 0y + 1v \\ -qy - pv \end{pmatrix}$.

In general, we will say that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

so if, as a second example, we revisit the linear cascade

$$\begin{cases} x' = ax \\ y' = cx + dy \end{cases}$$

which is already written as a system, we can represent the system as a matrix equation: $\vec{x}\,' = A\vec{x}$, where now the matrix is $A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$.

In this way, we see how a linear system is always written as $\vec{x}\,' = A\vec{x}$, and determined by the coefficient matrix A.

Let us now consider solutions to systems in the context of matrix notation. Perhaps the easiest possible example of a systems is the uncoupled system.

Example 4.2 (Uncoupled). Let k_1 and k_2 be constants and consider the system

$$\begin{cases} x_1' = k_1 x_1 \\ x_2' = k_2 x_2 \end{cases}$$

As before, we convert this system to the matrix form $\vec{x}\,' = A\vec{x}$, with $A = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$. As we know, the solutions to each ODE in the system can be solved independently to yield

$$\begin{aligned} x_1 &= C_1 e^{k_1 t} \\ x_2 &= C_2 e^{k_2 t}. \end{aligned}$$

Thus, the *vector* solution to the system $\vec{x}\,' = A\vec{x}$ is $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} C_1 e^{k_1 t} \\ C_2 e^{k_2 t} \end{pmatrix}$ which we can write as $\vec{x} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{k_1 t} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{k_2 t}$. As we will see momentarily, this decomposition is a good preview of the more general solution.

It is interesting to compare this solution to the analogous one-dimensional case: $x' = ax$ whose solution is $x = Ce^{at}$. The solutions to the system are sums of solutions of the form $\vec{x} = \vec{C}e^{k_i t}$, where now the "constant" is a vector, $\vec{C} = C\vec{v}$ for certain $\vec{v}$'s.

The next example is our main teaching example used to motivate the Method of Eigenvectors. This time, we begin with a matrix, but translate back to a familiar 2nd order equation.

Example 4.3 (Beginning with a matrix). We begin with the system of first order ODEs in the matrix form, $\vec{x}\,' = A\vec{x}$, where $A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$. Equivalently, we have the following system

$$\begin{cases} y' = v \\ v' = 4y. \end{cases}$$

In order to solve the system, we can convert it back to a second order differential equation $y'' = 4y$. Of course we already know how to solve the second order differential equation of the form $y'' + py' + qy = 0$. In this case, the characteristic polynomial is $r^2 - 4$, whose roots are $r_1 = -2$ and $r_2 = 2$ and the solution of $y'' - 4y = 0$ is $y = C_1 e^{2t} + C_2 e^{-2t}$.

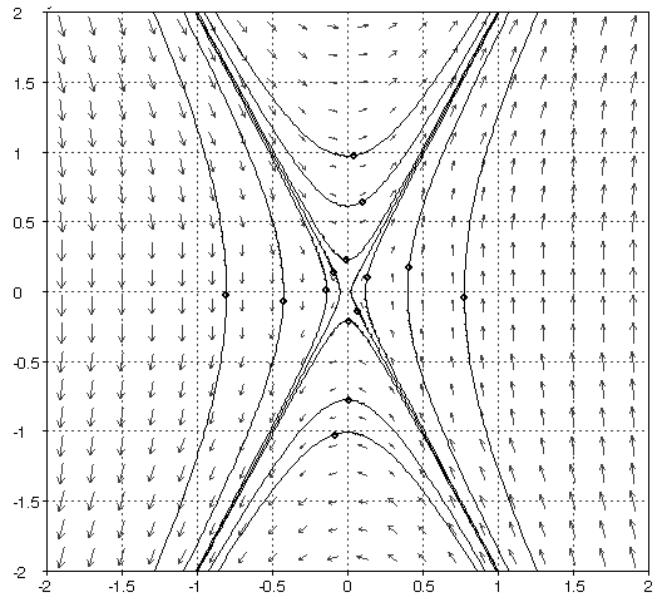

Figure 5: A big X made of eigenlines

The solution to the second equation in the system is just $v = y' = 2C_1e^{2t} - 2C_2e^{-2t}$. So, we can write the solution of the system in the matrix form

$$\begin{aligned}\vec{x} = \begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} C_1e^{2t} + C_2e^{-2t} \\ 2C_1e^{2t} - 2C_2e^{-2t} \end{pmatrix} &= C_1 \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \end{pmatrix} \\ &= C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t},\end{aligned} \tag{4.1}$$

decomposing the solution as in Example 4.2.

