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Teaching the Complex Numbers: What History and Philosophy of Mathematics Suggest

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Synopsis

The narrative about the nineteenth century favored by many philosophers of mathematics strongly influenced by either logic or algebra, is that geometric intuition led real and complex analysis astray until Cauchy and Kronecker in one sense and Dedekind in another guided mathematicians out of the labyrinth through the arithmetization of analysis. Yet the use of geometry in most cases in nineteenth century mathematics was *not* misleading and was often key to important developments. Thus the geometrization of complex numbers was essential to their acceptance and to the development of complex analysis; geometry provided the canonical examples that led to the formulation of group theory; and geometry, transformed by Riemann, lay at the heart of topology, which in turn transformed much of modern mathematics. Using complex numbers as my case study, I argue that the best way to teach students mathematics is through a repertoire of modes of representation, which is also the best way to make mathematical discoveries.

1. The Introduction of Complex Numbers

High school students, like Leibniz and other mathematicians of the early modern era, are often puzzled by the complex numbers. What could we possibly mean by $\sqrt{-1}$? Why shouldn't we worry that the use of such a paradoxical concept might not tempt us into the pursuit of nonsense? Historically, the existence and usefulness of complex numbers were not widely accepted until their geometric interpretation around 1800, which was formulated at roughly the same time by Caspar Wessel, the Abbé Buée, Jean-Robert Argand, and the great mathematician Carl Friedrich Gauss.[1] Gauss published a memoir about the geometric interpretation of complex numbers in 1832, which launched its wide acceptance in the mathematical world, aided also by the work of Augustin Louis Cauchy and Niels Henrik Abel [7].

In the geometric interpretation, every complex number is identified with an ordered pair of real numbers, (x, y)—which may also be written $x + iy$—and thus identified with a point on the Euclidean plane. The notation of Cartesian coordinates for points on the plane (x, y) suggests itself here naturally, and the number $z = x + iy$ is mapped onto the point $P = (x, y)$ with the real component x of z as abscissa, and the imaginary component y as ordinate [8, Ch.6]. However, the notation of polar coordinates is a more fruitful way of writing complex numbers under this geometric interpretation, where r is the nonnegative length of the segment joining (x, y) to 0, and θ is the angle from the x-axis to this segment. We call $r = |z|$ the *absolute value* or *norm* or *modulus* of the complex number z and θ its *argument*. Indeed, this was Argand's mode of presenting the complex numbers. The geometric interpretation, using this notation, immediately illuminates and is illuminated by Abraham de Moivre's formula (1730), because if we represent a complex number in this way:

$$z = |z|(\cos\theta + i \sin\theta),$$

and if we know from de Moivre's formula that:

$$(\cos\theta + i \sin\theta)^n = \cos(n\theta) + i \sin(n\theta),$$

then we know that the absolute value of a product of complex numbers is the product of the absolute values of the factors, and the argument is the sum of the arguments of the factors. Geometrically this means that complex multiplication corresponds to a dilation followed by a rotation.

And the same insight holds for Euler's formula (1748),

$$e^{i\theta} = \cos\theta + i \sin\theta,$$

from which de Moivre's formula can be derived, as can the beautiful truth we call Euler's identity,

$$e^{i\pi} + 1 = 0,$$

[1] Leonhard Euler invented the notation $i = \sqrt{-1}$ late in his life, but it was Gauss's use of i in his *Disquisitiones arithmeticae* in 1801, which resulted in its widespread adoption.

which in my experience never fails to enchant students of mathematics, at whatever level of study.[2] Euler's identity is also a special case of another general identity that states that the complex n^{th} roots of unity, for any n, add up to zero:

$$\sum_{k=0}^{n-1} e^{2\pi ik/n} = 0.$$

Students are delighted to discover that complex n^{th} roots of unity can be found using trigonometry, and indeed geometry, because the complex n^{th} roots of unity are the vertices of a regular polygon of n sides inscribed in the unit circle, $|z| = 1$ on the complex plane [11].

