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Approval Voting Theory with Multiple Levels of Approval

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Abstract

Approval voting is an election method in which voters may cast votes for as many candidates as they desire. This can be modeled mathematically by associating to each voter an approval region: a set of potential candidates they approve. In this thesis we add another level of approval somewhere in between complete approval and complete disapproval. More than one level of approval may be a better model for a real-life voter's complex decision making. We provide a new definition for intersection that supports multiple levels of approval. The case of pairwise intersection is studied, and the level of agreement among voters is studied under restrictions on the relative size of each voter's preferences. We derive upper and lower bounds for the percentage of agreement based on the percentage of intersection.

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Acknowledgments

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Chapter 1

Introduction

The goal of voting theory is to make a mathematical model that represents some aspects of real elections and voting. We care about issues of voting since it forms the cornerstone of any democratic process. The ability to vote gives people the ability to have a say in decisions that affect them, whether by a government or other organizations. Although other situations arise where voting is used, we will specifically investigate voting used to elect a person (a candidate) to some position.

A candidate has to have some set of beliefs, or positions, they hold. Following the terminology of Berg et al. (2010) we call each of these a *platform*. The set of all possible platforms is called the *spectrum*. Throughout this thesis we will assume that the spectrum is $\mathbb{R}$, so that the spectrum is a line. This could represent a scale of how, for example, liberal versus conservative that candidate is. It is also possible to use a spectrum of $\mathbb{R}^n$ if candidates can be ranked on more then one scale (for example fiscal and social).

In the United States a common type of voting for political candidates is known as plurality voting. In this system a voter will choose the one candidate they like best out of a list of potential candidates. However, there also exist other voting systems.

This thesis will be concerned with a generalization of *approval voting*. In approval voting each voter selects all of the candidates they approve (all the candidates they find "acceptable"). Therefore each voter can cast a vote for any number of candidates. The winner is the candidate that is selected by the greatest number of voters. Several organizations use approval voting to elect candidates, including the Mathematical Association of America and the American Mathematical Society. Taylor and Pacelli (2008) note that one advantage of approval voting is that it reduces the impact of *spoiler candidates*, candidates who cannot win, yet prevent another candidate from winning. One drawback is that it is not clear where a voter should make the distinction between approval and disapproval when the platforms of candidates are more complicated then the linear spectrum considered here.

Berg et al. (2010) have proven various results in approval voting theory relating to what percentage of voters can be proven to agree on some candidate given certain local conditions. It is possible to interpret a set of voters in approval voting theory as an *interval graph* (defined in Section 2.3) by giving each voter a vertex, and connecting two vertices with an edge if and only the corresponding voters overlap. Using this idea, Abbott and Katchalski (1979) proved a result for interval graphs that has direct implications for approval voting theory. This result relates the percentages of edges in an interval graph to the largest *clique* (subgraph that is also a complete graph) that must be in that graph. We show a similar interpretation for a set of voters with multiple levels of approval as a tolerance graph (described by Golumbic and Trenk (2004)). Chapter 2 provides more information about these previously proven results.

In Chapter 3 we provide new definitions for a model of approval voting theory with multiple levels of approval. We also give the details for the correspondence between a set of voters and a tolerance graph. This chapter also proves a generalization for a result from Berg et al. (2010).

In Chapter 4 we give generalizations for the results of Abbott and Katchalski (1979) regarding a lower bound on the number of voters that pairwise agree given how many voters agree overall. A corresponding upper bound in also provided in that chapter.

Chapter 5 shows how to gain stronger result about pairwise intersection by bounding the relative sizes of the different levels of approval. Motivating examples are provided to show why such bounds are desirable. Possible directions for future research are discussed in Chapter 6. Appendix A describes a computer program that creates random arrangements of voters. The source code is available through the Internet, and can be used to test or devise new theorems.

Chapter 2

Background

In this chapter we provide a summary of previous work that is related to approval voting theory. Many results rely on understanding results in number theory. These results are covered in the first section. Some theorems from the work of Berg et al. (2010) related to approval voting theory are presented. We explain the work of Abbott and Katchalski (1979) on interval graphs. It is shown how this work relates to the theory of approval voting. A background for tolerance graphs is also given. We will return to the idea of tolerance graphs as a representation of approval voting theory with two levels of approval in Section 3.2.

2.1 Useful Results from Graph Theory

In the introduction to their book on tolerance graphs, Golumbic and Trenk (2004) provide a definition for *interval graphs*.

Definition 2.1 (Interval Graphs)**.** *A graph $G = (V, E)$ is an interval graph if every vertex, v, can be associated with an interval in $\mathbb{R}$ such that two vertices are connected by an edge if and only if the corresponding intervals intersect.*

The concept of an interval graph is important to the theory of approval voting since each voter's approval region is an interval. Thus by assigning each voter a vertex and connecting vertices with edges if the corresponding voters intersect produces an interval graph. Various theorems about the properties of interval graphs play a key role in the proofs of the theorems discussed in the following sections.

A *clique* is a subset of the vertices of a graph such that there is an edge between every pair of vertices in the subset. A clique in an interval graph

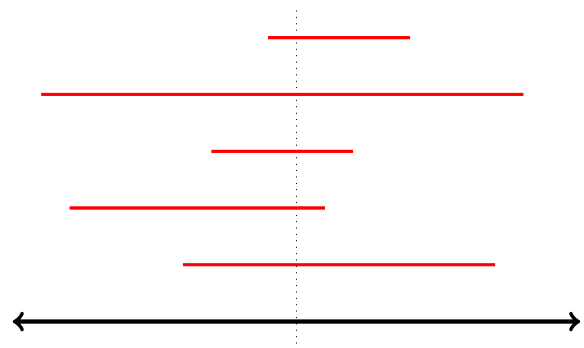

Figure 2.1 An example of pairwise intersecting intervals.

corresponds to a set of voters that pairwise intersect.

Definition 2.2 (Clique Number)**.** *The size of the largest clique in a graph G, the* clique number*, is denoted by $\omega(G)$.*

Theorem 2.1 (Helly's Theorem)**.** *Pairwise intersecting intervals have a common point.*

Helly's Theorem will be the basis for our first result from approval voting theory.

2.2 The Model of Berg et al.

Berg et al. (2010) consider the following model for approval voting. Each political position is called a *platform* and the set of all platforms is the *spectrum*. Each voter has a set of platforms they approve. These are modeled as closed intervals and are called the voter's *approval region*. Helly's Theorem implies that if every pair of voters' approval regions intersect, then there is a platform in every approval region. Berg et al. (2010) call this the *Super-Agreeable Society Theorem*.