Here we turn to our focus on graphical techniques in the course. What happens if we consider the phase plane diagram for this system? We use *pplane* to plot the vector $\begin{pmatrix} y' \\ v' \end{pmatrix}$ at each point $\begin{pmatrix} y \\ v \end{pmatrix}$ in the yv-plane. See Figure 5. Within *pplane*, we select initial values near the origin, the equilibrium point for the system. This allows the long term behaviour to be more apparent.

From this orbital portrait the two *eigenlines* are strikingly easy to see. Some students might guess, incorrectly, that these are nullclines, so it is useful to point out the location of the nullclines on the y- and v-axes, and to acknowledge that these new "eigenlines" are far more interesting. We explore these lines and develop the Eigenvalue Equation $A\vec{x} = \lambda\vec{x}$.

First, note that the eigenlines are multiples of the vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, respectively. (If students have seen flow lines in their MVCalc class, there is a nice connection here.) Regardless of the point at which we start on the eigenline associated with the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, the graph approaches the origin in a direct straight-line path. However, if we start anywhere on the eigenline associated with the vector $\begin{pmatrix}1\\-2\end{pmatrix}$, the orbit is directly away from the origin. Note that the special property here is that along the eigenline, the vectors, $\begin{pmatrix}y'\\v'\end{pmatrix}$ in the slope field, lie entirely on the eigenline.

Looking at our vector solution (4.1), let

$$\vec{x}_1 = C_1 \begin{pmatrix}1\\2\end{pmatrix} e^{2t} = \begin{pmatrix}C_1 e^{2t}\\2C_1 e^{2t}\end{pmatrix}.$$

Then since,

$$\vec{x}_1{}'(t) = \begin{pmatrix}2C_1 e^{2t}\\4C_1 e^{2t}\end{pmatrix} = A\vec{x}_1(t),$$

we see that $\vec{x}_1(t)$ is a solution to $\vec{x}\,' = A\vec{x}$. The question is: what is so special about the vector $\vec{v}_1 = \begin{pmatrix}1\\2\end{pmatrix}$?

Note that $A\vec{v}_1 = \begin{pmatrix}2\\4\end{pmatrix} = 2\begin{pmatrix}1\\2\end{pmatrix}$. Graphically, this corresponds to the fact that the vector $\begin{pmatrix}y'\\v'\end{pmatrix}$ placed *at* the point $\begin{pmatrix}1\\2\end{pmatrix}$ is $\begin{pmatrix}2\\4\end{pmatrix}$, a multiple of $\begin{pmatrix}1\\2\end{pmatrix}$ itself. This is why the (eigen)line along $\begin{pmatrix}1\\2\end{pmatrix}$ is so visually prominent.

One can ask: does this same phenomenon occur for $\vec{v}_2 = \begin{pmatrix}1\\-2\end{pmatrix}$? From our phase plane diagram, it must. Let us look algebraically. We get

$$A\vec{v}_2 = \begin{pmatrix}0 & 1\\4 & 0\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}-2\\4\end{pmatrix} = -2\begin{pmatrix}1\\-2\end{pmatrix}.$$

In both cases, we have vectors $\vec{v}_i$ satisfying $A\vec{v}_i = \lambda_i\vec{v}_i$ for some λ_i. In general, we call the equation:

$$A\vec{v} = \lambda\vec{v} \tag{4.2}$$

the Eigenvalue Equation. We define a non-zero vector $\vec{v}$ satisfying (4.2) for some λ to be an *eigenvector*. A constant λ for which an eigenvector exists is called an *eigenvalue*. (We note that while $\vec{v}$ must be non-zero to be considered an eigenvector, we may have eigenvalues λ equal to zero.)

Note that the general solution to the system $\vec{x}\,' = A\vec{x}$ is

$$x(t) = C_1\vec{v}_1 e^{\lambda_1 t} + C_2\vec{v}_2 e^{\lambda_2 t}, \tag{4.3}$$

where the λ's are the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -2$ and the $\vec{v}$'s are the eigenvectors $\vec{v}_1 = \begin{pmatrix}1\\2\end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix}1\\-2\end{pmatrix}$.

Now, let us explore the Eigenvalue Equation and its relation to solutions via a more general problem for which the solution is not already known.