Thus the testimony of history is that we should introduce students to the complex numbers as they were introduced to the world of mathematicians between the mid-16$^{\text{th}}$ century (when the Italians Ludovico Ferrari, Geronimo Cardan, Niccolo Tartaglia, and Rafael Bombelli made important discoveries about the algebra they had inherited from medieval Latin and Arabic texts) and the mid-19$^{\text{th}}$ century (when complex analysis, the theory of functions of a complex variable, flourished).[3] In this period, we find a suite of modes of representation offered for the complex numbers, and when the geometric interpretation is offered, mathematical research on the complex numbers explodes. So we have an analogy, between historical sequence and pedagogical sequence: but how shall we understand its significance? Here we might turn to the philosophers and the classroom teachers. In what follows, I am going to present a pedagogical "case study" by David Egan on a website that offers advice about how to teach the complex numbers to high school students. Then, since I am a philosopher of mathematics, I will elaborate and refine his suggestions with some philosophical interpretation of my own.

2. Imaginative Education

To find my case study, I went to the website of the Imaginative Education Research Group (`http://ierg.net`, accessed most recently on January 2, 2013) where I found a wealth of pedagogical materials, generated by a group

[2] A reader poll in the *Mathematical Intelligencer* in 1990 named it as the most beautiful theorem in mathematics [15].

[3] Reuben Hersh earlier made a similar suggestion in this Journal; see [10].

of researchers at Simon Fraser University, in Vancouver, Canada, among whom Kieran Egan is perhaps the most visible. He is the author of *The Educated Mind: How Cognitive Tools Shape Our Understanding* [3], *Thinking Outside the Box* [4], and *Learning in Depth* [5]. His way of thinking about education is shaped in part by the doctrines of Lev Vygotsky, whose work among Anglophone educators has recently enjoyed a resurgence, attested in the *Cambridge Companion to Vygotstky* [1]; it is also influenced by the writings of Ralph Waldo Emerson and John Dewey.[4]

Here is one of their descriptions of the project:

> Established in 2001, the Imaginative Education Research Group in the Faculty of Education at Simon Fraser University is dedicated to improving the quality of education by providing a conceptual framework, information, and practical materials designed to stimulate the imagination of teachers and learners. We aim to show how imaginative education can be implemented in everyday classrooms and to provide the resources that will support its routine achievement.
>
> Connecting the child's imagination with the world is the key to much successful teaching and learning. That connection is the focus of our work. We want nothing less than to make the learning experiences of all children in all schools more interesting, meaningful, and imaginatively engaging. By developing teachers' and students' imaginations, we believe we can transform the experience of schooling, and help students become more knowledgeable, and more creative in their thinking.[5]

What does this research group mean by "imagination," a notoriously indeterminate member of the collection of faculties, as one looks back through the history of philosophy? The opening description on the website says that the group intends to "build later forms of understanding on intellectual skills that are common in children in [traditional] cultures, such as story-telling,

[4]I might mention in this regard the well-received book *The Gleam of Light: Moral Perfectionism and Education in Dewey and Emerson* co-authored by Naoko Saito (Kyoto University) and Stanley Cavell, written in the tradition of the Harvard neo-Pragmatists Israel Scheffler, Hilary Putnam and Cavell [13].

[5]http://ierg.net/about/aims.html, accessed on January 2, 2013.

metaphor generation and recognition, image formation from spoken words, and so on. In the methods of teaching we have developed, such skills play a foundational role, and engage such students in learning more energetically than is common with more traditional methods."[6] In a further elaboration, written by Claudia Ruitenberg (formerly a graduate student at Simon Fraser, now Associate Professor of Educational Studies at the University of British Columbia), we find a critique of more common methods. According to her, one of the strengths of this method, "is that it corresponds more closely to how people acquire lasting understandings of the world. Many ideas about education rely on notions of storage and retrieval (or "banking"), where the main challenge for the learner lies in mentally storing as much correct information as possible, and then being able to retrieve that information when needed. Education is also sometimes thought of as an assembly-line process, in which the main challenge for the learner lies in the progressive accumulation of pieces of knowledge and skills."[7] In sum, the research group is seeking to promote educational methods that are not modeled on the assembly line or warehouse; the topics that recur in their writings are developmental topics: somatic, mythic, romantic, philosophic and ironic understanding. The idea is that the ascent up these levels of understanding should not abandon but maintain and integrate the modes of education used on the lower levels. (See [12]).

