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The Symbolic and Mathematical Influence of Diophantus's *Arithmetica*

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Synopsis

Though it was written in Greek in a center of ancient Greek learning, Diophantus's *Arithmetica* is a curious synthesis of Greek, Egyptian, and Mesopotamian mathematics. It was not only one of the first purely number-theoretic and algebraic texts, but the first to use the blend of rhetorical and symbolic exposition known as syncopated mathematics. The text was influential in the development of Arabic algebra and European number theory and notation, and its development of the theory of indeterminate, or Diophantine, equations inspired modern work in both abstract algebra and computer science. We present, in this article, a selection of problems from the *Arithmetica*, which have been rewritten for ease of reading, and consider Diophantus's advancements in mathematics and mathematical notation in the context of ancient Greek mathematics. In particular, we examine Diophantus's use of syncopated mathematics, most notably his use of generic solutions that present an algorithm for solving an entire class of equations through the application of that algorithm to a single representational example, and how these techniques suggest a more extensive use of concrete examples when approaching modern mathematics.

1. Introduction

Diophantus is best known today for being the namesake of Diophantine equations, a topic he was one of the first to study systematically. Although it was once considered one of the pillars of ancient Greek mathematics, Diophantus's principal work, the *Arithmetica*, is rarely read today, eschewed in favor of texts by Euclid and Archimedes. This is somewhat understandable, as the Greek approach to geometry much more closely resembles the way we teach mathematics today than do the algebraic problems in the *Arithmetica*. (Interested readers unfamiliar with the *Arithmetica* may wish to skim through §§3.3-3.4 to get a rough sense of its flavor before continuing.)

I believe this neglect is undeserved. The *Arithmetica* had a resounding historical effect on algebra and number theory, and it is difficult to understand the development of notation without considering how Diophantus wrote mathematics. It is true that the mathematical content of the *Arithmetica* seems inconsequential and banal to most modern readers, an attitude only partially due to the changes in mathematical thinking between antiquity and the present. Diophantus solves some computationally difficult problems, but does nothing that we today would regard as brilliant. No problem in the *Arithmetica* is especially useful or theoretically significant, although the text has inspired several deep mathematical problems. Even to a modern-day specialist in Diophantine equations, the text is primarily a historical curiosity. Yet I find reading the *Arithmetica* an exciting and rewarding experience. Diophantus's method of presenting his solutions is unfamiliar, but it should not be. The *Arithmetica* exemplifies the idea of understanding mathematics through examples and concrete experimentation, rather than limiting oneself to the most general and abstract case possible.

Despite his contributions to mathematics, we know very little about Diophantus's life. It is known that he lived in Alexandria, the center of learning in all disciplines in the Greek world, and by examining the works that he references and the works that reference the *Arithmetica*, we can date him absolutely to between 150 BC and 280 AD [9]. It is usually believed that his work fell towards the end of this range, because of an eleventh-century letter stating that another mathematician had dedicated a text to "his friend Diophantus" in 270 AD [7]. Although we lack information about Diophantus's life, we can deduce his age upon his death from an algebraic problem that was purported to have labeled his tombstone:

> His boyhood lasted $\frac{1}{6}$ of his life; his beard grew after $\frac{1}{12}$ more; after $\frac{1}{7}$ more he married, and his son was born five years later; the son lived to half his father's age, and the father died four years after his son [8].

By solving the equation

$$x = \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4$$

it is easy to conclude that Diophantus lived to the age of eighty-four.

Diophantus's only truly significant mathematical work is the *Arithmetica*, a text that treats the subject of finding solutions to indeterminate, or Diophantine, equations in two and three unknowns. We currently possess six of the original thirteen books, as well as four more books in Arabic that are attributed to Diophantus and overlap somewhat with the other six [2]. Arabic mathematicians recovered the *Arithmetica* from the AD 641 sack of Alexandria, and through them the text influenced both Arabic and eventually European mathematics. Perhaps most notably, the *Arithmetica* inspired some of Fermat's greatest number theory, which lay the foundation for the works of Euler and Gauss.

Diophantus also wrote a treatise titled *On Polygonal Numbers*, part of which survives. This text is in a traditional Greek style, and treats the topic of polygonal numbers in a geometric, instead of an algebraic, manner. We have a fragment of a proof in which Diophantus attempts to count the number of ways that a given integer can be expressed as a polygonal number, as well as an introduction and several smaller propositions. Diophantus gives complete proofs of five general statements in *On Polygonal Numbers* and states two general rules without example or proof [8].

In the *Arithmetica*, Diophantus references propositions from a text called the *Porisms* (roughly meaning *Lemmas*) which is lost and whose author is unknown [8]. A representative proposition from the *Porisms* is that the difference of any two cubes can also be expressed as the sum of two cubes [15]. Some scholars have attributed the *Porisms* to Diophantus, even placing it within the *Arithmetica*, but the current consensus is that it was most likely a separate text by another author [2].

In many ways, the *Arithmetica* is an unusual text in the context of Greek mathematics. It abandons the theorem-proof style of earlier Greek works, such as those by Euclid and Archimedes, and focuses on algebraic and number-theoretic problems instead of geometry. Thus it is more akin to the concrete and computational works of the Egyptians and Mesopotamians. Yet the text introduces radical innovations in mathematical symbology and presentation. The presentation of Egyptian and Mesopotamian mathematics was algorithmic and highly rhetorical. A scribe writing mathematics in one of these civilizations might need a paragraph to describe a problem we could write with a couple of short equations. For instance, here is a problem found in the Mesopotamian city of Larsa:

> Take the length and width of a rectangle. I have multiplied the length and width, thus obtaining the area. Then I added to the area the excess of the length over the width: result 183. I have added the length and the width: result 27. Find the length, width, and area [10].

In modern notation, letting x stand for the length and y the width, this problem is simply to find x and y given

$$\begin{aligned} &xy + x - y = 183 \\ &x + y = 27. \end{aligned}$$

By using symbols for unknowns and for small powers of rational numbers, Diophantus wrote solutions in a style that is similar in many ways to the symbolic notation familiar to the modern reader. The main obstacle to mastering the *Arithmetica* remains to be the non-proof-based format and the trial-and-error techniques Diophantus uses to solve Diophantine equations.

The problems in §§ 3.3-3.4 are selected from the *Arithmetica* both for their particular mathematical interest and for displaying Diophantus's methods. I have rewritten the solutions given in Thomas L. Heath's translation of Diophantus's works, originally published in 1885. Heath's solutions are a more literal translation of the Greek, and retain much of the original grammatical structure, though Heath adds elucidating annotations and modernizes much of the notation. However, due to the age of Heath's translation, the difficulty of reading a literal translation of ancient Greek mathematics, and the unconventional style of the *Arithmetica*, Heath's solutions may be laborious for a modern reader to follow. I hope that my paraphrases will be easier to read while retaining Diophantus's mathematical ideas and much of the idiosyncrasy resulting from his notation and methods of solution. The statements of the problems are given exactly as Heath translates them, for they illuminate how Diophantus thought about algebra and the relationships between quantities. I also provide a symbolic statement of the problem that should be more readily understood by the modern reader.