Example 4.4 (A more general matrix). Let us look at the system

$$\begin{cases} x_1' = 5x_1 + 4x_2 \\ x_2' = -8x_1 - 7x_2 \end{cases}$$

with the corresponding matrix

$$A = \begin{pmatrix} 5 & 4 \\ -8 & -7 \end{pmatrix}.$$

Given the solution to the previous example, we guess that the general solution will be of the form (4.3) where the $\vec{v}_i$'s and λ_i's satisfy the Eigenvalue Equation (4.2). So we are looking for non-zero vectors $\vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ that satisfy $A\vec{v} = \lambda\vec{v}$ for some λ. In other words, we need to solve

$$\begin{pmatrix} 5 & 4 \\ -8 & -7 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \lambda\alpha \\ \lambda\beta \end{pmatrix}$$

which translates to finding α, and β such that

$$\begin{cases} 5\alpha + 4\beta = \lambda\alpha \\ -8\alpha - 7\beta = \lambda\beta \end{cases}$$

or, equivalently,

$$\begin{cases} (5-\lambda)\alpha + 4\beta = 0 \\ -8\alpha + (-7-\lambda)\beta = 0. \end{cases} \tag{4.4}$$

No matter the value of λ, we are asking to solve two (linear) equations in two unknowns. Our intuition from high school algebra says that there should be a unique solution. Indeed, geometrically, we are asking to find the intersection of two lines in the plane, which, of course, consists of exactly one point. Unfortunately, these two lines both go through the *origin* $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. (Looking at equations (4.4), it is easy to see that $(\alpha, \beta) = (0, 0)$ is a simultaneous solution.) The resolution to this apparent problem can be seen if we observe what happens when we let $\lambda = 1$. Our equations become

$$\begin{cases} 4\alpha + 4\beta = 0 \\ -8\alpha - 8\beta = 0 \end{cases}$$

which, though technically still two equations in two variables, is easily seen to have a redundant equation. Our two-equations-in-two-unknowns is really just one equation in two unknowns. So any non-zero choice of α will yield a β so that $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is an eigenvector (with eigenvalue $\lambda = 1$).

For the purposes of completing our example, we choose $\alpha = 1$ to get the eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. And, one can check that $\vec{x} = \vec{v}e^t = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$ is a solution to the original system.

The more important observation, however, is that for *certain* λ's the pair of equations may lead to more than one solution. Geometrically, this means our two equations are describing the same line, so of course there are an infinite number of points of intersection. The question is now: how can we find such λ's, if they exist. Under what circumstances is are two linear equations, say,

$$\begin{cases} a\alpha + b\beta = 0 \\ c\alpha + d\beta = 0 \end{cases} \tag{4.5}$$

describing the same line? The answer is that one of the equations must be a constant multiple of the other. (Note: in order to correctly state the facts here in the most general terms, one needs to be careful due to the possibility that one of the equations might simply say $0x + 0y = 0$. In practice, as we will see, this all works out very nicely.) We claim that this occurs exactly when $ad - bc$ equals 0. In other words, if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we claim that $\det B = 0$ if and only if (4.5) has a non-trivial solution.

For us, this means that we can only find an eigenvector if the eigenvalue λ makes $\det \begin{pmatrix} 5-\lambda & 4 \\ -8 & -7-\lambda \end{pmatrix} = (5-\lambda)(-7-\lambda) - (4)(-8) = 0$. Multiplying out: we need to find λ such that $\lambda^2 + 2\lambda - 3 = 0$. This equation is called the *characteristic equation*. And in our case it factors as $(\lambda - 1)(\lambda + 3) = 0$ telling us that $\lambda = 1$ and $\lambda = -3$ are the only eigenvalues. For no other values of λ will there be any eigenvectors *and* for those values of λ there *will* be eigenvectors.

Substituting $\lambda = -3$ our equations (4.4) become

$$\begin{cases} 8\alpha + 4\beta = 0 \\ -8\alpha - 4\beta = 0. \end{cases}$$

As predicted, we get a redundant equation. So any (non-zero) choice of α yields a corresponding β so that $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is an eigenvector (with eigenvalue $\lambda = -3$). Again, we choose $\alpha = 1$ to get the eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. And, one can check that $\vec{x}_2 = \vec{v}_2 e^{-3t} = \begin{pmatrix} e^{-3t} \\ -2e^{-3t} \end{pmatrix}$ is a solution to the original system.