Theorem 2.2 (Super-Agreeable Society Theorem)**.** *If every pair of voters have intersecting approval regions then there is a platform in every voter's approval region.*

Figure 2.1 shows an example of five pairwise intersecting voters. As guaranteed by Theorem 2.2 we see that there is a platform in every interval.

The paper also presents a similar theorem with a weaker hypothesis. They call a set of voters (k, m)-*agreeable* if out of every m voters there are k voters who have a point in common. Thus Theorem 2.2 considers those sets of voters that are $(2, 2)$-agreeable. The more general theorem is called the *Agreeable Society Theorem*.

Theorem 2.3 (Agreeable Society Theorem)**.** *Let $2 \leq k \leq m$. Let n be the number of voters. If a set of voters is (k, m)-agreeable then there is a platform that has the approval of at least $n(k-1)/(m-1)$ voters.*

For example, this implies that if out of every three voters there are two with a common platform then there exists some platform that has the approval of at least half of the voters. If such a platform were adopted by a candidate they would have a good chance of winning the election.

2.3 Results of Abbott and Katchalski

Since voter's preferences can be thought of as closed intervals in the real line (in our model), interval graphs can be used to represent the intersections between voters. We create an interval graph from a set of voters by assigning each voter a vertex and letting there be an edge between two vertices if and only if the corresponding voters have a platform in common.

Abbott and Katchalski (1979) proved a lower bound for the largest clique in an interval graph based on the percentage of possible edges the graph has.

Theorem 2.4. *Given an interval graph G, let N be the number of vertices in G, and let E be the number of edges. Let $\alpha = E/\binom{N}{2}$ (the percentage of possible edges) and let $\beta = \omega(G)/N$. Then we have*

$$\beta \geq 1 - \sqrt{1-\alpha}.$$

We care about the size of the largest clique that must exist in a graph with N vertices and E edges because Theorem 2.2 implies that the voters that correspond to the vertices in this clique have a platform in common that they approve.

2.4 Tolerance Graphs

Tolerance graphs are a generalization of interval graphs. A *tolerance graph* is a graph where every vertex can be assigned a closed interval in the real

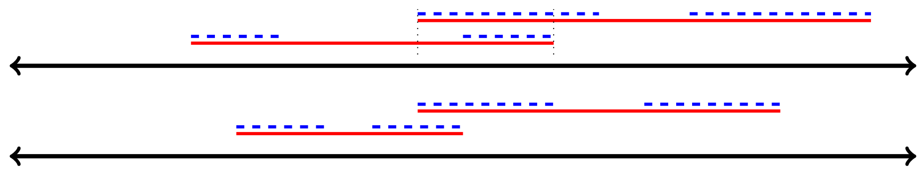

Figure 2.2 An example of intersecting and nonintersecting tolerance intervals.

line and a positive real number called the *tolerance* such that there is an edge between two vertices if and only if the length of the overlap of the corresponding intervals is at least the smaller tolerance (see Golumbic and Trenk (2004)). Thus the idea of tolerance graphs is a generalization of interval graphs. We will see later that tolerance graphs have an immediate application to approval voting theory with two levels of approval.

As an example, the interval $[-6, 2]$ with tolerance 2 and the interval $[-1, 9]$ with tolerance 4 would have an edge between their corresponding vertices since the overlap has length 3, which is greater then $\min(2, 4)$. However, the interval $[-5, 0]$ with tolerance 2 and the interval $[-1, 7]$ with tolerance 3 would not have an edge between their vertices since their overlap has length 1 and $1 < \min(2, 3)$. Figure 2.2 demonstrates these examples, with the first case shown at the top, and the second case shown at the bottom. The red (solid) lines represent the interval while the blue (dashed) lines represent the size of the tolerance.

We have seen that a set of voters can be modeled as an interval graph under the model of Berg et al. (2010). We will see in Chapter 3 that we can use a similar correspondence to model a set of voters in approval voting theory with two levels of approval as a tolerance graph.

Chapter 3

Accounting for Two Levels of Approval

In this chapter we give an interpretation of approval voting theory that allows for voters to have multiple levels of approval. As previously stated, approval voting is a system where each voter selects all the candidates they approve. We will now allow the voters an opportunity to waver on certain candidates. Each voter will still have some set of platforms they approve of, but there will also be a set of platforms that they weakly approve of. This could represent a region where they vote for each included candidate with a 50% probability. This chapter will show how to interpret a set of voters with two levels of approval as a tolerance graph. We also reinterpret Theorem 2.2 and discuss pathological cases for the arrangement of voters.

3.1 Definitions

We let the set of all voters be denoted $\mathbb{V}$. A voter v is described by four points L_v, l_v, r_v, and R_v with $L_v < l_v < r_v < R_v$. We call $[l_v, r_v]$ the voter's *approval region*, denoted by A_v. Note that this is still modeled as a closed interval as in approval voting theory. We call $[L_v, l_v) \cup (r_v, R_v]$ the voter's *maybe region*, denoted by M_v. If we need to be more specific, $[L_v, l_v)$ is know as the *left maybe region* and $(r_v, R_v]$ is known as the *right maybe region*. Together we call $[L_v, R_v] = A_v \cup M_v$ the voter's *interest region*, I_v, since it contains all the platforms that voter shows any interest in. We see that this represents a voter who strongly approves of a candidate taking any position inside some closed interval, and might approve of a candidate that is outside but close to the approval region. Figure 3.1 shows the four points

Figure 3.1 The four points that define a voter.

that define a voter. The approval region is shown in red (solid lines) and the maybe region is shown in blue (dashed line).

Each platform, p, is given a *value* with respect to each voter v. The value with respect to voter v is defined as

$$\mathfrak{V}_v(p) = \begin{cases} 1 & p \in A_V \\ 0.5 & p \in M_v \\ 0 & \text{otherwise.} \end{cases}$$

The overall value for a platform is

$$\mathfrak{V}(p) = \sum_{v \in \mathbb{V}} \mathfrak{V}_v(p).$$

Two voters, u and v, are said to *agree* or *intersect* if there is a platform that is approved by both voters, or is approved by one of the voters and is in the maybe region of the other. We provide the following precise definition:

Definition 3.1 (Intersection)**.** *Two voters intersect if either $I_u \cap A_v \neq \emptyset$ or $I_v \cap A_u \neq \emptyset$.*

We see that this is equivalent to the existence of a platform p such that $\mathfrak{V}_u(p) + \mathfrak{V}_v(p) > 1$.