2. Diophantus's notation

Diophantus's notation, which involves such strings of symbols as

$$K^{\tilde{\nu}}\bar{\alpha}\Delta^{\tilde{\nu}}\overline{\iota\gamma}\varsigma\bar{\epsilon}\mathring{M}\bar{\beta}$$

may intimidate modern mathematicians. In my experience, when mathematics students and teachers see such expressions for the first time, their reactions range from confusion to utter disgust. Yet hidden behind a foreign numeral system and unconventional order is a way of writing mathematics practically isomorphic to our own, and it is possible to begin reading and even manipulating algebraic expressions in Diophantus's notation in just a few minutes.

2.1. Symbols for numbers

Diophantus's symbols for integers were in standard Greek alphabetical notation. The integers from 1 to 10 were expressed by using the first ten letters of the ancient Greek alphabet, from α to ι, with a marking superscript. Hence $1 = \bar{\alpha},\ 2 = \bar{\beta}$, etc. Larger integers were written in a slightly modified base-10 system, using later letters of the alphabet for larger powers of 10 and integer multiples of those powers. In this system, Greek numerals could be written more compactly than in the number systems used in Egypt and Mesopotamia, but the system was still difficult to use, especially since there was no symbol for 0 [10].

Diophantus wrote fractions in several different ways depending on their forms, and the various manuscripts on Diophantus express the more complex fractions in a variety of ways. Unit fractions were expressed by writing the symbol for the number in the denominator and adding a sign, often $''$. For instance, as $\bar{\gamma} = 3$, $\gamma'' = \frac{1}{3}$ [3]. Diophantus used a special symbol, which resembled a script ω, for $\frac{2}{3}$, which is notable as that number was the only non-unit fraction used in ancient Egyptian mathematics [10, 3].

2.2. Symbols for unknowns

Diophantus used symbols for unknowns and powers of unknowns up to sixth powers. He had a single symbol for an unknown, ς. He describes this quantity as having an "undefined number of units" and referred to it as

"$\acute{o}$ $\alpha\rho\iota\theta\mu o\varsigma$," meaning "the number" [8].[1] Being able to use only a single symbol at a time to represent an unknown often resulted in expositions that seem unusual to the modern reader; for instance, see problem **I 20** below. While Diophantus only had a single name for an unknown he was solving for, he could refer to multiple unknowns at once by names such as "first unknown," "second unknown," etc. When solving for any of these unknowns, he would begin using ς instead.

Diophantus defined powers of unknowns not as a power of the unknown, but as an unknown with the property of being that power of a positive rational number; accordingly he refers to each power as a "species" of number [8]. For a square number or $\delta\grave{\upsilon}\nu\alpha\mu\iota\varsigma$, he used the sign $\Delta^{\tilde{\upsilon}}$. A cube or $\kappa\grave{\upsilon}\beta o\varsigma$ was abbreviated $K^{\tilde{\upsilon}}$. The signs for the other powers up to six were made by combining these two symbols in accordance with the basic rules of multiplication. Hence a fourth power, or $\delta\upsilon\nu\alpha\mu\iota\varsigma\delta\grave{\upsilon}\nu\alpha\mu\iota\varsigma$, was a $\Delta^{\tilde{\upsilon}}\Delta$, a fifth power was $\Delta K^{\tilde{\upsilon}}$, and a sixth $K^{\tilde{\upsilon}}K$ [8]. For the reciprocals of powers, what we think of today as negative powers, he added a χ: for instance, the reciprocal of $K^{\tilde{\upsilon}}$ was written $K^{\tilde{\upsilon}\chi}$ [1].

Diophantus also had a separate symbol to represent a constant term (or zeroth power) in an equation, $\mathring{M}$, which derives from the work $\mu o\nu\acute{\alpha}\delta\epsilon\varsigma$, meaning "units." [8]. Like the symbols for unknowns, $\mathring{M}$ was always accompanied by a coefficient.

Diophantus's treatment of unknowns in both statements and solutions is especially relevant in light of recent mathematics education research concerning mathematical conceptions of the idea of "variable." Because Diophantus's solutions are more truly algorithms than proofs, the statements of problems are written with the expectation that readers will apply his results to solve concrete, specific problems rather than engage in more theoretic pursuits. For instance, problem **II 8**, below, starts with the assumption of a value for a single, concrete number (in this case a square). The entire statement and solution is written under the assumption that we have picked a square and are solving the problem based on that very square. This differs from the

[1] There is some speculation as to why Diophantus chose this letter to denote his unknown. The most common theory is that ς is the last letter of its name, $\alpha\rho\iota\theta\mu o\varsigma$ [8]. However, it has also been suggested that ς was the only letter not used in the ancient Greek numeral system [3]. In fact, $\bar{\varsigma}$ sometimes denoted 6, but $\overline{\sigma\tau}$ was also used for that number at various points in history.

more rigorous but less concrete treatment of the modern approach to the same problem, which would begin with a number a^2 and proceed from there with the fullest possible generality.

Once Diophantus has begun to solve a problem, his idea of variable becomes slightly more flexible. Using ς and the various symbols for powers, he is able to name and manipulate numbers without knowing their values. Nonetheless, these numbers do not possess the ability to change value, and their value is predetermined by Diophantus's choice of given values. Under the language of the 3 Uses of Variable Model, discussed in [19], Diophantus works only with unknowns. Thus he provides an interesting extreme to the way algebra students today may struggle assigning meaning to variables, particularly parameters or other variables whose values can change [18]. Diophantus's given values are an obvious example of a number that is concrete, but can be symbolized and thus become a parameter. Unknowns solved for later in the problem are related variables, tied to initial parameters by the steps of Diophantus's algorithms.

2.3. Mathematical expressions

In addition to Greek numerals and unknowns, Diophantus possessed symbols for subtraction ($\pitchfork$) and equality (ι^σ, a shortened form of the word for "equals," $'\grave{\iota}\sigma o\zeta$) [11]. Because he had no symbol for +, he represented addition by concatenation. Hence the equation we would write as $x^3 - 2x^2 + 10x - 1 = 5$ was expressed by Diophantus as

$$K^{\tilde{v}}\bar{\alpha}\varsigma\bar{\iota} \pitchfork \Delta^{\tilde{v}}\bar{\beta}\mathring{M}\bar{\alpha}\iota^\sigma\mathring{M}\bar{\epsilon}.$$

Note that the positive terms $K^{\tilde{v}}\bar{\alpha}$ and $\varsigma\bar{\iota}$ appear together, followed by $\pitchfork$ and then the negative terms $\Delta^{\tilde{v}}\bar{\beta}$ and $\mathring{M}\bar{\alpha}$. Diophantus used no sign for multiplication, hence he would only state the result of a multiplication.