So if our guess is correct then the solution of the system should be

$$\vec{x}(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t}.$$

We check this both graphically and algebraically. If we plot the phase diagram for this system, as in Figure 6, we observe behavior similar to the previous example. We *clearly* see the eigenlines following the vectors $\vec{v}_1$ and $\vec{v}_2$ in our phase plane diagram.

What else do we need in order to apply this method for any (2×2) linear system? One additional piece of notational convenience is the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then we can see that the key step is to write down

$$\det(A - \lambda I) = 0.$$

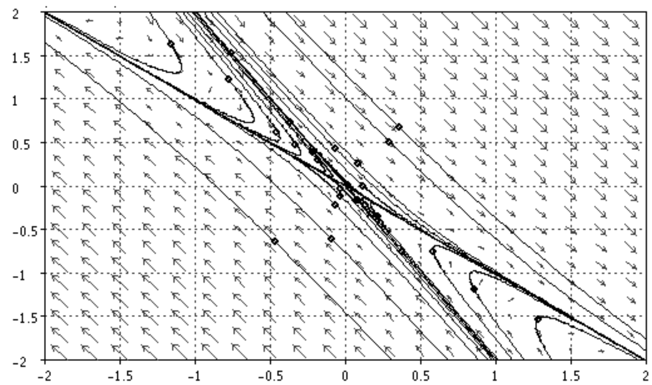

Figure 6: Two clear eigenlines again

to get the characteristic equation. The solutions to this equation are the eigenvalues. For each eigenvalue we can find an eigenvector. And, together, the eigenvalues and eigenvectors allow us to write down the general solution to the linear system.

At this point the students are ready to try this on their own with homework exercises. The process is straight-forward, particularly if we only give them distinct real eigenvalues for 2×2 matrices. They can be reassured that, if they have found the eigenvalues correctly, then they will *always* only have one equation in two unknowns to solve. They can return to the earlier examples in this section to confirm that this method works there.

5 Follow up Topics in Linear Systems

Having presented the series of examples in the previous section, we must evaluate the efficacy of our lesson. First, we note that although the Method has been demonstrated, above, for a saddle, the computation works the same for other types of behaviors (nodes, spirals, etc.) even if the graphics are less helpful. The method of finding real solutions when the eigenvalues are complex remains the same as one would teach it in a course that assumes linear algebra.

Next, we should continue the discussion that $B\vec{x} = \vec{0}$ has a non-trivial solution iff $\det B = 0$. In the case of a 2×2 system, we have already sketched a proof of this fact, for students without a linear algebra background, by rephrasing in terms of familiar planar geometry facts. Of course a student is likely to believe, without proof, that this generalizes to higher dimension (as, in this case, they should). They can certainly experiment with the 3×3 proof on their own (with leading questions in their homework, perhaps). As instructors, though, we should encourage the student to seek out a full, valid proof *and a follow up course in Linear Algebra is precisely where they should look.*

Nevertheless, it can be helpful to demonstrate a 3×3 example. The purpose, here, is to give the students a sense of what additional issues may arise, not to open up a huge can of worms. We found this example to be reasonably representative and not overly complicated.

Example 5.1 (A three variable case). Consider

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}.$$

If we can find three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, then the solution to the system $\vec{x}\,' = A\vec{x}$ will be, as before,

$$\vec{x} = C_1\vec{v}_1 e^{\lambda_1 t} + C_2\vec{v}_2 e^{\lambda_2 t} + C_3\vec{v}_3 e^{\lambda_3 t}.$$

Here we remind the student how to take a 3×3 determinant to get the characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = -(\lambda - 1)^2(\lambda - 5).$$

We proceed to find eigenvectors by solving $(A - \lambda I)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The $\lambda = 1$ case is interesting in that it yields three copies of the equation $a + 2b + c = 0$. This leads to a 2-dimensional eigenspace spanned by $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

For $\lambda = 5$ we get

$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which is interesting in that it does not have a completely obvious redundancy. Students need to do some elementary row reduction (or use high school algebra techniques) to find that $a = b = c$, so the eigenvector can be chosen to be $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Of course, it is very easy to check that $A\vec{v}_3 = 5\vec{v}_3$.

Even the question of deficient eigenspaces can be addressed in a reasonably straightforward manner in a 2×2 example, though one does need to be careful not to assume too much.