We note that two voters' maybe regions could have a point in common without the voters intersecting, as shown in Figure 3.2. We see that the value of the platform at the dotted line is 1 since the value of the platform for each voter is 0.5. Since this is not strictly greater than 1, we see that the voters also do not intersect by the alternate definition. The alternate definition is provided since it would be easy to generalize to more than two levels of approval.

3.2 Connection to Tolerance Graphs

As we have seen previously tolerance graphs are based on the intersections of intervals with tolerances. If we consider a voter's interest region to be an

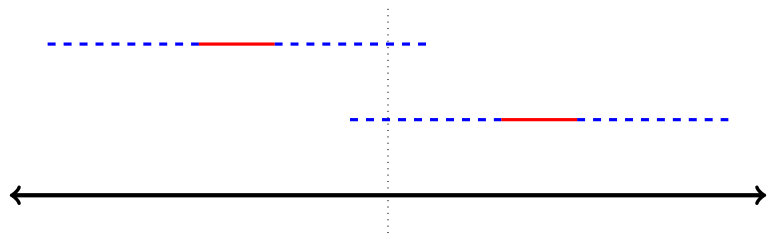

Figure 3.2 An example of nonintersecting voters.

interval and consider the size of a voter's maybe region to be the tolerance, then it can be shown the resulting tolerance graph is the same as the intersection graph of the voter. Notice, however, that this does require that the sizes of the two maybe regions for every voter be the same. Since, in this limited case, the graphs are the same, we see that any result for tolerance graphs is also true for intersection graphs.

Of special importance is the fact that tolerance graphs are perfect, since the fact that intersection graphs (for one level of approval) are perfect played a major role in the proof of Theorem 2.3 in Berg et al. (2010). The restriction that the size of the maybe regions must be the same can be lifted if we consider bitolerance graphs (where there is a separate left and right tolerance), which are also discussed in Golumbic and Trenk (2004).

3.3 Intersection Theorem

We were able to prove an initial result regarding agreement among voters that pairwise intersect.

Theorem 3.1. *If every pair of voters intersect, then there is some platform that is in every voter's interest region, and at least one voter's approval region.*

Proof. Since every pair of voters intersect we know that their interest regions pairwise intersect as well. Since the interest regions are closed intervals Theorem 2.2 tells us that there is some platform, p, that is in every voter's interest region. If p is in some voter's approval region then we are done. Therefore assume that p is in every voter's maybe region.

Let

$$a = \max\left(\{L_v, l_v, r_v, R_v | v \in \mathbb{V}\} \cap (-\infty, p]\right).$$

That is, a is the closest boundary of a maybe region to p that is also to the left of p. If $a = l_u$ or $a = r_u$ for some voter u then a is the desired point.

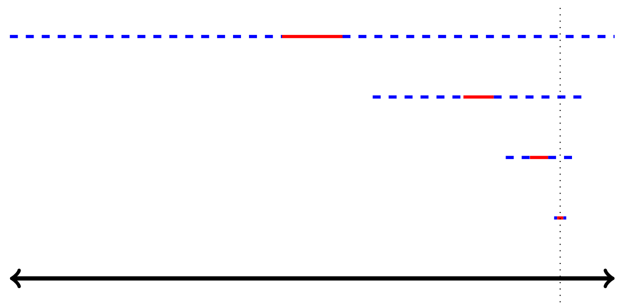

Figure 3.3 Example for intersection theorem.

Since a is the closest boundary point to the left a is still in the interest region of every voter except u and a is in the approval region of u (a might be in the approval region of other voters if a is a boundary point for more than one voter). We see that a cannot equal R_u for any voter u since that would mean that $p \notin I_u$. However, a could equal L_u for some voter u.

In this case, let

$$b = \min\left(\{L_v, l_v, r_v, R_v | v \in \mathbb{V}\} \cap (p, \infty)\right).$$

If b equals l_w or r_w for some voter w we are done. We note that b cannot equal L_w for any voter w since then $p \notin I_w$. If $u = w$ then $b = l_u$ since $L_u \leq p < l_u$, and b is the desired point. Otherwise $u \neq w$ and $b = R_w$ for some voter w. Since this would imply that $r_w < L_u \leq p < R_w < l_u$ we see that voters w and u do not intersect, which is a contradiction. Therefore the only valid cases show that there is a platform in every voter's interest region, and at least one voter's approval region. □

We see that we cannot guarantee any better result by examining Figure 3.3. In this figure we add new voters that are completely contained inside the previous voter's right maybe region. This guarantees that any point that is in the interest region of the last voter added will be in the maybe region of all the other voters. Thus we can find a platform that is in the approval region of some voter, and the maybe region of all other voters. We cannot, however, find a platform that is in the approval region of more than one voter and the maybe region of all other voters. We can have an arbitrary number of voters in this arrangement.

Chapter 4

Bounding Maximum Value by Percentage of Edges in Intersection Graph

In this chapter we consider the platform with the maximal amount of approval in terms of how many edges are in the intersection graph for the voters. An intersection graph with relatively many edges represents a set of voters for which there is a large amount of agreement. Thus we will show that a high percentage of possible edges will lead to a platform that is in the interest regions of a large percentage of voters. Both a lower and an upper bound for these values are determined.

Let N denote the number of voters. Let the number of edges in the intersection graph divided by $\binom{N}{2}$ be called the *pairwise agreement proportion* (denoted α). We see that this is the percentage of possible edges. Let the maximum value of $\mathfrak{V}(p)$ divided by N be called the *agreement proportion* (denoted β). We see that this is the percentage of possible value.

We will prove upper and lower bounds involving α and β. The development of these results is motivated by Abbott and Katchalski (1979) who proved a similar result for interval graphs.

4.1 Bounds

Theorem 4.1 (Lower Bound)**.** *We first establish a lower bound for β. The agreement proportion β satisfies the lower bound*

$$\beta \geq \frac{1-\sqrt{1-\alpha}}{2},$$

where α is the pairwise agreement proportion.

Proof. We will interpret our voter's interest regions as intervals so that we can apply the Abbott and Katchalski result. Let G be the intersection graph, and let E be the number of edges in G.

Consider the interval graph that arises from letting each voter's interest region be an interval. We call this graph G' and call its number of edges E'. Let α' be the percentage of edges for this graph, and let β' be the percentage of value for the intersection of these intervals.