The *Arithmetica* used symbols to a far greater extent than any earlier surviving work, and marks an important transition between rhetorical methods of writing mathematics and our current highly symbolic notation. Its style has often been referred to as "syncopated" or "syncoptic" mathematics (see for instance [15]). Syncopated algebra is an intermediate stage in the development of mathematical symbolism, and uses some symbols to abbreviate otherwise lengthy expressions [3]. Diophantus's notation is a major advance over the previous rhetorical manner of exposition, and was a primary influence of later, progressively more symbolic notation (see §4.2). However, Diophantus's syncopated notation is not as easy to manipulate as modern symbolic notation, since it inherits its overarching structure from rhetorical algebra. For instance, Diophantus allows at most one subtraction in any expression and his conventions for naming unknowns force the user to be careful to avoid confusing two distinct quantities with the same abbreviation.

The translations presented in this paper preserve Diophantus's style, although because modern notation wisely prefers x, y, and z to "first, second, and third unknowns", these translations are more compact, more symbol-dense, and hopefully more readable than the original.

3. The *Arithmetica*

3.1. The structure of the Arithmetica

The *Arithmetica* is divided into 189 problems [2]. All of the problems in the first five books are purely algebraic, and the problems in the sixth are algebraic despite a geometric veneer. Unlike expository Egyptian or Babylonian texts, Diophantus phrases problems in purely abstract terms, and does not associate numbers with any real-world objects, (although as is usually the case with Diophantus, there is an exception to the rule: Problem **V 30** deals with the price of measures of wine [8]).

Diophantus begins each problem with the statement of a task: to find a number or a set of numbers satisfying a property. Usually this property is of the form "given a set of numbers, find another set of numbers satisfying a property." He then provides a specific solution to the problem, but this solution can usually be generalized to find an infinite class of solutions. For instance, if the task is to find two numbers with a given relation, he will assign a specific value to one of the numbers and then find the other one from that piece of information. However, if the initial choice of value is changed, Diophantus's method can be repeated to get a new set of solutions. Hence, while Diophantus does not provide explicit demonstrations of how to solve the equation with general parameters, his steps are easily retraced with new parameters. Diophantus also usually makes choices for intermediate values, so that answers can be generalized beyond choosing new given unknowns. In fact, some problems, such as **VI I** below, have no given values, so that the infinite class of solutions comes from choosing new values for the intermediate values Diophantus uses in the solution. In some more complex problems, Diophantus's method can be generalized to an infinite class of solutions, but there are additional classes of solutions that cannot be found using his method [8].

3.2. The preliminary

The introduction of the *Arithmetica* dedicates the text to one Dionysius, who is seeking to learn mathematics, and sets forward Diophantus's mathematical methods and some of his notation. This establishes the didactic tone of the *Arithmetica*, and throughout the text Diophantus suggests that his readers should master the techniques that he presents to solve problems of their own [11]. As we have seen, Diophantus names each power and each negative power up to six, his symbol for an unknown, and his symbols for equality and subtraction. Diophantus also states the concept of $\lambda\epsilon\iota\psi\iota\varsigma$, meaning deficiency, and the rules that:

> deficiency multiplied by deficiency yields availability; deficiency multiplied by availability yields deficiency [2],

which is easily recognized as a statement that the product of two negative numbers is positive, and the product of a negative and a positive number is negative. Diophantus also gives the definition of $\acute{o}\ \alpha\rho\iota\theta\mu o\varsigma$ as a positive rational number. His refusal to accept negative numbers as solutions resembles the approach of the Mesopotamians, who accepted only positive solutions to quadratic equations even when they were aware that those equations had two solutions [10]. Nonetheless, Diophantus is willing to manipulate negative numbers to obtain solutions; for instance, he freely calculates that $(x^2 + 4x + 1) - (2x + 7) = x^2 + 2x - 6$, though he considers the equation $4x + 20 = 4$ to be absurd, since it gives the solution $x = -4$[8]. His use of negative numbers also marks a step away from the geometric perspective of earlier Greek number theory and algebra, in which numbers were manifestations of geometric lengths, areas, and volumes, and were therefore constrained to positivity. Diophantus also uses powers beyond cubes, and freely adds and subtracts terms of different degrees. Such manipulation is natural for an algebraic way of doing mathematics, but does not make sense in a geometric context where adding length to area, for instance, is absurd.

3.3. Selected algebraic problems from the Arithmetica

In the following problems, I denote given values by $a, b, \ldots,$ the final solutions by $x, y, \ldots,$ and intermediate unknowns solved for by Diophantus by $m, n, \ldots.$ Though I employ different symbols for each intermediate unknown, the reader should remember that Diophantus used ς for each of these. And he referred to the final solutions as "the first number," "the second number," and so on. Recall that the problems are stated exactly as they were in the Heath translation [8].

> **I 20.** *To divide a given number into three numbers such that the sum of each extreme and the mean has to the other extreme a given ratio:* $x + y + z = a,\ x + y = bz,\ y + z = cx.$
>
> *Solution.* Let the given number be 100. Let the three numbers we need to find be $x, y,$ and z and let $x+y = 3\cdot z$ and $y+z = 4\cdot x$. (Here x and z are the extremes and y is the mean.) Set $z = m$. Thus $x + y = 3m$ and $100 = x + y + z = 4m$. Hence $m = 25$, and $x + y = 75$. Let x be n. Then $y + z = 4n$, so $5n = 100$ and $y = 20$. The required parts are 20, 55, 25. □

Diophantus needed two unknowns to complete the problem, but possessed only ς. Because the need for m is dropped once n is introduced, he uses $\alpha\rho\iota\theta\mu o\varsigma$ for each in turn. When he refers to the sum of certain of the solutions, he refers to them by their order among the solutions, but he does not need to use ς to refer to them until he needs to solve for their value. The statement $x + y = 3z$ is a given assumption, so x is referred to as the first unknown. However, when the time comes to solve for this unknown, Diophantus refers to it as ς instead.

Sometimes Diophantus poses problems that only have solutions under certain conditions. In these cases, he names these conditions. Remarkably, almost every one of the conditions he names is both necessary and sufficient. Often, these conditions demonstrate surprising number-theoretical knowledge; for instance, Diophantus knew that any integer can be expressed as the sum of four integer squares, a fact which was not proved until 1770 [8]. However, it is very unlikely that Diophantus had any theoretical understanding of these facts beyond repeated trial and error, much less a proof.

