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Harmonics in the Library

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Synopsis

Students of traditional calculus courses can discover significant mathematics original to themselves, especially if these courses are taught in a way that allows shafts of mathematical light to shine through. We tell a story of such an incident in the form of a dialogue between two fictional students. Our students, on their own, discover (or rediscover) a well-known problem based on the harmonic series. We believe opportunities for such discoveries are greater if students have had some experience with inquiry-based learning prior to entering a traditional course. More broadly, we aim to demonstrate what can occur when students feel no inhibition exploring and even creating mathematics on their own.

Many mathematics instructors are experimenting with Inquiry Based Learning (IBL), a teaching technique which encourages students to discover mathematics on their own or with only minimal guidance [5, 7]. The discovery may be a proof of a theorem or, as in the scenario presented here, a simple case of the book overhang problem [8]. In mathematics the most well-known proponent of what is today called IBL was R. L. Moore, a professor of mathematics at the University of Texas at Austin during the years 1920–1969. Moore taught using an extreme version of IBL, which is today called the Texas Method or the Moore Method, that provided almost no guidance to the learner during the discovery period.

In the dialogue we present here, the two students have no real guidance, either. However, the spirit of discovery may have been encouraged in some prior mathematics class; perhaps, we might assume, a lesson learned was not forgotten. More broadly, our goal in this essay is to demonstrate what can occur when students feel no inhibition exploring and even creating mathematics on their own.

Some practitioners of IBL avoid using the standard names and labels of concepts and theorems (e.g., continuity, the Heine-Borel Theorem) in their teaching. This is done so as to minimize intimidation, especially when students are solving difficult problems or proving theorems on their own. Technical words and terminology may carry unintended cognitive noise. Avoiding them, students might have a better chance to discover solutions or proofs on their own. In the following then, it is in this spirit that the reader does not see the words "harmonic series" or "the book overhang problem" used until the students discuss their discovery with their professor.

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This is a story of two students who make a marvelous discovery, not only of a fact but of a solution original to them. We call them Toby and Anne. Anne is a physics major, and Toby has recently switched to a major in mathematics. Their calculus professor's classes are very much like the guided drill calculus courses typical in most universities; however, he allows some "shafts of mathematical light" to shine through from time to time while lecturing. We call him Dr. Bradford.

Eventually, and most importantly, Anne and Toby discover and solve the problem on their own. Moreover, they finally learn what divergence means, at least in the case of the harmonic series, even though they have seen the material before in their calculus course. A poor memory may actually help them. Only after they successfully apply themselves to the problem do they present their discovery to their calculus teacher. Much of the time, students do not really learn a concept until they become intimately engaged with an appropriate problem.

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We meet Toby and Anne, as part of their study routine, in the library doing their homework for Dr. Bradford's calculus class. Let us listen in.

Anne: I've done so many of these problems. I could do them in my sleep.

Toby: I hear you!

As you might expect from talented students, they are bored with routine drill work. They become distracted. Their minds search for a more interesting topic.

Anne: (Glancing at a set of reference books nearby) There's an old set of encyclopedias.

She begins playing with them, placing one on top of the other but offset a little from the edge of the table. Real discovery starts with some mind doodling. In our years of teaching, we sometimes say that doing mathematics starts back in the sandbox on the playground.[1] In that spirit, Anne's play seems natural, as we are about to see.

Anne: Could we stack some of these books so that by displacing each a little from the one underneath, the stack arches out from the table so much so that the top book is totally above the floor? (See Figure 1.)

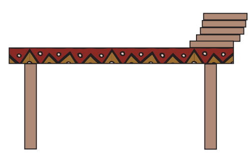


Figure 1: Anne's stack of books.

Toby: Is that possible?

Anne: It might work. Let's see ... from physics, the center of gravity of the stack must be above the table or the books will fall.

Just before it would tip and fall off the table, she places one book half on and half off the table as you can see in Figure 2(a) on the next page. The center of gravity of the one book is just above the right edge of the table. Notice that she keeps her approach simple in the beginning. We will continue to

[1]The author makes this statement from time to time to help motivate students when they need to dig a little deeper to create proofs. For more on how children's play influences their way of doing mathematics, see [3, 4].

