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Cover Page Footnote

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How Can Mathematics Students Learn to Play?[1]

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Synopsis

When we teach mathematics, we strive to teach students to think like mathematicians. In this paper we discuss one particular mathematical habit of mind that students do not naturally display. More specifically our study of voting patterns in data collected from classroom voting questions indicates that the undergraduate students who were in the classes using these questions did not understand the significance of counterexamples to statements, or lacked the ability to construct them, or both. Searching for counterexamples to disprove statements is a natural habit of mind for professional mathematicians. In this paper we give examples, and make some recommendations. We believe that if our students get used to routinely seeking out counterexamples, as they play with various mathematical ideas, they may also end up enjoying their mathematical experiences more.

As mathematics faculty, we are different from the majority of our students. Sometimes those differences are clear and well-known to us, and other times we do not even realize how different we are. In this paper we discuss our findings regarding a difference we had not been aware of.

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College mathematics faculty teach and do research, and the connections between the two can be very important. As teachers, we are constantly challenged to find new ways to engage our students and to improve our teaching methods. As researchers, we are constantly exploring new mathematical statements, ruling out those that are false and seeking to establish the truth of those we think may become theorems. One of the goals of a mathematics teacher is to pass on the joy of exploring unfamiliar mathematical statements to our students. One aspect of mathematics that we may not be emphasizing enough in our classrooms is providing students with sufficient opportunities to explore and enjoy mathematics by wrestling with unfamiliar mathematical statements.

Before reading on, ask yourself this question: You are given a mathematical statement and asked to determine if it is a true statement or a false statement. Are you likely to be more confident in your answer if you report true or if you report false?

As part of the Project MathVote National Science Foundation grant, we have collected a large number of responses by students to true/false questions in which they were asked if a mathematical statement was true or false and in which they had an opportunity to rate themselves as very confident or not very confident. We found some interesting results in this data that highlight an important way in which we as professional mathematicians differ from our students. We asked the questions during class strictly as learning tools, not as assessment tools, and no grade was attached to the results. An example of one of these questions is Question 1 below.

Question 1.
True or False: If $f'(x) = g'(x)$ then $f(x) = g(x)$.

(a) True, and I am very confident.
(b) True, but I am not very confident.
(c) False, but I am not very confident.
(d) False, and I am very confident.

All of the true/false questions we asked have these four options, so we automatically learned about student confidence levels as they answered these questions. We have data for 385 (non-distinct) questions. This data was recorded by ten faculty members teaching thirty-two distinct sections at seven different institutions, including one community college. The courses covered include liberal arts mathematics, business mathematics, precalculus, calculus, multivariable calculus, linear algebra, differential equations, and analysis; although most of the sections are calculus.

When we use these questions in class, the usual process is to present a question, ask students to discuss the question in small groups, and then ask students to "vote" electronically (with a "clicker") to select the correct answer. Students cannot see how their peers are voting while the vote is open. After the voting period is closed, the instructor shows the class the tabulated voting results and then leads a class-wide discussion on the question. Ideally members of the class raise and successfully resolve the key mathematical issues inherent in the question. Some instructors modify this technique slightly and do two rounds of voting before the class-wide discussion: for the first round, students work on the question individually, then after the first vote, which may or may not be shown to the students, students discuss in small groups and vote again. (For more on classroom voting see, for example, [1]. For more on student confidence and voting, see [2].)

Overall, we find that students are generally quite confident. The average percentage voting "very confident" is 73.2%. We note that students are slightly more confident than they are correct, with an average of 69.7% voting correctly.

As mathematicians, however, the result we find most interesting is tied to whether students voted true or false on the question. Because a counterexample can often be quickly constructed and definitively demonstrates the falseness of a statement, we believe that most mathematicians would be more confident saying a statement is false than saying it is true, when asked to decide within a relatively short timeframe. In fact, when we gave a presentation on this topic at the 2013 Joint Mathematics Meetings, nearly all of the 30 or so members of the audience said they would be more confident voting false than true when we described the situation. However, students voting results do not mirror this predilection. Regardless of the actual truth of the statement, when students voted true, an average of 75.2% voted "very confident;" whereas, when they voted false, only 70.9% voted "very confident." Note that false statements slightly outnumbered true statements in the set of questions this data is from, with the correct answer being "true" for 184 questions and "false" for 201, so student confidence in answers of "true" should not be due to that "usually" being the right answer.

These results are fairly robust across different courses and different levels of courses. Fifteen of the thirty-one sections had voting results recorded for at least ten true/false questions, and we looked at the data for these sections separately. While numbers varied from section to section, in all but three of the fifteen sections students voted more confidently when voting true than when voting false. There was no discernible unifying characteristic in the three sections where the results were reversed: two were calculus sections and one was linear algebra. In one of the calculus sections students voted substantially more confidently when voting false (80% are confident given they voted false versus 60% are confident given then voted true), but in this section correct answers of false were twice as common as correct answers of true. We do note that the one section of analysis stood out as having the lowest student confidence, with students voting confidently just 57.6% of the time, even though on average 70.0% of the students voted correctly.