Since intersection for voters is more restrictive than intersection for intervals, we see that $E \leq E'$. Thus $\alpha = E/\binom{N}{2} \leq E'/\binom{N}{2} = \alpha'$. Therefore we see that

$$1-\sqrt{1-\alpha} \leq 1-\sqrt{1-\alpha'}.$$

Abbott and Katchalski's results tells us that

$$1-\sqrt{1-\alpha'} \leq \beta'.$$

All that remains to show is that $\beta' \leq 2\beta$. If the value of β' comes from platform p, then we have that $2\mathfrak{V}(p) \geq N\beta'$ since each voter that contributes one to $N\beta'$ contributes at least $1/2$ to $\mathfrak{V}(p)$. Since β is the maximum percentage, $2N\beta \geq 2\mathfrak{V}(p)$ which completes the proof. Figure 4.1 shows the graph of this lower bound. □

Before providing an upper bound for β we prove the following lemma.

Lemma 4.1. *If there is a platform that is in the left maybe set of two voters then those two voters intersect.*

Proof. Let the voters be u and v. Consider $\min\{l_u, l_v\}$. Since this point is in the approval region of one of the voters, and is in either the maybe region (if $l_u \neq l_v$) or the approval region (if $l_u = l_v$) of the other we see that these voters intersect. □

A similar proof shows the same result when there is a platform that is in the right maybe set of two voters.

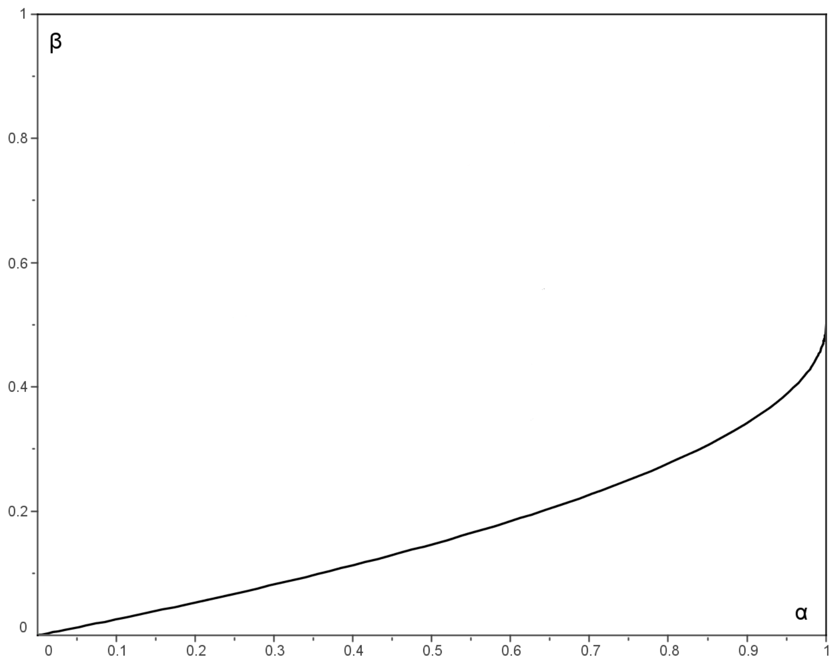

Figure 4.1 Graph of lower bound.

Theorem 4.2 (Upper Bound)**.** *The upper bound of the agreement proportion with respect to the pairwise agreement proportion satisfies the equation*

$$\binom{N}{2}\alpha \geq \binom{N\beta}{2}$$

where N is the number of voters.

Proof. Let G be the intersection graph, and let E be the number of edges in G. Assume that the platform, p, that maximizes the value be in the approval region of l voters and the maybe region of k voters (so that $\mathfrak{V}(p) = l + k/2$).

We will count the number of edges guaranteed to exist among these $l + k$ voters. Of the k maybe voters, let p be in left maybe region of k_1 of them, and in the right maybe region of k_2 of them. We see that we have $k_1 + k_2 = k$. The l voters that have p in their approval region all intersect with each other for $\binom{l}{2}$ edges. The l voters that have p in their approval region also intersect with each of the k voters that have p in their maybe region for lk edges. The k_1 voters that have p in their left maybe region all intersect with each other by Lemma 4.1 for $\binom{k_1}{2}$ edges. Likewise the k_2 voters that have p in their right maybe region all intersect with each other for $\binom{k_2}{2}$ edges. Thus

$$E \geq \binom{l}{2} + lk + \binom{k_1}{2} + \binom{k_2}{2}.$$

Expanding this out gives

$$\begin{aligned} E &\geq \frac{l(l-1)}{2} + lk + \frac{k_1(k_1-1)}{2} + \frac{k_2(k_2-1)}{2} \\ &= \frac{l^2 - l + 2lk + k_1^2 - k_1 + k_2^2 - k_2}{2}. \end{aligned}$$

Recall that $-k_1 - k_2 = -k$. We therefore have that

$$E \geq \frac{(l^2 + lk/2 + lk/2 - l - k/2) + lk + k_1^2 + k_2^2 - k/2}{2}.$$

Consider $lk + k_1^2 + k_2^2 - k/2$. We have $lk \geq 0$. Since $(k_1 + k_2)^2 = k_1^2 + k_2^2 + 2k_1k_2$, we see that $k_1^2 + k_2^2 = k^2 - 2k_1k_2$. Since k_1 and k_2 are subject to the constraint $k_1 + k_2 = k$ we see that the maximum value of k_1k_2 is $k^2/4$. Thus $k_1^2 + k_2^2 \geq k^2 - k^2/4$. It is easy to verify that

$$k^2 - \frac{k^2}{4} - \frac{k}{2} \geq \frac{k^2}{4} \quad \text{for } k \geq 2.$$