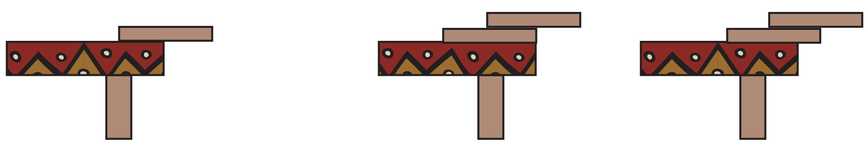
see that in subsequent steps. In that spirit, we share the following statement from "Challenge in the Classroom:" [2]

> **"***It is only the clearest of minds that are the first to think of something which when once thought of is clear to everybody.***"**

It is important to stress thinking simply, which Anne, apparently, has already begun to learn.

Anne: If I place a second book on top of the first but a little to the right, then the center of gravity of the two books would be off the edge of the table. Of course, the stack would fall to the floor. That won't do!

Toby: (Again, thinking simply) Why not place the second book directly beneath the first? Then, then ... slide the two books to the right so that the stack almost tips over. Now, you've got something! (See Figure 2(b).)



(a) Anne's idea. (b) Toby's idea.

Figure 2: Anne and Toby's attempts.

Anne: That's a great idea! Will your idea work for three books?

Even in mathematics, excitement may abound, especially, when students discover ideas on their own.

Toby: Let's give it a try.

[2] "Challenge in the Classroom" is a documentary film about the teaching methods of R. L. Moore. Produced in 1966 by the American Mathematical Association, it is available from the MAA's publication division with the new title of "The Moore Method: A Documentary on R. L. Moore."

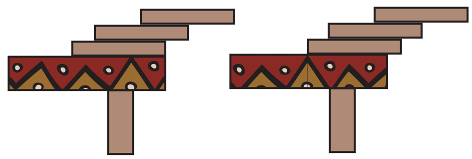


Figure 3: Anne and Toby's third try.

Anne places a third book directly under the stack of the first two, placing it where the right end of the book coincides with the edge of the table. Then, she slides the entire stack to the right until the stack is at a tipping point. (See Figure 3.)

Anne: The overhang distances are getting smaller and smaller real fast. It looks good but how many books will we need, assuming this all works?

Note that Toby and Anne are ingenuously using the previous stack of n books and place the $(n+1)$st book beneath that stack. An algorithm is developing.

Toby: Should we try one more?

Anne: No. Not yet, anyway. I want to see how far out we can stack them.

Anne is thinking like a scientist, searching for a generalization or a theory.

Toby: I don't think we can stack them with the top book completely hanging over the table.

Toby is expressing some healthy doubt but is open to what Anne is attempting. Next he will confidently begin to apply some mathematical thinking to the problem at hand.

Toby: (With a chuckle) Anyway, we don't have enough books to answer your question. So let's do some math. We'll need some notation. Let's have L be the lengths of the books and put the origin at the right end of the table.

Anne: I like putting the origin at the right edge of the table, but that would put the first book with its right edge at $L/2$ and its left edge at $-L/2$. Then we will have to deal with the factor $L/2$ throughout. Why don't we instead have the length of each book be $2L$? That way the right edge of the first book is at L and the left edge is at $-L$.

Toby and Anne are making a conscious transition from a kinetic/intuitive way of thinking about the problem to a more mathematical formulation, one that will add universality and useful abstraction.

Anne: We need a way to calculate the center of gravity of the stack when we know each book's center of gravity.

Toby: (Lifting a book from a nearby shelf) I think I remember but ... Ah, here is a physics book! Let's look up center of gravity. Here we are ... page .... Ah, here's what it says for a one-dimensional arrangement ... (Reading aloud)

> ... the case of n particles, strung out along the x-axis. The location of the center of mass is
>
> $$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{M}$$
>
> The symbols $m_1, m_2, \ldots, m_n$ denote the n masses with centers of gravity $x_1, x_2, \ldots, x_n$, respectively. The symbol M denotes the mass of the total system; that is, $M = m_1 + m_2 + \cdots + m_n$. The collective center of gravity of the entire system is denoted by x_{cm}.