These results point to an important difference between the mindsets of students in mathematics classes and professional mathematicians, and thus provide us with an opportunity to think about how to bridge this gap. Whereas most professional mathematicians approach new statements with skepticism and will likely search for a quick counterexample before attempting a proof, students tend to be more trusting of mathematical statements. They also tend to have significantly lower standards for "proof," and they often struggle to find counterexamples or do not even think to look for one.

As an example, let us return to the voting results from Question 1, which are given in Table 1.

	a. T w/conf.	b. T not conf.	c. F not conf.	d. F w/conf.
Section 1	12%	21%	8%	50%
Section 2	8%	12%	33%	46%
Section 2 revote	0%	0%	8%	92%

Table 1: Voting results for Question 1 from two different class sections. Section 2 voted twice on this question: once individually without discussion and again after peer discussion. The correct answer is false.

We see that there is some variation, but overall students tend to vote correctly and they mostly vote with confidence. However, even here where the results look rather good, the class discussion following the vote shows that most students do not have counterexamples to support their vote. They do usually have a good understanding that two functions could have the same slopes everywhere yet not be the same functions. Leading students to describe the actual relationship between two such different functions takes more work, and more urging is needed before a student in the class will volunteer actual functions f and g that demonstrate this statement is false.

While Question 1 comes from a first calculus class where students likely have not had much experience with finding counterexamples, we find the same phenomenon in later courses such as linear algebra and multivariable calculus. In Questions 2 and 3, below, we see students are more confident in their answers of "true" than in their answers of "false," even though in both cases the correct answer is "false." Question 2, from linear algebra, deals with commutativity in matrix multiplication, which is something students ought to be very alert to by the time this question is asked, but the issue arises in a new situation, being paired with matrix inversion.

Question 2.
True or False: Suppose that A, B, and C are square matrices, and $CA = B$, and A is invertible. This means that $C = A^{-1}B$.

Properly skeptical students ought to be able to construct a counterexample quickly, and any student who actively tries to construct a counterexample should succeed. Many other students will think the statement is plausible based on their years of experience with real-number arithmetic and will not look further. Table 2 displays the results of the voting on this question.

	a. T w/conf.	b. T not conf.	c. F not conf.	d. F w/conf.
Section 1	23%	27%	35%	15%
Section 2	30%	30%	20%	20%

Table 2: Voting results for Question 2 from two different class sections. The correct answer is false.

In Question 3, asked in multivariable calculus classes, students need to think about special cases to find the counterexample of three collinear points.

Question 3.
True or False: Any three points in 3-space determine a unique plane.

This emphasizes the important concept that in order for a statement to be true, it must be true in all cases, not just in "many" cases. Experienced mathematicians naturally explore the "boundary" conditions inherent in a premise, but students do not naturally exhibit this habit. More experience with types of special cases may help students develop the instinct necessary to find counterexamples to many other types of statements as well. Table 3 displays the results of the voting on this question.

	a. T w/conf.	b. T not conf.	c. F not conf.	d. F w/conf.
Section 1	38%	0%	19%	43%
Section 2	50%	15%	7%	27%
Section 3	29%	24%	29%	19%
Section 4	65%	5%	15%	15%

Table 3: Voting results for Question 3 from four different class sections. The correct answer is false.

Each of the three questions we have showcased here has instructive points for us to consider as mathematics instructors. In Question 1, developing a counterexample is very straightforward as soon as one explicitly decides to seek a counterexample. Question 2 requires some more imagination and time spent working to construct a counterexample, but student effort in this direction will be rewarded. Finally, Question 3 highlights the role "boundary" or "degenerate" conditions can play as a source of counterexamples for overly broad statements. We believe that part of the path to becoming a sophisticated mathematical thinker is developing the skills to dispense with all three of these questions as a matter of routine.

We need to find places at all levels of the mathematics curriculum to allow students to explore mathematics, to ask their own questions, to pose their own conjectures, and then to search for counterexamples or proofs of these conjectures. That, after all, is what professional mathematicians do, and that is the mindset we all hope to pass along to our students. At the very least, we should make a habit of asking students to construct counterexamples, developing and nurturing that skill early in their undergraduate careers. Not only will such experiences enhance students' mathematical thinking, they may even have fun along the way! We conclude with a challenge to each of us: find ways to integrate into each of our courses more opportunities for students to engage in mathematical play.

References

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[2] Ann Stewart, Christopher Storm, and Lahna VonEpps, "Analyzing Student Confidence in Classroom Voting with Multiple Choice Questions," PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, Volume 23, Issue 8 (2013), pages 718 - 732.