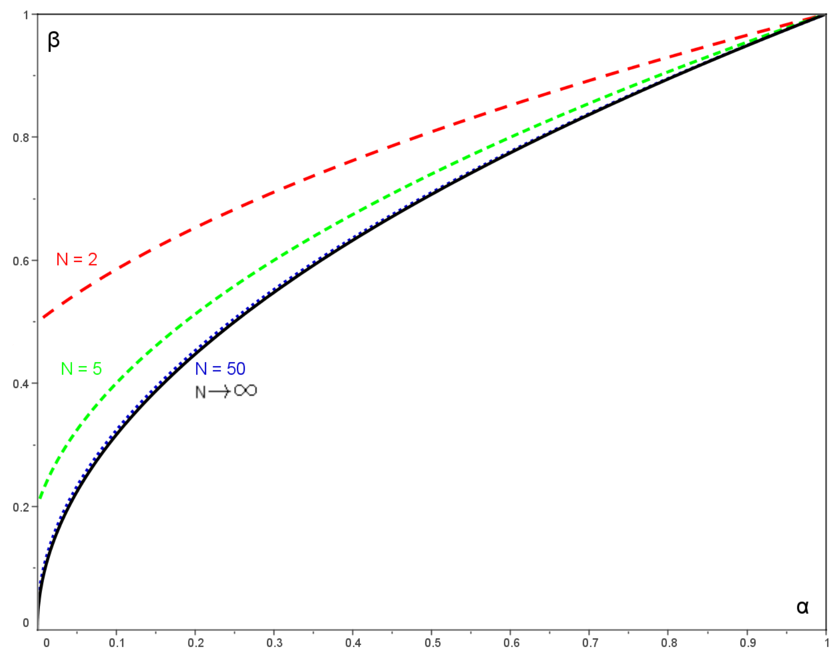

Figure 4.2 Graph of upper bound for selected values of N.

For $k = 1$ we have, without loss of generality, $k_1 = 1$ and $k_2 = 0$ so $k_1^2 + k_2^2 - k/2 = 1/2 \geq 1/4 = k^2/4$. For $k = 0$ we have $k_1 = k_2 = 0$, so $k_1^2 + k_2^2 - k/2 = 0 \geq 0 = k^2/4$. Thus for all k we have $lk + k_1^2 + k_2^2 - k/2 \geq k^2/4$.

This leaves us with

$$E \geq \frac{l^2 + lk/2 + lk/2 + k^2/4 - l - k/2}{2} = \frac{(l + k/2)(l + k/2 - 1)}{2} = \binom{l + \frac{k}{2}}{2}.$$

Since $E = \binom{N}{2}\alpha$ and $l + \frac{k}{2} = \mathfrak{V}(p) = N\beta$ this completes the proof. □

Figure 4.2 shows this bound for selected values of N. We start with $N = 2$ since α is not clearly defined for $N = 1$. The graph shows that for large values of N this bound is nearly identical to $\alpha \geq \beta^2$.

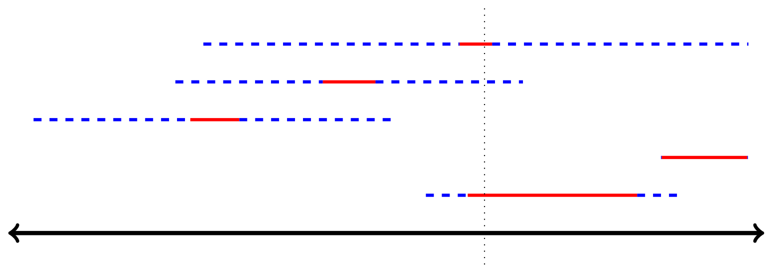

Figure 4.3 Example to illustrate the demonstrated bounds.

4.2 Example of Bounds

Consider the set of voters shown in Figure 4.3. The dotted line shows a platform that has maximal value. We see that this collection has five voters ($N = 5$), seven edges in the intersection graph, and $\mathfrak{V}(p) = 2.5$. Thus for this example $\alpha = 7/\binom{5}{2} = 7/10$ and $\beta = 2.5/5 = 1/2$. To see that these values of α and β satisfy the bounds we note that

$$\frac{1}{2} \geq \frac{1 - \sqrt{1 - \frac{7}{10}}}{2} \approx 0.226 \quad \text{and} \quad \binom{5}{2}\frac{7}{10} = 7 \geq \binom{5\frac{1}{2}}{2} = \frac{15}{8}.$$

Chapter 5

Application of Bounds on Relative Sizes

We saw previously in Figure 3.3 that allowing the interest regions to get arbitrarily small or large caused us to be unable to make strong claims about intersections of approval regions. In this chapter we look at the consequences of placing bounds on the relative sizes of the different regions.

5.1 Permissiveness and Wavering Ratios

For a set of voters, $\mathbb{V}$, we define the *permissiveness ratio*, $p(\mathbb{V})$, as

$$p(\mathbb{V}) = \max_{u,v \in \mathbb{V}} \left\{ \frac{r_u - l_u}{r_v - l_v} \right\}.$$

Since we assume the set of voters is finite this maximum will exist. Intuitively this is the ratio between the longest and shortest approval regions, which is a measure of how much more permissive some voters may be than others. Note that the corresponding minimum would be given by $1/p(\mathbb{V})$, so this bounds the ratio between the approval regions both above and below. We also define the *wavering ratios*. Let

$$w(\mathbb{V}) = \min_{v \in \mathbb{V}} \left\{ \frac{l_v - L_v}{r_v - l_v}, \frac{R_v - r_v}{r_v - l_v} \right\}.$$

Intuitively this represents how small a maybe region can be compared to the approval region for each voter, how unwavering some voters may be.

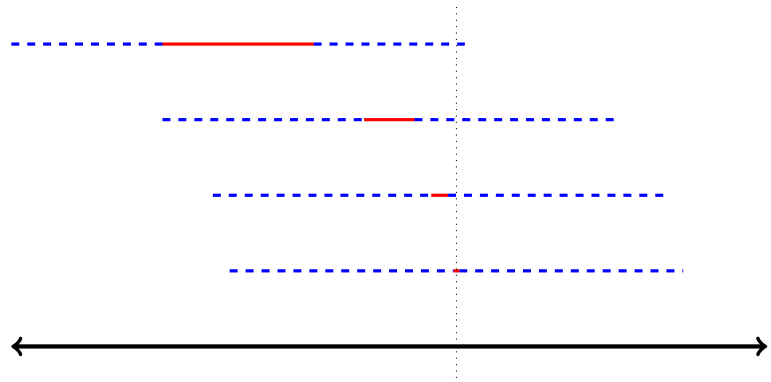

Figure 5.1 Variable sizes for approval regions.

For the following discussion we will keep the sizes of the left and right maybe regions the same for each voter, in which case we can simply write

$$w(\mathbb{V}) = \min_{v \in \mathbb{V}} \left\{ \frac{l_v - L_v}{r_v - l_v} \right\}.$$

We also let

$$W(\mathbb{V}) = \max_{v \in \mathbb{V}} \left\{ \frac{l_v - L_v}{r_v - l_v} \right\}.$$

This represents how large a maybe region can be compared to the approval region for each voter.