(No longer reading but speaking) For the stack, the top book is book 1, the second book is book 2 and so on to the bottom book. The origin at 0. We want the center of gravity of the stack to be at the edge of the table. So ... $x_{cm} = 0$. Since the books have the same mass, I get

$$0 = x_1 + x_2 + \cdots + x_n$$

We now zero in on Anne and Toby's calculations. In the following, the numbers $w_1, w_2, \ldots, w_n$ denote the respective overhangs of the n books for $1, 2, \ldots, n$. The sum of all overhangs is going to tell us how far the top book arches out over the edge of the table.

For a stack of just one book, we see that $w_1 = L$, as in Figure 2(a).

For two books, initially as in the first diagram in Figure 2(b), we can see that the first book counting from the top has center of gravity $x_1 = 0$, while the second book has center of gravity $x_2 = -L$. The overhang of the first book is $w_1 = L$. When we shift the stack of two books to the right, reaching a tipping point as in the second diagram in Figure 2(b), the centers of gravity of the books increase by w_2, the second overhang, so that $x_1 = w_2$, $x_2 = -L + w_2$. Thus, the center of gravity for the stack of two books is $0 = w_2 + (-L + w_2)$. Solving for w_2, we have $w_2 = L/2$. This in turn gives us $x_1 = L/2$, $x_2 = -L/2$. Since $w_1 = L$, $w_2 = L/2$, the total overhang of a stack of two books is $w_1 + w_2 = L(1 + 1/2)$.

For three books, initially as in the first diagram in Figure 3, the center of gravity of the first, second, and third books (counting from the top) are $x_1 = L/2$, $x_2 = -L/2$, and $x_3 = -L$, respectively, with overhangs $w_1 = L$ and $w_2 = L/2$ for the first two. Shifting the stack of three books to the tipping point as in the second diagram in Figure 3, the new centers of gravity are $x_1 = L/2 + w_3$, $x_2 = -L/2 + w_3$, $x_3 = -L + w_3$, where w_3 is the overhang for the third book. Therefore we have $0 = (x_1 + x_2 + x_3) = -L + 3w_3$. Solving for w_3, we get $w_3 = L/3$. Since $w_1 = L$, $w_2 = L/2$, $w_3 = L/3$, the total overhang of the stack of three books is $w_1 + w_2 + w_3 = L(1 + 1/2 + 1/3)$.

Moving from the simple to the general ....

Toby: Good! The total overhang is $L(1 + 1/2 + 1/3 + \cdots + 1/n)$ ... uh ... n is the number of books.

Anne: So, we *can* answer my question! We *can* figure out how many books it would take for the top book to be suspended above the floor. Wow!

Anne realizes the significance of this result and the power of generalization, a sign of a maturing scientist.

Toby: Yes! Let's see ... We need a total overhang that is greater than the length of a book. Umh ... we need $L(1 + 1/2 + 1/3 + \cdots + 1/n) \geq 2L$. And canceling L, we get $1/2 + 1/3 + \cdots + 1/n \geq 1$. Then, (Calculating) $1/2 + 1/3 + 1/4 \approx 1.08$. We are good with only four books!

Anne: Ahhh .... We almost had it! One more book would have done it. (She quickly builds the stack of four books with the top book situated completely above the floor.)

Toby: But, now we can answer how far out we can stack the books. Remember! Nothing in our derivation limits the number of books.

Do you remember Dr. Bradford talking about $1+1/2+1/3+\ldots$? He emphasized it a lot. He even got excited about it! We have something like that, but ours stops at n steps. We have $1+1/2+1/3+\cdots+1/n$.

Anne: What's the difference?

Toby: Well, Dr. Bradford said that $1+1/2+1/3+\ldots$ is actually infinitely large, whatever that means ... (His voice trails off) And, I can see that $1+1/2+1/3+\cdots+1/n$ stops at n, and $1+1/2+1/3+\ldots$ doesn't actually stop at any n.

Anne: (She has a lightbulb moment.) Oh! I know what it's like. When someone stands on a railroad track where the ground is real flat and seems to go on forever like in West Texas, you get the feeling that the tracks are coming together, but, you actually know they never meet. So, there is a difference, the tracks never meet ... they just seem to meet.