> **IV 34.** *To find three numbers such that the product of any two together with the sum of those two makes a given number:* $xy + x + y = a,\ xz + x + z = b,\ yz + y + z = c.$
>
> *Necessary condition: each [of the three given] number[s] must be 1 less than some square.*

Diophantus proceeds by letting the three given numbers be $8 = 3^2 - 1$, $15 = 4^2 - 1$, and $24 = 5^2 - 1$. However, this given condition is excessive. If we let a, b, and c be the given numbers, it is only necessary that $(a+1)(b+1)(c+1)$ be a square. Then the first indeterminate is given by

$$\sqrt{\frac{(b+1)(c+1)}{a+1}} - 1$$

and the others likewise [8]. Even this condition is only necessary to guarantee solutions in $\mathbb{Q}$, as of course there are irrational solutions in $\mathbb{R}$ for any values of a, b, and c.

> **II 8.** *To divide a given square into two squares:* $a^2 = x^2 + y^2$.

> *Solution.* Let the given square be 16, and let one of the required squares be x^2. Then $16 - x^2$ must be a square.
>
> Let $16 - x^2$ be a square of the form $(mx - 4)^2$, where m is any integer and $4 = \sqrt{16}$. For instance, take $m = 2$ and set $(2x - 4)^2 = 16 - x^2$. Then $4x^2 - 16x + 16 = 16 - x^2$, or $5x^2 = 16x$, and thus $x = \frac{16}{5}$. Therefore the required squares are $x^2 = \frac{256}{25} = \left(\frac{16}{5}\right)^2$ and $16 - x^2 = \frac{144}{25} = \left(\frac{12}{5}\right)^2$. □

Often Diophantus assumes an incorrect value for an unknown, then uses the incorrect equation generated by this value to adjust the assumed value and arrive at the correct answer. This technique, known as false position, was commonly used by the Egyptians to solve linear equations, and is a natural method of solving equations in a mathematical culture without symbolic notation [10]. An example is problem 26 of the Ahmes Papyrus, which dates to around 1650 BC:

> Find the quantity so that when its quarter is added to it, it becomes 15.

Using false position, the problem is easily solved without having a concept of "the unknown" that can be manipulated algebraically:

> Suppose the answer is 4. But 4 plus its quarter is $4 + \frac{4}{4} = 5$, not 15. Multiplying 5 by 3 makes it 15, as desired. Therefore, multiplying 4 by 3 gives the answer, 12 [10].

The commonly taught modern method of solving linear equations in one unknown is highly algorithmic and exploits the capabilities of symbolic notation, whereas false position is a quasi-algebraic way to manipulate an equation. Assuming an incorrect value as a solution is anathema to most modern students of mathematics, but I would argue that occasional use of false position before a student is capable of immediately solving a linear equation mentally can both provide another intuitive handle to linear equations and develop skill in accurate estimation. Diophantus uses false position in a more sophisticated way than the Egyptians, for instance in the following problem, in which he uses this technique to solve equations of higher degree.

> **IV 24.** *To divide a given number into two parts such that their product is a cube minus its side: $x + y = a, xy = u^3 - u$ for some u.*
>
> *Solution.* Let the given number be 6, and let the first and second parts be x and $6 - x$. We need $6x - x^2$ to be of the form of a cube minus its side.
>
> Take a cube with side $mx - 1$, say $2x - 1$. Then $(2x - 1)^3 - (2x - 1) = 8x^3 - 12x^2 + 4x$, which we need to be equal to $6x - x^2$. If the coefficient of x were the same on both sides, then we could reduce this to an equation of the form $ax^3 - x^2 = 0$, which can easily be solved by a positive rational x. To be able to do this, we must change the assumed value of m from 2 to another number. The solution to $3m - m = 6$ is 3, therefore let $m = 3$. (Note that 6 was the original given number.)
>
> Then $(3x-1)^3-(3x-1) = 6x-x^2$ or $27x^3-27x^2+6x = 6x-x^2$, which has the solution $x = \frac{26}{27}$.
>
> Thus the two parts are $x = \frac{26}{27}$ and $6 - x = \frac{136}{27}$. □

As the previous problem demonstrates, Diophantus is able to solve linear equations (in this case $3m - m = 6$) but he needs to use false position to derive this linear equation from a more complicated problem. Using modern notation, we could arrive at this linear equation as the key component by expanding $(mx-1)^3-(mx-1)$ and setting its linear coefficient equal to 6. Diophantus's solution also implies considerable experimentation and analysis in the choice of a cube with side $mx-1$. Because much of Diophantus may appear obvious or overcomplicated to the modern reader, it can be hard to tell which problems originally demanded more insight from Diophantus or his predecessors.

In the following problem, Diophantus deals with two simultaneous quadratic equations in a single unknown, which he seeks to make consistent. By manipulating the unknown using false position, he finds a value which makes the problem solvable.

> **IV 15.** *To find three numbers such that the sum of any two multiplied into the third is a given number:* $(x+y)\cdot z=a$*,* $(x+z)\cdot y=b$*, and* $(y+z)\cdot x=c$*.*
>
> *Solution.* Let the numbers be x, y, and z, and let $(x+y)\cdot z=35$, $(x+z)\cdot y=32$, and $(y+z)\cdot x=27$. Let $z=m$. Then $x+y=\frac{35}{m}$. Assume $x=\frac{10}{m}$ and $y=\frac{25}{m}$. Then substituting using the other two equations we have
>
> $$\frac{250}{m^2}+10=27$$
> $$\frac{250}{m^2}+25=32$$
>
> which is not consistent. However, these equations would be consistent if $25-10=32-27=5$. Therefore we need to divide 35 into two parts n and $35-n$ such that $m-(35-m)=5$, and replace 25 and 10 with these parts. The parts are 15 and 20. [Here Diophantus refers to the very first problem of the *Arithmetica*, which is to divide a given number into two parts with a given difference.]
>
> Therefore let $x=15/m$ and $y=20/m$. Then we have the consistent equations
>
> $$\frac{300}{m^2}+15=27$$
> $$\frac{300}{m^2}+20=32$$

We solve the resulting quadratic $12m^2 = 300$ to find $m = 5$. Returning to the equations above, $x = 15/5 = 3$, $y = 20/5 = 4$, and $z = x = 5$. □

The preceding problem illustrates Diophantus's usual technique in solving Diophantine equations: reducing to a known determinate equation that he was able to solve (usually a linear or quadratic equation, or a specialized form of the cubic [8]). To be able to reduce every equation to these few solvable forms necessitated great inventiveness and flexibility from Diophantus, hence it is often said that because every algorithm and method of solution is so distinct, knowing the method of solution to every preceding problem will be no help in following the next. However, many problems involve substitutions or transformations that can be classified in terms of more general methods of algebraic geometry [1]. Diophantus solved many quadratic and cubic Diophantine equations by methods that are equivalent to using secant and tangent lines to find rational points on a curve.

IV 18. *To find two numbers such that the cube of the first added to the second gives a cube, and the square of the second added to the first gives a square:* $x^3 + y = u^3$, $y^2 + x = v^2$ *for some* u *and* v.