This sketch of a pedagogical method made sense to me because one of my children is dyslexic. Although he is very intelligent, I noticed early on that he wasn't reading books as I expected he would given his keen curiosity about the world around him. During a sabbatical year in Paris that my family spent in 2004–2005, at the suggestion of my friend Cinda Agnew Musters, he worked with the educator Carol Nelson, using the Davis Method [2]. After our return to Pennsylvania, he worked with tutors at the Masonic Learning Center in State College, Pennsylvania, one of many nationwide. All of these Centers use the Orton-Gillingham Method [6]. Both the Davis Method and the Orton-Gillingham Method use pedagogical approaches that pull against the linear, symbolic learning methods employed in most public schools. Dyslexic students are typically visual, haptic, and spatial in their understanding of the world; they are often characterized as "intuitive" and

[6] `http://ierg.net/about/aims.html`, accessed on January 2, 2013.
[7] `http://ierg.net/about/whatis.html`, accessed on January 2, 2013.

"multi-sensory." They have trouble processing linear strings of words and punctuation, and abstract words that are not associated with pictures. Both the Davis and the Orton-Gillingham methods are multi-sensory, using visual, auditory, kinaesthetic and tactile approaches.

At a meeting of the Masonic Learning Center that I recently attended, the director Marsha Landis noted that perhaps one in five children suffer from dyslexia. If that is the case, then dyslexia isn't a limited problem for special needs children, but a global problem that affects a high proportion of the population. I myself can't spell very well; I'm left-handed; and I have always preferred geometry and topology to algebra and number theory. Like Cinda Musters and Carol Nelson, Marsha Landis has often observed that the methods that work for dyslexic children could (and should) probably be generalized and brought into every classroom. In sum, I recognize the methods recommended and tested in case studies by the Imaginative Education Research Group as in some respects similar to those employed by the researchers and teachers who have so successfully helped my dyslexic child.

3. Pedagogical Case Study

On the website of the Imaginative Education Research Group, under Teacher Resources, there are lesson plans for a variety of scientific and mathematical topics, including differential calculus, decimalization, infinity, angles, and complex numbers. The unit *Complex Numbers*, by David Egan, is aimed at 16-20 year old students, and lasts 2-3 weeks.[8] In Section 1, it identifies powerful underlying ideas: "It is very difficult to wrap the mind around the idea of imaginary numbers. Despite this bafflement, imaginary numbers fit quite sensibly into a system of complex numbers." Egan adds: "The experience of learning about complex numbers reinforces the tremendous power of abstract thinking, and the mathematical tools that facilitate it." That is, our earlier "intuitive," or "visual" grasp of numbers must be revised and extended by the abstract tools of mathematical thinking. One of these tools is the Argand diagram; if we express complex numbers in the form $z = a + bi$, Egan notes, students will discover that the real numbers are actually exceptional; they are "the exceptional set of cases where $b = 0$."

[8] http://ierg.net/lessonplans/unit_plan.php?id=23, accessed January 2, 2013.

The teacher can also use this notation to introduce the Fundamental Theorem of Algebra: the complex numbers allow us, finally, to find ourselves in an algebraically closed number system.

Section 2 is entitled "Organizing the content into a theoretic structure;" it is the most philosophical and so is where I am most critical of the author, for reasons I explain in the final section of this essay. Egan draws a distinction between a "mathematical" and an "intuitive" approach to numbers, and adds that an intuitive approach depends heavily on visualization and conceptualization in terms of concrete examples. Thus, we understand a negative number intuitively when we visualize it as the left-hand side of the real number line (left of zero), and conceptualize it as a bank account. By contrast, our arithmetic methods for multiplying large numbers we might call "blind" conceptualization: though we can't "see" what it means to multiply $435,678$ by $963,271$ we can easily carry out the calculation on paper, using the wonderful positional method bequeathed to us by Indian and Arabic mathematicians, and we trust the results even though we don't have the "safety net" of intuition to check them.