Consider the set of voters, $\mathbb{V}$, shown in Figure 4.3. Some computation shows that $p(\mathbb{V}) \approx 5.22$, $w(\mathbb{V}) \approx 0.01$ and $W(\mathbb{V}) \approx 7.91$.

If we allow sets of arbitrary numbers of voters to be constructed without considering these three bounds then we can arrive at arrangements of voters that limit the number of overlapping approval regions. The example from Figure 3.3 has bounds for $w(\mathbb{V})$ and $W(\mathbb{V})$ but not for $p(\mathbb{V})$. Figure 5.1 shows an example of such an arrangement for a bound on $p(\mathbb{V})$ but no bound on $w(\mathbb{V})$. This example, which can easily be generalized to an arbitrary number of voters, shows that situations can arise with only one approval region in the intersection of all interest regions.

Assuming that all voters pairwise intersect we now look at how many approval regions are guaranteed under specific values of the three bounds.

Theorem 5.1. *Let $\mathbb{V}$ be a set of N voters such that $p(\mathbb{V}) = 1$ and $w(\mathbb{V}) = W(\mathbb{V}) = \mu$. That is to say, all the approval regions are the same size and each* *maybe region is μ times as big as the approval region (also guaranteeing they are the same for all voters). If all voters pairwise intersect then there is some platform that is in the approval region of*

$$\left\lceil \frac{N}{1+\lceil \mu \rceil} \right\rceil$$

voters.

Proof. Consider any $\lceil \mu \rceil + 2$ voters in $\mathbb{V}$. I claim that there is a point that is in the approval region of two of them. Since they all have approval regions of the same size let that size be denoted x. Then we see that the size of each maybe region is μx. Let u be the voter for which l_u is smallest. Let v be the voter for which r_v is greatest. Voters u and v must still intersect so these l_u and r_v can be no further than $x + \mu x + x$ apart. Equality occurs when $R_u = l_v$ (or equivalently $r_u = L_v$).

Since these were the extremal voters all other approval regions fall between these points. Since $(\lceil \mu \rceil + 2)x \geq (\mu + 2)x$ we see that there is no way to place $\lceil \mu \rceil + 2$ approval regions into the given space without two overlapping (remember endpoints are included in the approval regions). Thus we see that the approval regions of the voters are $(\lceil \mu \rceil + 2, 2)$-agreeable. Recall that Theorem 2.3 tells us that there is therefore a platform in at least

$$\frac{N(2-1)}{(\lceil \mu \rceil + 2) - 1} = \frac{N}{1 + \lceil \mu \rceil}$$

approval regions. Since there is a platform in at least this many approval regions, and since the platform must be in an integer number of approval region, we can add a ceiling function to the expression. □

We take some time here to interpret this result. The first thing to note is the discrete jumps that occur in the bound at each integer. The easiest way to visualize a situation where the is no platform in a greater number of approval regions is to place all of the approval regions into columns of approximately equal size. Figure 5.2 shows that when $\mu = 3$ we cannot have five columns of approval regions that don't overlap, because endpoints are included in approval regions.

Thus the greatest number of columns that could be created is four, which is consistent with the result from the theorem. However, if μ were any bigger than 3 we could place a little bit of a gap between each of the endpoints of the approval regions in the columns. Thus if μ is any larger than 3 we can construct examples with five columns. We will not be able to have six

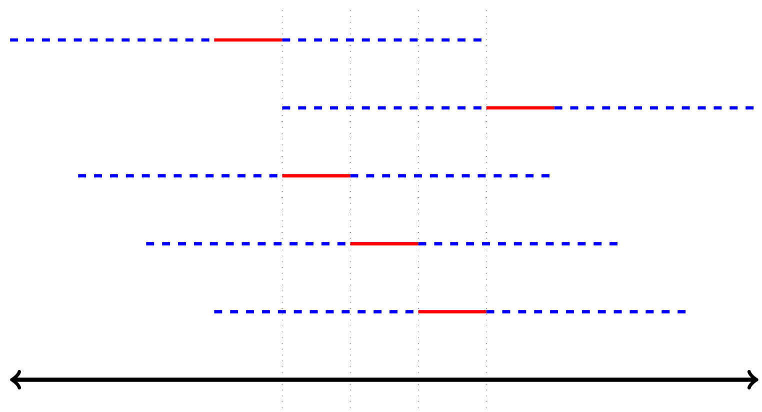

Figure 5.2 An example of five voters when $\mu = 3$.

columns until μ exceeds 4. Thus the bound only depends on $\lceil \mu \rceil$. If we were to create an alternate definition for approval regions where endpoints are not included this would not eliminate the discrete jumps, but would only cause the integer values of μ to be on the other side of the jump. We now consider the case where there are maybe regions of different sizes.

Lemma 5.1. *Let $\mathbb{V}$ be a set of N voters such that $p(\mathbb{V}) = 1$ and $W(\mathbb{V}) = \mu$. That is to say, all the approval regions are the same size and each maybe region is no more than μ times as big as the approval region. If all voters pairwise intersect then there is some platform that is in the approval region of*

$$\left\lceil \frac{N}{1 + \lceil \mu \rceil} \right\rceil$$

voters.

Proof. Since all voters have approval regions of the same size, let that size be denoted x. Note that if we were to extend the maybe regions of all voters to be of length $n \cdot x$ then all voters will still pairwise intersect, and the approval regions are unchanged. We can then apply the result of Theorem 5.1 to see that there is a platform in the approval region of at least $\lceil \frac{N}{1+n} \rceil$ voters. □

This lemma tells us that allowing some voters to have smaller maybe regions will not decrease the number of approval regions that overlap. Since smaller maybe regions limit the potential for a voter to intersect all other voters we would expect that in many cases we could guarantee even more overlapping approval regions than this lower bound. We consider some conditions where there are maybe regions of different sizes among the voters, but for which this bound cannot be improved.

Theorem 5.2. *For* $1 \leq i \leq n$ *let* $\mathbb{V}_i$ *be a set of voters such that* $w(\mathbb{V}) = W(\mathbb{V}) = i$. *Let* $\mathbb{V} = \bigcup_{i=1}^{n} \mathbb{V}_i$ *be a set with* N *total voters. If* $p\mathbb{V} = 1$, *all voters pairwise intersect, and* $|\mathbb{V}_i| \geq \frac{|\mathbb{V}_1|+\cdots+|\mathbb{V}_{i-1}|}{i}$ *then there is a platform in the approval region of*

$$\left\lceil \frac{N}{1+n} \right\rceil$$

voters, and no better bound exists.