Example 5.2 (Deficient Eigenspaces). Consider the first order system $\vec{x}\,' = A\vec{x}$, where

$$A = \begin{pmatrix} 0 & 1 \\ -9 & 6 \end{pmatrix}.$$

Letting the first variable be y and the second be $v = y'$, this system corresponds to the 2nd order equation $y'' - 6y' + 9y = 0$, which we know (from earlier in the course) has the general solution $y = C_1 e^{3t} + C_2 t e^{3t}$. Thus the vector solution to the original system is

$$\vec{x} = \begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} C_1 e^{3t} + C_2 t e^{3t} \\ 3C_1 e^{3t} + C_2(e^{3t} + 3te^{3t}) \end{pmatrix}$$

which can be written, intriguingly, as

$$\begin{aligned}
\begin{pmatrix} y \\ v \end{pmatrix} &= C_1 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} + C_2 \begin{pmatrix} te^{3t} \\ (e^{3t} + 3te^{3t}) \end{pmatrix} \\
&= C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + C_2 \left(\begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) e^{3t} \\
&= C_1 \vec{v} e^{3t} + C_2 (\vec{v} t + \vec{u}) e^{3t}.
\end{aligned}$$

where $\vec{v}$ is the eigenvector the student would find using earlier techniques and $\vec{u}$ is a *generalized eigenvector*. The students can be shown that in order to find $\vec{u}$ they need to solve

$$(A - 3I)\vec{u} = \vec{v} \quad \text{or, equivalently,} \quad \begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

or, rewritten, they need to solve $-3a + b = 1$.

The ideas and examples in this section are presented in our course as “advanced topics:” theory, higher dimensions, and degenerate cases. The point we want to emphasize is that all of this is interesting and a great motivator for a follow up course in Linear Algebra. The advanced student should begin to wonder about concepts such as linear independence, and the relation between eigenspace dimension and multiplicity of the eigenvalue in the characteristic equation.

6 The Rest of Our Course

One could object to only working with very low dimension and ask if we are essentially misleading our students by restricting ourselves to no more than 3×3 (perhaps 4×4 as a challenge homework assignment?) cases. We respond by considering how often $n \times n$ systems are genuinely taught in any ODE course, for large n? In looking through Boyce and DiPrima [3] we find that while nth order equations are frequently discussed, there are just seven references to specific differential equations of order greater than four; all are in exercises. Within the chapter, Systems of First Order Linear Equations, there are only six references to four-dimensional problems (and *none* to higher dimensions). Only one of those references has to do with solving a 4×4 system. (It is a long example with complex eigenvalues $\lambda = \pm i, \pm 2i$.) We note that *of course* there are few examples or exercises with $n > 4$, or even $n = 4$. It gets genuinely harder and messier to solve these larger problems even *with* the full linear algebra background. We are no more misrepresenting the full complexity of linear systems with a low-dimensional approach than we do when we pretend that one can easily find the roots of an nth degree characteristic polynomial for $n > 4$ in those higher dimensional problems. The “correct” approach would be to progress to a course in numerical analysis to approximate eigenvalues. But quite simply, we have left the students in a place where, we hope, they are *motivated* to take courses in linear algebra and beyond so that they can see what happens in more subtle circumstances.

In fact this initial presentation of eigenlines (and eigenvectors and, subsequently, eigenvalues) all works extremely well as motivation for a full discussion in such a follow-up course.

Additionally, by restricting ourselves to two and three dimensions, we have, hopefully, allowed ourselves a bit more time to fulfill the promise of showing the students some applications of linear systems. Having demonstrated the analytic methods for solving linear systems, we move on to applications, such as “love affairs” found in Strogatz’ book [6].

The next large portion of our course focuses on *non*-linear systems. An analysis of non-linear planar systems, we feel, should be possibly the greatest “take-away” for the students in the course. Here, we rely on *pplane* even more—to introduce attracting and repelling cycles, Hopf bifurcations, and the characterization of behavior near equilibrium points, etc. One highlight is the linearization of non-linear systems, for which the students need to take partial derivatives. This unit of the course culminates in the Poincaré-Bendixson Theorem for autonomous *planar* systems. A discussion of the (three-dimensional at last!) Lorenz system is, of course, interesting to the students due to the chaotic behavior it demonstrates. But it also provides a way for the student to see the limitations of Poincaré-Bendixson.