Toby: (After some reflection) I remember that when he tried to convince us that $1+1/2+1/3+\ldots$ was infinitely large, he did something rather neat, he grouped terms in an interesting way.

Toby writes the following on a piece of paper for Anne to see:

$$\begin{array}{rcl}
1+\frac{1}{2} & > & \frac{1}{2} \\
\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4} & = & \frac{1}{2} \\
\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} & = & \frac{1}{2} \\
\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16} & > & \frac{1}{2} \\
\vdots & &
\end{array}$$

Anne: I remember now. It didn't make sense to me at the time.

Anne then writes the following for Toby:

$$\begin{aligned}
w_1 + w_2 &> L \cdot \frac{1}{2} \\
w_1 + w_2 + w_3 + w_4 &> L \cdot 1 \\
w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8 &> L \cdot \frac{3}{2} \\
&\vdots
\end{aligned}$$

Anne: For stacks of two books, four books, eight books and so on, the total overhang follows a pattern.

Toby: (Thinking out loud) Like, for stacks of 2^n books, the total overhang is greater than $Ln/2$.

Anne: (With excitement) So ... whatever distance I pick, and if I had enough books, I could have created a stack that would have arched over the floor to that amount.

In other words, if d is the distance we want to reach, then we can find an n large enough so that $Ln/2 > d$.

Toby: (Laughing) We could stack them to the moon!

Anne: Get real! I don't think so. (Sarcastically) You need to take some more physics!

Toby: Do you want to tell Dr. Bradford what we found out? Or wait, maybe we should go to lunch? I'm starving!

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After lunch, Anne and Toby are discussing their discoveries with Dr. Bradford. We overhear their discussion after it has begun.

Toby: So ... the problem we described to you is not new?

Dr. Bradford: That's correct. But, you did the work yourself. That is what matters. You learned much more about the harmonic series than I was able to explain in class. You were blazing your own trail rather than following a guide. If you have ever been on a hike in a national park, you might remember that you had a hard time finding your way back to base camp although the guide knew the way back. On the contrary, if you had been your own guide, you would know how to get back to the base camp or ... (laughing) ... you would be lost. After all, creativity is a high-risk enterprise.

Dr. Bradford lets them down easily, underlining the importance of what they have done.

Anne: If I understand you, $1 + 1/2 + 1/3 + \dots$ is the harmonic series and $1+1/2+1/3+\cdots+1/n$ for $n = 1, 2, 3, \dots$ are approximating sums of the harmonic series. And we say the harmonic series diverges because the approximating sums don't converge?

Dr. Bradford: Yes! So do you both understand?

Anne: (Looking at Toby) We're good! Was the harmonic series discovered by one person or several people?

Dr. Bradford: That is a very good question! When Toby telephoned me to see if I would be available for an office visit, he told me what you both had done. So I reminded myself of some the facts concerning the overhang problem and the harmonic series. The harmonic series was first studied by Nicole Oresme in the fourteenth century, and for some reason, not considered until Pietro Mengoli, Johann Bernoulli, and Jacob Bernoulli in the seventeenth century [1]. It is like much of mathematics. Many people make contributions over a long period of time. The history of π is a great example. Check out the book "A History of π" [2] for a thorough development of π. Even calculus was created or invented by many, many people, from the ancient Greeks to the time of Isaac Newton and Gottfried Leibniz in the seventeenth century and beyond, through the nineteenth century when many theoretical questions were answered. An excellent source is "The Historical Development of the Calculus" [6].

Anne: Why the use of the word "harmonic"? Sounds like music but this is math.

Toby: I know that one! I looked it up when Dr. Bradford talked about harmonic series in class the other day. It comes from the study of overtones. That's another name for harmonics in music. These are the wavelengths of the overtones of a vibrating string, which are 1/2, 1/3, 1/4, and so on of the string's fundamental wavelength. (See Figure 4.)