In Section 2.2, concerned with organizing the unit, Egan asks: "What meta-narrative provides a clear overall structure to the lesson or unit?" Here the teacher asks the students to wrestle with the problem of how to understand five-dimensional Euclidean space; to see finally that they cannot visualize it; and then to resort to expressing it as $\mathbb{R}^5$, with points tagged by five real coordinates (v, w, x, y, z). They discover that although intuition has failed them, this notation will allow them to formulate and solve problems about the geometry of five-dimensional Euclidean space. And so by analogy with complex numbers. The romance of this narrative, Egan surmises, is that mathematics can bravely venture where intuition fears to tread. And here he brings in the historical narrative: "A survey of the history of complex numbers shows the strong, and sometimes furious, opposition with which the idea of complex numbers was met ... Many prominent mathematicians refused to accept complex numbers, and they only became widely accepted in the nineteenth century."

Section 3 is entitled, "Developing the tools to analyze the theoretical structure." Here students learn to express complex numbers in the form $z = a + bi$, and then learn the rules for their addition, subtraction, multiplication, and division.

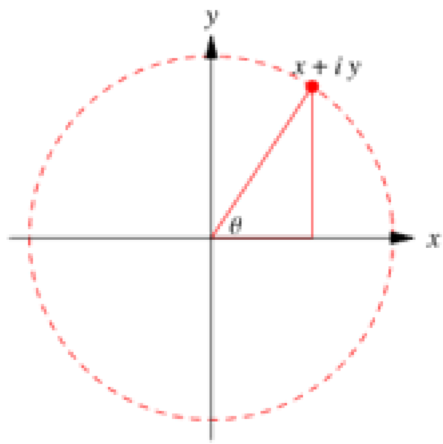

Figure 1: The Argand diagram displaying the point $z = a + bi$ on the complex plane.

Egan observes that the Argand diagram (see Figure 1) makes complex numbers easier to visualize, and also provides a demonstration of the power of abstract, mathematical thinking: the one-dimensional number line has become a part of a two-dimensional plane. (He adds that this is like the moment in a chess game, when a pawn is promoted to a queen.) The diagram can then be used to introduce the expression of complex numbers in polar coordinates, which might then lead to an introduction of their use in mechanics. This train of thought continues into Section 4, which continues the historical narrative by pointing out that Gauss, at the age of 21, proved the Fundamental Theorem of Algebra: every algebraic equation has a solution in the set of complex numbers! This is also a good moment to point towards the world of complex numbers expressed as vectors using polar coordinates (using some linear algebra and trigonometry), and perhaps also to the world of functions of a complex variable.

Egan concludes, in the last three sections, that the point of the Complex Numbers unit is to encourage students to be bold, to reach out beyond their own "intuitive" comfort zone to go beyond restrictions and limitations, "to navigate this fantastic world with the mathematical tools they have acquired." And he adds that the romantic approach works nicely with adolescents, who as we all know are keenly interested in voyages of exploration, like Odysseus and Robert Louis Stevenson. Thus too students can recognize retrospectively that the natural numbers, the integers, the rational numbers, the algebraic numbers, the real numbers, and the complex numbers form a surprising but inevitable sequence, and that the complex numbers provide striking closure for that development.