Solution. From the first equation, we have that y is a cube minus x^3, say $8 - x^3$. Then $x^6 - 16x^3 + 64 + x =$ a square number $= (x^3 + 8)^2$, say. Then $32x^3 = x$, or $32x^2 = 1$. There would be a rational solution to this equation only if $32 = 4 \cdot 8$ were a square. Therefore we must substitute for 8 a cube number which is a square when multiplied by 4. Hence if the cube is v^3, $4v^3 =$ a square $= 16v^2$, so $v = 4$ is a valid choice. Therefore, if we choose 64 instead of 8, we can let the two numbers we need be x^3 and $64 - x^3$. Using the assumption that the square of the second added to the first is a square, $x^6 - 128x^3 + 4096 + x =$a square$= (x^3 + 64)^2)$, say. Then $256x^3 = x$, so $x = \frac{1}{16}$. Therefore the numbers are $\frac{1}{16}$ and $64 - \left(\frac{1}{16}\right)^3 = \frac{262143}{4096}$. □

Here Diophantus is aware that there is an irrational solution to the equation $32x^2 = 1$, but persists in finding a rational answer. Despite his concept of "number" being more advanced than that of most Greek mathematicians up to his time, his unwillingness to accept irrational solutions is very much in the Greek tradition [12].

The following problem demonstrates the technique Diophantus called $\pi\alpha\rho\iota\acute{o}\tau\eta\tau o\varsigma\ \alpha\gamma\omega\gamma\acute{\eta}$, which is most nearly translated as "the method of approximation by bounds." In the following problem, Diophantus uses this method to find two square numbers whose sum is 13, while each is individually as close as possible to $\frac{13}{2}$.

V 9. *To divide unity into two parts such that, if the same given number be added to either part, the result will be a square:* $x + y = 1$, $a + x = u^2$, $a + y = v^2$.

Necessary condition:[2] *The given number must not be odd and the double of it plus 1 must not be divisible by any prime number which, when increased by 1, is divisible by 4. [In other words, let* a *be the given number and* p *a prime dividing* $2a + 1$*. Then we must have that* $a \not\equiv 1 \mod 2$ *and* $p \not\equiv 3 \mod 4$*.]*

Solution. Let the given number be 6. The problem is equivalent to dividing $13 = 2 \cdot 6 + 1$ into two squares $x^2 + y^2 = 13$ such that both x^2 and y^2 are squares greater than 6, or equivalently, $|x^2 - y^2| < 1$. Then $x^2 - 6$ and $y^2 - 6$ will satisfy the initial conditions of the problem.

Since $\frac{13}{2}$ is not a square, one of the two squares will be larger than the other. To find this square, first find a small square $\frac{1}{m^2}$ such that $\frac{13}{2} + \frac{1}{m^2}$ is a square. Because $\frac{13}{2} + \frac{1}{m^2} = \left(\frac{1}{2}\right)^2 \left(26 + \frac{4}{n^2}\right)$, this is equivalent to finding $n = 2m$ such that $26 + \frac{1}{n^2}$ is a square.

Assume that

$$26 + \frac{1}{n^2} = \left(5 + \frac{1}{n^2}\right)^2 \text{ so that } 26n^2 + 1 = (5n + 1)^2.$$

Solving this quadratic equation[3] gives that $n = 10$ and hence

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[2]In the surviving text of the *Arithmetica*, the statement that the given number must be odd is clear, but the rest of the condition is corrupt. Fermat gave the necessary condition on primes dividing the given number, although it seems unlikely that this would have exactly been Diophantus's original condition, since Diophantus shows no awareness of modular arithmetic beyond parity modulo 2.

[3]The choice of $r = 5$ in the square $(5 + \frac{1}{r})^2$ comes from solving the equation $n = 2r/(26 - r^2)$. Hence to minimize $\frac{1}{n}$ we must have $r \geq 5$, for else $n^2 > 1$.

$\frac{1}{m^2} = \frac{1}{400}$, and $\frac{13}{2} + \frac{1}{400} = \left(\frac{51}{20}\right)^2$.

Therefore, we wish to divide 13 into two parts $x^2 + y^2 = 13$ such that x and y are both as close as possible to $m = \frac{51}{20}$.

Because of the initial conditions on the given number 6, we can write 13 as the sum of two squares $13 = 2^2 + 3^2$. Note that $3 - \frac{51}{20} = \frac{9}{20}$ and $2 - \frac{51}{20} = -\frac{11}{20}$, but we cannot take $x = 3 - \frac{9}{20}$ and $y = 2 + \frac{11}{20}$, for then

$$x^2 + y^2 = \left(\frac{51}{20}\right)^2 + \left(\frac{51}{20}\right)^2 = \frac{2601}{400} + \frac{2601}{400} = \frac{5202}{400} > 13.$$

Therefore, let $x = 3 - 9r$, $y = 2 + 11r$, where r is very nearly $\frac{1}{20}$. Then

$$(3 - 9r)^2 + (2 + 11r)^2 = 13$$

which is a quadratic equation. Solving it, we have $r = \frac{5}{101}$.

Hence $x = 3 - 9r = \frac{257}{101}$ and $y = 2 + 11r = \frac{258}{101}$, and so the answer to the original question is that the numbers are

$$x^2 - 6 = \frac{4843}{10201}$$
$$y^2 - 6 = \frac{5358}{10201}.$$

Indeed, $\frac{4843}{10201} + \frac{5358}{10201} = 1$ and when 6 is added to either number, the result is a square. □

3.4. Selected problems on right-angled triangles from the Arithmetica

The final book of the *Arithmetica* deals with problems related to right-angled triangles. All twenty-four of the problems in the book ask the reader to find a right-angled triangle with rational sides and some given property relating some measurements of the triangle, including the area, hypotenuse, base, and perpendicular (the remaining side of the triangle). Diophantus did not see triangles as purely geometric objects, as they had been considered in much of the previous Greek mathematical tradition. Instead, the variety of interconnected measurements that can be obtained from a right-angled triangle gave him fertile ground for many problems, many of which are among the most difficult in the *Arithmetica*.

Note that Diophantus speaks not of the hypotenuse or the length of the hypotenuse, but of *ο εν τη υποτεινουση*, meaning "that in the hypotenuse" or slightly less literally "that number represented in the hypotenuse." He uses similar terminology for area, other sides, and so on. Diophantus often refers to the "triangle formed by" two numbers or an unknown and a number. By this he means that given positive rational numbers m and n, the triangle formed by m and n is the right triangle with sides given by Euclid's result that

$$m^2 - n^2,\ 2mn,\ m^2 + n^2$$

is a Pythagorean triple. (In the following, the problems are once again stated exactly as they were in the Heath translation [8].)