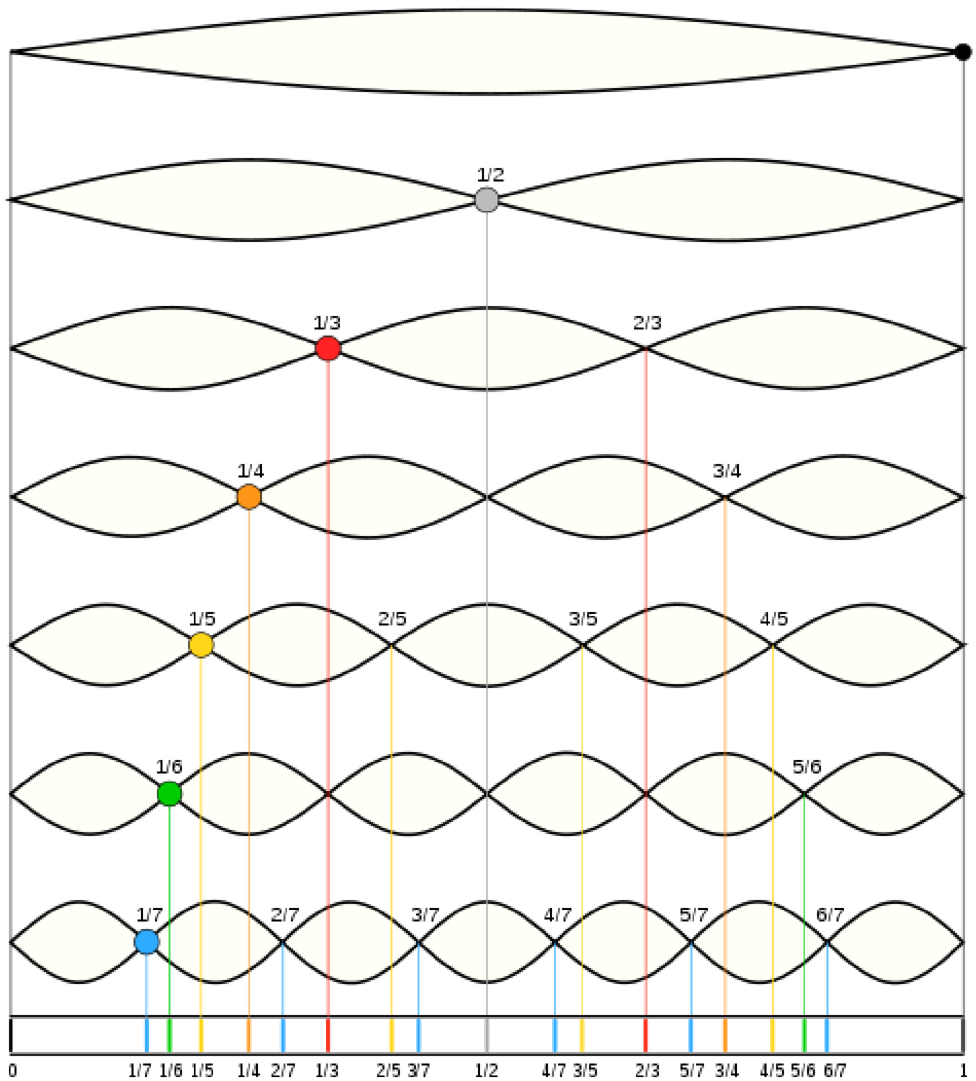

Figure 4: Nodes of a vibrating string (from Wikimedia Commons), available at `https://en.wikipedia.org/wiki/File:Moodswingerscale.svg`, accessed July 18, 2013.

Dr. Bradford: (To Anne but nodding to Toby) Just to add to what Toby has said, if you want an easy but extensive explanation of harmonics with some very interesting piano exercises, you can just go ahead and study *Perspectives in Mathematics* by David Penney [9], starting on page 70. And here is a historical fun fact about harmonic sequences: Especially in the Baroque period, architects used them to establish harmonic relationships between interior and exterior architectural details of churches and palaces.

Anne: That's very interesting! Thank you so much for your time and new ideas and ... encouragement. We have a new perspective on mathematics. It is more than a tool but is more like an art. I like that!

Toby: I agree! I'm glad I'm in mathematics.

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On this note, we end, with the following quote from [10]:

> "*The mathematician is an artist whose medium is the mind and whose creations are ideas.*"

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