4. Philosophical Reflections

Having criticized at length elsewhere the use of the term "intuition" in the writings of Descartes, Kant, and Brouwer [8, Ch. 2.1 and Ch. 9.2], I will not be shy about criticizing its use in this lesson plan, but I will also be careful about giving my reasons. Twentieth century mathematics was dominated by abstract algebra, and twentieth century philosophy of mathematics was dominated by logic. Thus it has become commonplace to identify algebra and logic (and by association arithmetic) with mathematical reason, and geometry with intuition. The narrative about the nineteenth century that many philosophers of mathematics favor is that geometric intuition led real and complex analysis astray into confusion and contradiction until Cauchy and Kronecker in one sense and Dedekind in another guided mathematicians out of the labyrinth through the arithmetization of analysis. While there is some truth to this particular myth, the other side of the story is that the use of geometry in most cases in nineteenth century mathematics was *not* misleading and in many cases it was the key to important developments. As we have seen, the geometrization of complex numbers was essential to their acceptance and to the development of complex analysis; geometry provided the canonical examples that led to the formulation of group theory; and geometry, revolutionized by Riemann, lies at the heart of topology, which from the end of the nineteenth century throughout the twentieth century transformed much of modern mathematics.

In my recent book, *Representation and Productive Ambiguity in Mathematics and the Sciences* [8], I admit that in certain cases the demand for "purity of method" and the restriction of notation to one kind (the "ontological parsimony" beloved by many philosophers, some mathematicians, and some educators) may be helpful for mathematical research and pedagogy; in words the mathematician proposes, let's see how much we can produce or how far we can proceed with very limited means, and then think about what this reveals. However, in many cases, and indeed I believe in the most fruitful cases, we find mathematicians juxtaposing and even superimposing a variety of notations or more generally "modes of representation." That is, we find them multiplying rather than restricting their "paper tools," as Ursula Klein calls them [9].

In order to counter and clarify the various uses of the term "intuition" as opposed to reason, I prefer to use the terms "iconic" and "symbolic," borrowed from the American philosopher C. S. Peirce. Some mathematical modes of representation are iconic, that is, they picture and resemble what they picture; others are symbolic and represent by convention, "blindly" and without much resemblance. In many cases, problems in mathematics are most successfully understood, addressed and solved when the problematic things that give rise to them are represented by a consortium of modes of representation, some iconic and some symbolic. Both kinds do important conceptual work: symbols typically help to analyze and distinguish, and icons help to unify and integrate. We need to do both at the same time in order to identify, reformulate and solve problems.

Thus, I would revise Egan's lesson plan. As I see it, the reason why the geometric interpretation of complex numbers moved mathematical research forward historically and why it aids students pedagogically is because it gives us a repertoire of modes of representation that can be used in concert to understand what complex numbers are and how to use them. Students are *not* leaving behind the timid formulation of i as the square root of -1, or as the solution to the equation $x^2 + 1 = 0$ given in the original context of an algebra of arithmetic transformed by its use in analytic geometry. They are using it *together with* the Argand diagram, which is neither more nor less "intuitive" and "mathematical" than the algebraic representation. But it is certainly more iconic and spatial, whereas the algebraic representation is more "blind" and symbolic. The Argand diagram naturally suggests two different symbolic formulations, one in Cartesian and one in polar coordinates; the latter immediately brings in the notation of trigonometry and the transcendental functions, originally so foreign to analytic geometry, that underlie it. This combination of modes of representation, by the way, reveals our old friend the circle in an entirely new way, as the home and factory of the sine and cosine functions. All of mechanics, we might say, lies folded up in the circle.

Thus, contrary to how Egan characterizes the situation in Section 2, I would re-write the romance of complex numbers. What we find historically is the addition of an important iconic, geometric representation of complex numbers to the existing algebraic representation, which in turn suggests a trigonometric representation that lends itself to mechanics. In the end, we'll have at least four modes of representation on the page, to help us think through problems! We could call this a kind of laboratory work, investigating mathematical things on the "combinatorial space" of the page (as Jean Cavaillès called it) with "paper tools" [14]. So what we are teaching students is not how to leave behind the "intuitive" (whatever that is) for the "mathematical," but rather how to profit from and think together a new range of representations, all of them mathematical, some of them iconic and some of them symbolic, in order to investigate complex numbers and complex functions more effectively. The pedagogical point is still innovation, critical thinking, and originality, and the goal is teaching students how to make conceptual breakthroughs. But under this framework, I would argue, we are guided by a more accurate reading of history, and by philosophical ideas that do not promote algebra and logic at the expense of geometry.

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