VI 1. *To find a rational right-angled triangle such that the hypotenuse minus each of the sides gives a cube: $z - x = u^3$ for some u, $z - y = v^3$ for some v, (x, y, z) is a Pythagorean triple.*

Solution. Let the triangle be formed by m and 3. Then the hypotenuse is $m^2 + 9$, the perpendicular is $6m$, and the base is $m^2 - 9$. We want $m^2 + 9 - (m^2 - 9) = 18$ to be a cube, but it is not. Because $18 = 2 \cdot 3^2$, we must replace 3 by n such that $2 \cdot n^2$ is a cube. Let $n = 2$. Therefore, let the triangle be formed from m and 2, so we obtain $(m^2 - 4, 4m, m^2 + 4)$, which satisfies one condition since $(m^2 + 4) - (m^2 - 4) = 8$ which is a cube. To satisfy the other condition we need $m^2 - 4m + 4 = (m - 2)^2$ to be a cube, so let $(m - 2) = 8$. Then $m = 10$ and the triangle is $(40,\ 96,\ 104)$. □

The previous problem supports the claim that Diophantus intended his solutions to demonstrate a general method, rather than find a single answer. If he were content with a single solution, we would expect him to find the simplest solution and move on. In the preceding problem, he easily could have let $m - 2 = 1$, and then have arrived at the smaller solution $(5, 12, 13)$. This trend holds when we examine other problems as well: Diophantus usually chooses given values and intermediate unknowns so that calculations are reasonable but the result is not minimal.

In most of Diophantus's solutions, he works with small integers and relatively simple fractions, a wise choice given his lack of computational devices and a Greek number system not well-suited to dealing with large numbers. Nevertheless, he does tackle some difficult computations in the problems that demand them, such as the following.

> **III 19.** *To find four numbers such that the square of their sum plus or minus any one singly gives a square: find* w, x, y, z *such that the eight numbers*
>
> $$\begin{aligned}&(w+x+y+z)^2 \pm w\\&(w+x+y+z)^2 \pm x\\&(w+x+y+z)^2 \pm y\\&(w+x+y+z)^2 \pm z\end{aligned}$$
>
> *are all squares of rational numbers.*
>
> *Solution.* First, note that for any right-angled triangle with bases m and n and hypotenuse h, $h^2 \pm 2mn$ is a square, for by the Pythagorean theorem $h^2+2mn = (m+n)^2$ and $h^2-2mn = (m-n)^2$. Therefore it is sufficient to find four right-angled triangles [with rational side lengths] with the same hypotenuse. [Then $h^2 = (w+x+y+z)^2$ and $w+x+y+z$ will be twice the product of the bases of each of the four triangles respectively.] But this problem is equivalent to finding a square [the square of the hypotenuse] which can be expressed as the sum of two squares in four ways, and problem **II 8** above gives us a method to express a square as a sum of two squares in an infinite number of ways.
>
> Take two right-angled triangles, say $(3, 4, 5)$ and $(5, 12, 13)$. Then
>
> $$13 \cdot (3, 4, 5) = (39, 52, 65)$$
>
> and
>
> $$5 \cdot (5, 12, 13) = (25, 60, 65)$$
>
> are two right-angled triangles with hypotenuse 65. This gives two divisions of 65 as the sum of two squares, and $65 = 4^2 + 7^2 =$

1^2+8^2 are two more such expressions.[4] By using Euclid's formula for generating Pythagorean triples, we find that the triangle generated from 4 and 7 is $(33, 56, 65)$ and the triangle generated from 1 and 8 is $(16, 63, 65)$. This gives us four expressions of 65^2 as the sum of two squares. Now let $w + x + y + z = 65p$, and because each triangle has hypotenuse 65 and each unknown is twice the product of the bases of a triangle, we have

$$\begin{aligned} w &= 2 \cdot 39 \cdot 52 \cdot p^2 = 4056p^2 \\ x &= 2 \cdot 25 \cdot 60 \cdot p^2 = 3000p^2 \\ y &= 2 \cdot 33 \cdot 56 \cdot p^2 = 3696p^2 \\ z &= 2 \cdot 16 \cdot 63 \cdot p^2 = 2016p^2 \end{aligned}$$

Then by adding the right-hand sides of the previous equations, we have that $65p = (4056 + 3000 + 3696 + 2016)p^2 = 12768p^2$, thus $p = \frac{65}{12768}$. Therefore

$$\begin{aligned} w &= \frac{17136600}{163021824} \\ x &= \frac{12675000}{163021824} \\ y &= \frac{15615600}{163021824} \\ z &= \frac{8517600}{163021824}. \end{aligned} \qquad \square$$

The solution of the following problem is among the most extended in the *Arithmetica*. Diophantus assumes a value, disproves its validity, and finds a valid solution to an equation multiple times in the course of the problem.

VI 14. *To find a right-angled triangle such that its area minus the hypotenuse or minus one of the perpendiculars gives a square:*

[4] Diophantus notes that these latter two "natural" divisions follow from the fact that $65 = 13 \cdot 5 = (1^2 + 2^2)(2^2 + 3^2)$. Indeed, if $k = (\alpha^2 + \beta^2)(\gamma^2 + \delta^2)$, then

$$\begin{aligned} k &= (\alpha\gamma + \beta\delta)^2 - (\alpha\delta - \beta\gamma)^2 \\ &= (\alpha\delta + \beta\gamma)^2 + (\alpha\gamma - \beta\delta)^2 \end{aligned}$$

find x*,* y*,* z *such that* $\frac{xy}{2} - z = u^2$ *for some* u*,* $\frac{xy}{2} - z = v^2$ *for some* v*, and* (x, y, z) *is a Pythagorean triple.*

Solution. Let the triangle be $(3m,\ 4m,\ 5m)$. Then both $6m^2 - 5m$ and $6m^2 - 3m$ must be squares. If we write $6m^2 - 3m$ as a square of the form n^2m^2, we have $m = \frac{3}{6-n^2}$, where $n^2 < 6$. Substituting $m = \frac{3}{6-n^2}$ into $6m^2 - 5m$, we have that the quantity

$$\frac{54}{n^4 - 12n^2 + 36} - \frac{15}{6 - n^2} = \frac{15n^2 - 36}{(n^2 - 6)^2}$$

must be a square, which it is only when $15n^2 - 36$ is a square. But this is impossible, for 15 is not the sum of two squares.[5] Therefore we must change the assumed triangle. If the given triangle is $(3, 4, 5)$, then $15n^2$ is the product of n^2 (where $n^2 <$ the area of the triangle), the hypotenuse, and the perpendicular. Also, 36 is the product of the area, the perpendicular, and the difference between the perpendicular and the hypotenuse. Let the three sides of the triangle be $h,\ b,\ p$ and let $n^2 < 6$. Then we must find $h,\ b,\ p,\ n$ such that

$$n^2hp - \frac{pb}{2} \cdot p(h - p) \tag{1}$$

is a square. Let us form the triangle from two numbers $r,\ s$, so that $p = 2rs$. Then if we divide equation (1) through by the square $(r - s)^2 = h - p$, we must find a square $u^2 = m^2/r - s)^2$ such that