Other topics, such as Laplace Transforms or Series Solutions, may require a bit more calculus skill on the part of the student, but nothing from Linear Algebra.

7 Repositioning ODEs

With technological advances, ODEs has the ability to move *earlier* in the curriculum, before linear algebra. The Method of Eigenvectors should not be seen as a purely algebraic technique, but also, as we have emphasized in our presentation, as a geometric one. The focus *should* be on planar and spatial systems, where software exists to help us ask appropriate questions. Mathematical modeling is central to our course, not just tacked on. We think it is critical (as do our partner disciplines) for students to see “real world” applications of linear systems. Of course, it is a students ability to synthesize several approaches to the same concept that is the true goal of mathematical pedagogy.

Our main point is a positive one: *use* the graphics to *motivate* the eigenvalue equation. Frankly, this is the material that should be motivating a student to care about and learn the abstract generalizations found in a course in linear algebra.

8 Our Syllabus

Below is the syllabus that we used for our course, including a day-by-day list of topics. We have removed listings for exam days. The sections refer to [2], which is the text we used for our course. The material presented in this paper corresponds approximately to Days 12 and 13.

Differential Equations and Modeling
Spring 2013

Learning Objectives: This is an introductory course in the study of ordinary differential equations (ODEs). Standard analytic techniques for solving first and second order differential equations will be presented, as well as for systems of linear differential equations. In addition, a dynamical system approach will be taken, in order to develop a qualitative understanding of solutions to families of non-linear systems. Properties of equilibrium solutions and limit cycles, including stability and bifurcation will be developed. Throughout the course the differential equations and systems of equations will be associated with applications through standard modeling techniques.

Learning Outcomes:

A. Students will learn the basic vocabulary of differential equations (including what it means to be a solution to an ODE or an initial value problem (IVP)). Students will learn to classify a differential equation by type and order.

B. Students will learn fundamental techniques for modeling physical and biological phenomenon by differential equations and systems of differential equations.

C. Students will learn standard techniques for solving first ODEs and IVPs, including separation of variables and integrating factors.

D. Students will solve and interpret solutions to systems of linear systems with constant coefficients using eigenvalues and eigenvectors.

E. Students will understand the fundamental questions surrounding the existence and uniqueness of solutions to IVPs.

F. Students will learn to create state (phase) plane diagrams and to interpret such diagrams. They will become familiar with the terminology associated to such diagrams.

G. Students will learn to classify the behavior of solutions near equilibrium points and steady state solutions, both in linear and non-linear systems.

H. Students will be learn to recognize some basic types of bifurcations in planar systems.

I. Students will acquire the ability to read, write, listen to, and speak about mathematics within the framework of differential equations and modeling.

J. Students will learn how to engage in nontrivial mathematical problem solving, both on an individual basis and as part of small groups.

Text: Borrelli and Coleman, *Differential Equations: A Modeling Perspective*, 2nd Ed.

Day	Topic	Section
Day 1	Introduction to Modeling	Intro, 1.1, 1.2
Day 2	Examples, Integrating Factor	1.3, 1.4, 2.1
Day 3	Interpreting solutions, Existence and Uniqueness	2.2, 2.3
Day 4	Slope fields	2.3, 2.4
Day 5	Separation of Variables, Predator-Prey	2.5, 2.6
Day 6	Lotka-Volterra, Long Term Behavior	2.6, 2.7
Day 7	State Lines, Bifurcations	2.8. 2.9
Day 8	Hooke's Law, 2nd Order ODEs	2.9, 3.1
Day 9	Characteristic equations, types of solutions	3.2, 3.3
Day 10	Direction fields	3.3
Day 11	The Pendulum, orbits and their behavior	4.1, 6.1
Day 12	Eigenvalues and eigenvectors	6.2, 6.3
Day 13	Real vs. Complex eigenvalues	6.3, 6.4
Day 14	Orbital portraits	6.5
Day 15	The matrix exponential, steady states	6.6, 6.7
Day 16	Non-linear systems, nullclines	7.1, 7.2
Day 17	Modeling interacting species	7.2, 7.3
Day 18	Stability in linear and non-linear systems	8.1, 8.2
Day 19	Cycles	8.2, 9.1
Day 20	Long term behavior in non-linear systems	9.2, 9.3
Day 21	Bifurcations	9.3
Day 22	Chaos	9.4
Day 23	Laplace Transforms	Chapter 5
Day 24	Series Solutions	Chapter 11

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