Proof. We see by Lemma 5.1 that we can find a platform in the approval region of

$$\left\lceil \frac{N}{1+n} \right\rceil$$

voters. We now demonstrate an arrangement of voters in $\mathbb{V}$ that does not have a platform in more approval regions. We will show by induction on n that we can place these voters into approximately equal columns so that no column has more than $\lceil \frac{N}{1+n} \rceil$ voters in it.

When $n = 1$, we can place all voters (almost) equally into two columns which hold $\lceil \frac{N}{2} \rceil$ and $\lfloor \frac{N}{2} \rfloor$ voters. Thus the bound cannot be improved. Assume that we have a total of N voters in k types and that they can be placed in $k+1$ columns as described. We see that no column has more than $\lceil \frac{N}{1+k} \rceil$ voters in it. Now consider adding a set $\mathbb{V}_{k+1}$ of N' voters. Since these voters have a longer maybe region they can be placed in a separate column so that they will still intersect all other voters. Since N' is an integer and by assumption $N' \geq \frac{|\mathbb{V}_1|+\cdots+|\mathbb{V}_k|}{k+1}$ we see that we have at least as many voters in $\mathbb{V}_{k+1}$ as were originally in the most populated column. We initially place these voters into the new column and distribute the remaining voters among the columns so that any columns with one less voters are filled first. When we are done we see that there will only be two possible size for columns, differing by one only if we cannot fill all the columns equally. Since we have $N + N'$ voters evenly split among $k+2$ columns we see that the most voters that can be in any column is

$$\left\lceil \frac{N+N'}{1+(k+1)} \right\rceil.$$

Thus we see that for the given constraints there is an arrangement such that no platform is in more than $\lceil \frac{N}{1+n} \rceil$ approval regions. □

Figure 5.3 shows an example of the construction described in Theorem 5.2 when $|\mathbb{V}_1| = 3$, $|\mathbb{V}_2| = 5$, $|\mathbb{V}_3| = 8$ and $|\mathbb{V}_4| = 6$. These numbers satisfy the constraints of Theorem 5.2. We can also easily verify that the voters pairwise intersect. The dotted lines show the "columns" created by the construction. No approval regions in different columns overlap. We see that the platform in the greatest number of approval regions is a platform in one of the first two columns. These are in

$$\left\lceil \frac{3+5+8+6}{1+4} \right\rceil = \lceil 4.4 \rceil = 5$$

approval regions, as desired.

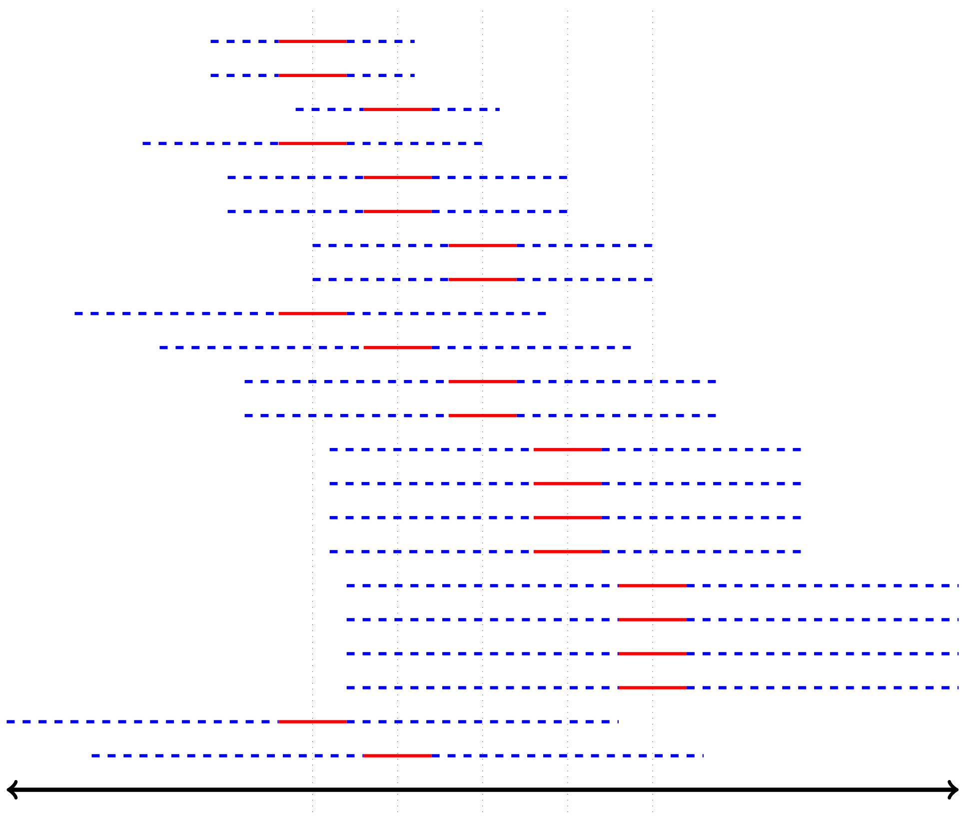

Figure 5.3 Example for Theorem 5.2.

Chapter 6

Future Work

Many questions are left to be answered about approval voting with multiple levels of approval. In Chapter 5 different bounds can be explored. Of particular interest are bounds where approval regions are allowed to have different sizes among voters.

In all the theorems currently proven we assume that all the voters pairwise intersect. I wish to reinterpret the Agreeable Society Theorem (Theorem 2.3) to voting with two levels of approval by imposing some form of a (k, m)-agreeable condition. I conjecture that this additional constraint will give a ratio that acts multiplicatively with the ratio given in Theorem 5.1.

It is yet to be seen if the connection between approval voting with two levels of approval and tolerance graphs can yield new theorems. Lastly, the theory can be extended to account for more than two levels of approval. With arbitrary numbers of levels of approval each voter's preferences would look closer to a continuous function describing how much they approve of each platform. The most general form would be to extend the results to arbitrary functions.

Appendix A

Simulations

Source code is available at http://www.math.hmc.edu/seniorthesis/archives/2012/cburkhart/cburkhart-2012-data.zip to create a random arrangement of voters that can be used to test ideas and create new hypotheses. The programing language used is Ruby. The program works by choosing two random points between -5 and 5 (using the built in random number generator) to be L_v and R_v. Then a random point is chosen between 0 and $(R_v - L_v)/2$ to represent the distance between the endpoint and the end of the maybe regions (so that l_v is defined as L_v added to this random value). Certain statistics, such as α and β from Chapter 4, are also computed.