$$u^2hp - \frac{1}{2}pb \cdot p \tag{2}$$

is a square. Substituting through with r and s, we obtain that

$$u^2(r^2 + s^2)2rs - rs(r^2 - s^2)2rs$$

[5]In general, the equation $Ax^2 - c^2 = y^2$ has a solution if and only if A can be written as the sum of two squares. Indeed, if $x, y \in \mathbb{Q}$ are solutions to the equation, we have

$$A = \left(\frac{c}{x}\right)^2 + \left(\frac{y}{x}\right)^2.$$

must be a square. This statement is true if $u^2 = rs$, for the equation then reduces to

$$\begin{aligned} u^2(r^2+s^2)2rs - rs(r^2-s^2)2rs &= \\ 2r^2s^2(r^2+s^2) - 2r^2s^2(r^2-s^2) &= \\ &4r^2s^4 \end{aligned}$$

which is a square. Accordingly, let $r = 1$ and $s = 4$, making $u^2 = 4$ and satisfying equation (2). Then our triangle is $(8, 15, 17)$, and the condition for n becomes that

$$n^2 \cdot 17 \cdot 8 - \frac{8 \cdot 15 \cdot 8 \cdot (17-8)}{2} = 136n^2 - 4320 \tag{3}$$

must be a square. By hypothesis, $n^2 = u^2(r-s)^2 = 3^2 \cdot 1 \cdot 4 = 36$, satisfying equation (3). Because the triangle formed from $(1, 4)$ is $(8, 15, 17)$, the triangle we need is of the form $(8m, 15m, 17m)$. To find m, solve $60m^2 - 8m = 36m^2$, which has the solution $m = \frac{1}{3}$.

Therefore, the required triangle is $\frac{1}{3}(8,\ 15,\ 17) = (\frac{8}{3},\ 5,\ \frac{17}{3})$. Indeed $\frac{20}{3} - \frac{8}{3} = 4$ and $\frac{20}{3} - \frac{17}{3} = 1$. $\square$

By this point I hope that the reader is more comfortable with Diophantus's mathematical style, particularly his use of algorithmic example rather than generalized proof. His algebraic techniques, especially his extensive use of false position, are somewhat less approachable, since they differ so greatly from the methods ingrained into most modern mathematicians. But in most cases, Diophantus's use of specific numbers actually makes his solutions easier to read. The solutions presented above are already filled with unknowns (for instance, the preceding problem has eight.) It can be difficult enough to remember the meanings of each of these many unknowns, whose names only vaguely suggest their purpose, and this task would have been even more challenging for a reader of the original text who had to manage endless "first unknowns," "second unknowns," and widely varying uses of ς. Rational numbers provide guiding lights in this alphabetical sea. While it can be unclear, even in context, what quantity a numeral stands for, the more thoroughly explained exposition of syncopated mathematics, along with the reader's subconscious ability to estimate, can usually fill in this information with little mental exertion from the reader.

Much of current mathematics education research stresses the importance of working with examples to more easily grasp proofs and general results

[20]. In the context of modern mathematics, Diophantus's solutions can be called *generic proofs*, an appellation for a demonstration that applies a proof which holds for a general class of objects to a particular one of those objects [13]. It is argued in [16] and others, and I have found in my own experience, that students gain better understanding and conviction of proofs when they consider a generic proof before being shown a formal demonstration. Diophantus's solutions are perfect examples of generic proofs. Diophantus goes through every necessary step for his reader to find the solution, but does so with a particular arbitrary number with no distinguishing properties – in other words, his proof would work equally well no matter what the initial choice of number. It seems improbable that Diophantus deliberately chose this method of presenting his solutions to make them easier to read, and instead he most likely took it as another facet of the Mesopotamian and Egyptian style that he in many ways inherited. Nevertheless, I hope that the reader has gained some appreciation for this now-unusual exposition, and perhaps even found it easier to follow than the same problems solved in a fully general manner.

4. The Spread of the *Arithmetica*

4.1. Translation and influence in Arabic mathematics

When the Library of Alexandria was destroyed in 641 A.D., Diophantus's works were among those to survive. The *Arithmetica* was first translated into Arabic in the ninth century by Qusta ibn Luqa, and its mathematics and notation significantly affected the development of Arabic mathematics [2, 11].

An especially noteworthy book on algebra is *Al-Fahkri*, written around 1000 A.D. by al-Karaji. The introduction of the *Al-Fahkri* is very similar to Diophantus's preface, although its mathematical depth is greater. For instance, al-Karaji introduces infinitely many positive powers and infinitely many negative powers, where Diophantus needed and used symbols for only six of each sign [2]. Al-Karaji also formulated general rules for multiplying and dividing exponents of the same number, thus demonstrating greater flexibility in manipulating powers of an unknown. He made extensive use of irrational numbers, and saw them as valid arithmetical objects, unlike Diophantus [2]. The body of the *Al-Fahkri* contains all of the Diophantine equations that Diophantus solved and that were translated into Arabic, accompanied by Diophantus's own solutions. In addition, al-Karaji adds other problems on solving indeterminate equations that may have originated in the Egyptian or Arabic tradition of such problems [11].

Arabic mathematics is notable for making algebra an independent subject. The mathematician and poet Omar Khayyám defined it as "the art whose subject is absolute number and measurable magnitudes that are unknown but refer to some known thing that makes it possible to determine them" [2]. However, despite their mathematical advances over Diophantus, Arabic mathematicians continued to write in a literal, not syncopated, notation [17].

4.2. Spread into Europe

The first notable European text inspired by Diophantus was *Algebra*, written by Rafael Bombelli over several decades but first published in 1572 [2]. After Bombelli had written a manuscript of *Algebra*, the mathematician Antonio Maria Pazzi directed him to a manuscript in the Vatican library by a "certain Diophantus." This copy of the *Arithmetica* had a great effect on Bombelli, and he was inspired to translate it "in order to enrich the world with such a remarkable book" [2]. He did not complete the translation, but his own *Algebra* was fundamentally influenced by the *Arithmetica*. Indeed, the *Algebra* includes 143 problems from the *Arithmetica*, many including extended solutions by Bombelli. Just as Arabic mathematicians were spurred on by Diophantus to new levels of generality, so Bombelli innovated in the cause of solving equations. *Algebra* is the first text to deal extensively with complex numbers and to use them to solve algebraic equations [17]. Bombelli also stated general arithmetic rules for working with imaginary numbers, including a result equivalent to what is now known as the *de Moivre formula*, that for an angle θ

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

This and other statements involving trigonometric functions owe much to Arabic mathematicians, who thoroughly developed both the theory and the application of trigonometry. Bombelli was also the first mathematician since Diophantus to use a syncopated style, and the notation of the *Algebra* closely resembles that of the *Arithmetica* [2]. For instance, for $x^6 - 10x^3 + 16 = 0$, Bombelli writes

$$1.\underline{6}\ m.10\ \underline{3}\ \tilde{p}.16\ \textit{eguale a}\ 0.$$

Note the use of Arabic numerals, and that the abbreviations for $-$ and $=$ come from Latin, whereas Diophantus's symbols $\pitchfork$ and ι^σ originated in Greek.

