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The Triex: Explore, Extract, Explain

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Often as I walk into my classroom and run through what I will do, a recurrent image haunts me. I think of that cliche: Chinese must be an easy language since in China ,so many children can speak it. I know that if those same children had been born here and wereto study Chinese in high school or college, many would flunk out and give up. This image reminds me that students who fail with my approach might do very well under a radically different regimen.

For this reason I am open to alternatives to the standard lecture approach, where the focus is too much on the teacher. I would like to provide an environment in mathematics analogous to that which helps Chinese children learn Chinese: pernaps the Moore method is the way, or semester-long projects, or greater one-to-one contact. But with the boundary conditions that I face, I cannot turn to any of them.

Instead, over the past few years I have been gradually developing a modest alternative, which I will describe even though I am sure it has been often proposed and just as often been disregarded. I know that education is like death or love -- the important things that can be said about it have already been said. One need only browse through old volumes of the *Journal of the National Education Association* to be convinced of this. For instance, in its 1935 volume we find the report of an experiment in which pupils did not begin arithmetic until the sixth grade [1]. It turns out that within one year they caught up with pupils who had a three-year head start. If such a major discovery can vanish with scarcely a trace, I am sure that my proposal, couched in different terms, lies similarly abandoned in the archives.

My suggestion is "humanistic" in that I believe a humanistic education develops a student's ability to read, write, and analyze -- in short, to think. Of course, such an education should also develop an awareness of the origins of civilizations, East and West, and a love of classical music and literature, but I would be content if it just produced a critical self reliance.

However, no college catalog that I've read places reading, writing, or thinking at the core of its curriculum. Instead it lists courses and their topics, for instance, Linear Algebra: vector space, base, dimension, linear transformation, eigenvalue. The textbooks also reflect this inversion of emphasis. By the time we prepare our first lecture of the semester, most of us -- no matter how vehemently we have cried, "This is not a trade school," and, "Facts are secondary," -- lose sight of our real goal and rush to get through the syllabus. As surely as the debased coin displaces the good coin, routine problems drive out significant problems. By a significant problem I mean one that gives students a chance to explore on their own, to develop self confidence and independence, to carry on, so to speak, a miniature research project, and to write up their conclusions in complete sentences.

I am not going to propose a vast reform. I will describe only what I have been doing, timidly at first but recently more boldly. When I presented my ideas at one conference [3], the older participants said, "Old hat," but the younger ones asked such questions as: How do you make up this type of problem? How many problems do you give? How much time do you allow? Do they count in the grade? If so, how?

I propose that we offer our classes what I call "triex" problems; "triex" stands for, "Explore, Extract, Explain." Such problems do not begin with, "show that," "prove that," or "verify that." Instead, they begin with an opportunity for experimenting. The experiments should suggest a plausible conjecture to most students, even though at first glance the answer should not be at all evident. The conjecture should be easy enough for many of the students to prove. Even students who do not complete the third step of a triex are at least primed to appreciate the explanation when given by the instructor or by another student.

Such an exercise puts the emphasis on exploring, extracting, explaining, writing. Therefore the exercise need not relate directly to the course in which the student is enrolled. For instance, in the second or third semester of calculus, it may be drawn from the first semester. The focus is on process, not on fact; the center of responsibility moves from the teacher to the student. In [2] and [3] I presented several examples. Now I will describe three more in some detail to make the idea of the triex more concrete.

> **Example 1**: Let $y = f(x)$ be a nondecreasing function defined on $[0,1]$, with $f(0) = 0$ and $f(1) = 1$. Let R be the region below the graph of f and above the x axis. How low can the centroid of R be? How high? How far to the left? How far to the right?

Note first that the student cannot immediately guess the answers. However, there are accessible experiments, for instance, testing the curves $y = x^n$ or step functions. The first part, exploring, is not hard, though students must get used to accepting this responsibility. (If students get stuck, a hint may get them out of a rut.) The second stage is "extract." (It turns out that x is between 1/2 and 1 and that y is between 0 and 1/2.) The final step, "explain," involves only a symmetry argument. (As a follow-up one could ask whether the centroid of R lies in R.)

> **Example 2:** Diocles, in the year 190 B.C., in the book *On Burning Mirrors*, studied the reflecting property of a spherical surface that subtends an angle of 60°. When this surface is aimed at the sun, the rays of light arrive parallel to the axis, bounce off the inner surface of the sphere, and pass through the axis. How much of the axis is illuminated by the reflected rays?

The solution involves nothing more than trigonometry or, perhaps l'Hopital's rule (depending on how the problem is solved). As a follow-up, which requires the derivative, one could ask, "Describe the variation in the amount of light that strikes in the vicinity of each point of the illuminated part of the axis "

The next example is appropriate in an elementary discrete mathematics course, for it requires an induction or the use of binomial coefficients.

> **Example 3:** How many ways can you list the integers 1, 2, . . ., n such that each integer after the first one you list differs by 1 from an integer that you have already listed? (For n = 5, 32415 is one such list.)

This exercise satisfies the triex criteria: the answer is not immediately obvious; exploration through examples is feasible; the resulting conjecture is simple; the proof is not difficult and its write-up requires exposition, not just a string of equations.

I may require that a triex be turned in at the next meeting or perhaps in a week. If I am not satisfied with the solution or a student is stuck, I will comment on the paper and return it for further work. There may be a class discussion of Step 1 to catch errors which were interfering with Step 2. How often I assign such problems depends on the size of the class and the time I have to read the papers. These problems are separate from the regular homework which is read by an undergraduate.

Often a standard exercise can easily be reworded to become a triex problem. Consider the exercise, "Show that for any odd integer n, the number $n^2 - 1$ is a multiple of 8." As it stands, Steps 1 and 2 of a triex are missing. Such an exercise minimizes the involvement and responsibility of the student. It alienates by insinuating that mathematics is discovered by an elite and is merely checked by the masses. However, that same problem, rephrased, easily turns into this triex: "What is the largest fixed integer that divides $n^2 - 1$ for all odd integers n?" Clearly the three steps are now present.

Even the simple exercise, "Prove that $x + 1$ divides $x^n + 1$ for every positive odd integer n," can be transformed to a triex, namely, "For which positive integers n does $x + 1$ divide $x^n + 1$?" This triex, in turn, generalizes to, "For which positive integers m and n does $x^m + 1$ divide $x^n + 1$?" and to, "For which positive integers m and n does $x^m - 1$ divide $x^n - 1$?"

A triex creates the environment of a miniature research problem, whether applied or theoretical. It puts more responsibility on the student. I expect (but have not tried to prove) that it develops self reliance and self esteem. It certainly exploits a key feature of mathematics, which such disciplines as physics and history lack: all the cards can be laid on the table -- the student need not depend on facts transmitted by an authority. The use of the triex may reduce the alienation and passivity which develop through years spent on plug-in problems.

The triex is one of my responses to that image of little children speaking fluent Chinese. Through it I try to place process above fact. I suggest that more teachers try a few triexes in class in order to become familiar with them and their implications.

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Noesis

Lee Goldstein

Emication of thought is not love,
Because it has no exteriority;
Yet whatever is muted willfully
Has a countenance;
Nay, the autoptic ----- relativistic range of things
Can be a beauteous species,
If thought can be transmuted,
Even as in a mirror,
By law, and homologically into strings.