A.1 Limitations

Although this code is an interesting tool to examine some of the situation that can arises, it does posses a number of limitations. For every voter this code creates the size of the two maybe regions is equal. This is a natural restriction when we wish to use a tolerance graph approach, but is not a restriction that needs to be adhered to in general. The boundary of -5 and 5 is probably not a restriction since any situation with a finite number of voters can be rescaled to fit into this range. However, the limits on how small variables can be on a computer could prevent significantly small regions for the voters, such as would be required for constructions like Figure 3.3.

Since all the voters are created using the same random process, for large number of voters the outputs from this program look very similar. As such, there tend to be no data to explore the extreme cases close to the upper and lower bounds on α and β. This is demonstrated in Figure A.1 which shows a plot of α versus β for twenty random configurations of 10 (red), 25 (green)

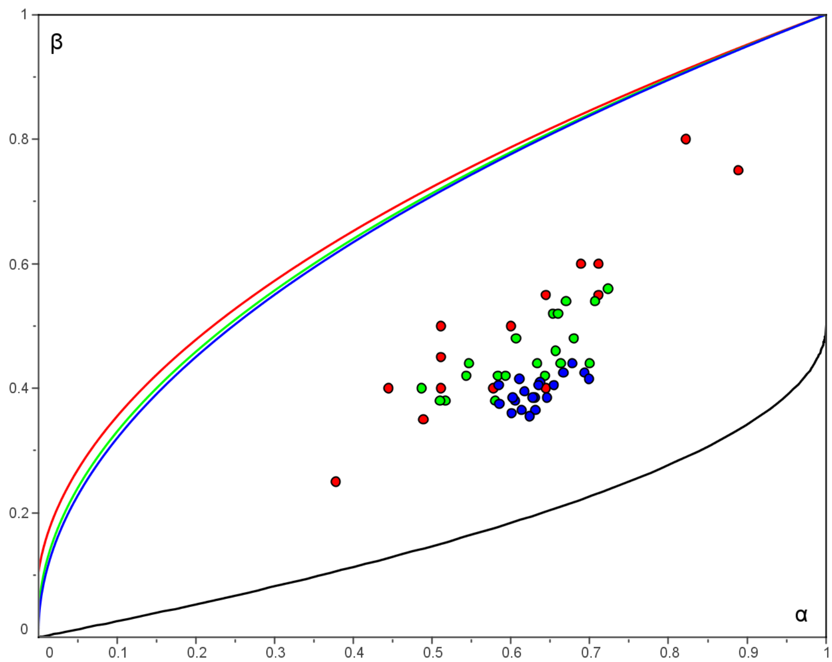

Figure A.1 Plot of α versus β for randomly generated voters.

and 100 (blue) voters produced by the program. It can be seen that as the number of voters increases the output from the program is increasingly similar. Also shown in the figure are the upper and lower bounds derived in Theorems 4.1 and 4.2.

A.2 Examples

The following is an example output of this code with fifteen voters. Figure A.2 shows the voters displayed with the spectrum. As we can see a variety of different lengths for the approval and maybe regions were produced. There are also voters at varying positions along the spectrum. Figure A.3 shows the intersection graph of the voters. In the intersection graph each vertex represents a voter and two vertices are joined by an edge if and

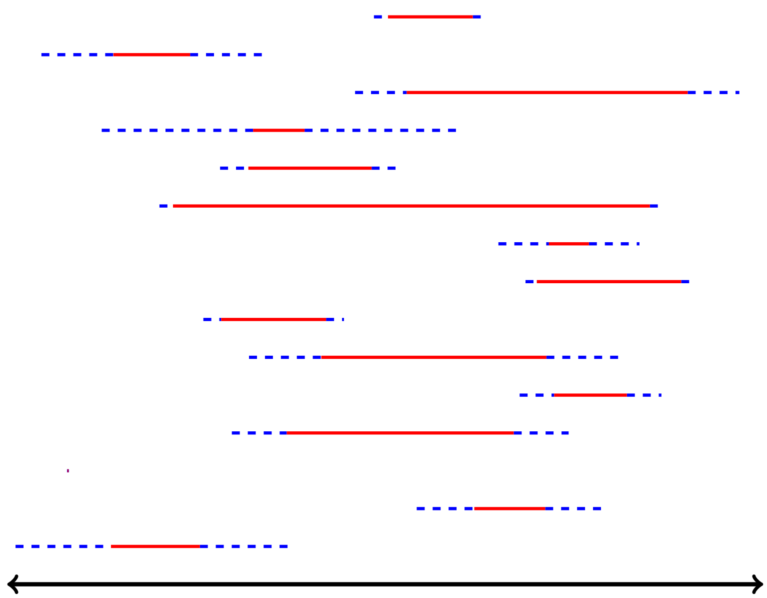

Figure A.2 An example of a fifteen-voter output from the code.

only if the corresponding voters intersect. The code also tells us that for this graph $\alpha \approx 58.1\%$ and $\beta = \frac{6}{15}$.

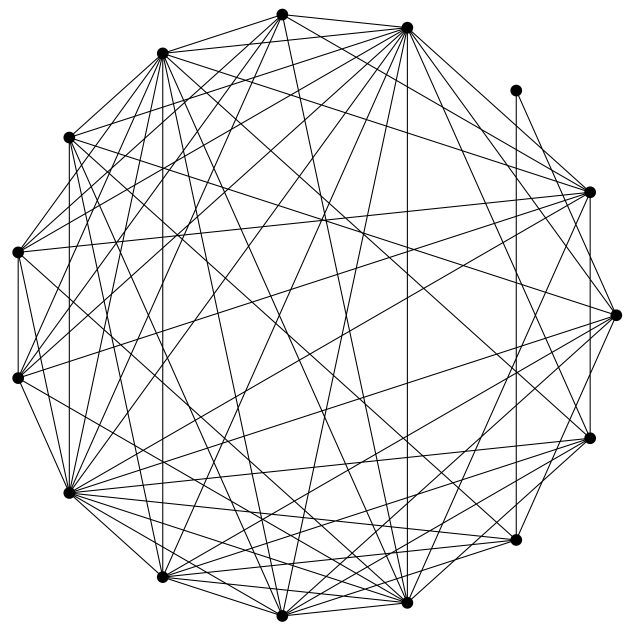

Figure A.3 The intersection graph for the fifteen voters shown in Figure A.2.

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