The next major leap forward, in both algebra and symbolism, came from the French mathematician François Viète. Viète was the first mathematician to use symbols for unknowns and given parameters, allowing him to write general formulas and results instead of needing to use specific numbers when any would do. Viète made progress in solving both Diophantine and determinate equations by creative substitutions and by combining algebra and geometry [2]. However, despite his use of irrational and complex numbers, Viète permitted only positive solutions to equations.

4.3. Fermat and Euler's number theory

It is well-known that the French mathematician Pierre Fermat wrote the statement of his famous "Last Theorem" in the margin of his copy of the *Arithmetica*, next to problem **II 8**, given above [8]. In fact, Fermat wrote comments next to many other problems in the *Arithmetica*, which inspired much of his work in number theory. He also validated many of Diophantus's claims that previous mathematicians had not been able to verify [2].

Leonhard Euler wrote a treatise on Diophantine equations which was inspired by the *Arithmetica*. In it, he solved several of Diophantus's problems in a more symbolic way, and found all general solutions instead of focusing on a specific case [8]. He greatly advanced the theory of Diophantine equations by setting down general rules for when many forms are and are not solvable. Euler's expository style, particularly in his letters, may also have been influenced by Diophantus. Particularly notable is his 1742 letter to Goldbach concerning quadratic reciprocity, in which he presented many individual subcases of the law (one of which is completely incorrect) and then two theorems (which he did not prove, but gave many concrete examples of) which resemble the statement of quadratic reciprocity familiar to modern mathematicians. Writing on this letter, Harold Edwards summarizes Euler's exposition thus:

> Euler first states a number of special theorems covering the prime divisors of $a^2 + Nb^2$ (a, b relatively prime) for $N = 1, 2, 3, 5, 7, 11,$

> $13, 17, 19, 6, 10, 14, 15, 21, 35, 30$ before he states the general theorems. This style has the advantage that the reader, far from having to struggle with the meaning of the general theorem, has probably become impatient with the special cases and has already made considerable progress toward guessing what the general theorem will be [5].

Euler's style in this letter resembles Diophantus's exposition in two major ways: first, the statement of theorems as fact without proof (although Euler does add that he desired a proof of these theorems), and second, the extensive use of examples to guide the reader. The statement of quadratic reciprocity is intimidating and nonintuitive. When I was first exposed to quadratic reciprocity, it was stated in terms of Legendre symbols, and I had to process the statement several times to derive significant meaning from it. But when it is presented gradually in terms of subcases with adequate examples, quadratic reciprocity becomes much easier to grasp. More importantly, this method of exposition allows the reader to understand each aspect of the result individually and make deductions throughout the process, in contrast to a theorem-proof approach in which the reader must first grapple with an unfamiliar statement and only then attempt to understand why it is true.

4.4. Hilbert's tenth problem

Over the centuries, many classes of Diophantine equations were generally solved. In 1900, David Hilbert gave a lecture in Paris in which he proposed twenty-three problems to be solved in the new century. The tenth problem was to find a method to solve all Diophantine equations in any number of unknowns [6]. In 1900, this wording was not well-defined, but eventually both Alonzo Church and Alan Turing gave precise definitions of an algorithm. Thus this statement of what was originally called *Arithmetica*, then algebra, and was later approached with geometrical ideas, became a question in logic and computing. The eventual proof of Hilbert's problem was inspired by a computer-theoretical question known as the *halting problem*, which asks if there is a procedure to find if a given mathematically idealized "program" will stop running. One way of thinking about this problem is to consider a set of integers S. If the answer to the halting problem is that there is such a procedure, then for every S there is a machine which takes in an integer and in a finite number of steps returns an answer of whether or not that natural number is in S. Turing proved in 1936 that there is no such procedure.

In 1970, Yuri Matiyasevich proved that for every $S \subset \mathbb{N}$, there exists a Diophantine equation such that a natural number is a solution of the Diophantine equation if and only if it is a member of S [14]. His proof was based on work by Julia Robinson and others, who showed that if a Diophantine equation existed whose solutions grew exponentially in a certain sense, then it would be possible to find an appropriate Diophantine equation for every subset of integers. By finding Diophantine equations whose solutions are related to the Fibonacci numbers, which grow exponentially, Matiyasevich showed that these Diophantine equations exist. But if it were possible to solve every Diophantine equation by a general algorithm, there would also be a procedure that would solve the halting problem for every set. Therefore, because of the impossibility of solving the halting problem, there is no general algorithm for solving Diophantine equations.

The solvability of Diophantine equations has also been related to the consistency of axiomatic systems, and can be used to give an alternative proof of Gödel's theorem of the inconsistency of sufficiently powerful axiomatic systems [6].

5. Conclusion

For the modern reader, the *Arithmetica* is an unusual work, particularly in its exposition. When you begin reading the text, you are met with mathematics that may at first seem uninteresting and not especially readable. Justifying the study of the *Arithmetica* by its historical impact is easier, and it was by examining the legacy of the work that I was able to gain a deeper appreciation for the mathematics itself. The significance and impact of the *Arithmetica* comes not just from its advances in notation, but from a thorough examination of its subject, overlaying a sophisticated understanding of number theory. Through its readers, the *Arithmetica* has inspired many of the ancestors of today's notation, the use of complex numbers to solve polynomial equations, and what was the most famous conjecture of modern mathematics. For me, however, one of its greatest strengths is the aspect of the text that seems most foreign to a modern reader: the concrete generic proofs. After a few problems, reading the *Arithmetica* in this style becomes natural and the easier way to process Diophantus's unusual (to us) mathematics. The *Arithmetica* exemplifies the strengths of doing mathematics not only through proof, but by example, and I hope that it inspires modern students of mathematics to do the same.

References

[1] I.G. Bashmakova, *Diophantus and Diophantine Equations*, volume 20 of *Dolciani Mathematical Expositions*, Mathematical Association of America, Washington DC, 1997.

[2] I.G. Bashmakova and G.S. Smirnova, *The Beginnings and Evolution of Algebra*, volume 23 of *Dolciani Mathematical Expositions*, Mathematical Association of America, Washington DC, 2000.

[3] Florian Cajori, *A History of Mathematical Notations*, Dover Publications, Mineola, NY, 1993.

[4] Douglas M. Campbell and John C. Higgins, editors, *Mathematics*, volume 2, Wadsworth International, Belmont, CA, 1984.

[5] Harold M. Edwards, "Euler and Quadratic Reciprocity," *Mathematics Magazine*, Volume **56** Issue 3 (1983), pages 285–291.

[6] Timothy Gowers, editor, *The Princeton Companion to Mathematics*, Princeton University, Princeton, 2008.

[7] Stephen Hawking, editor, *God Created the Integers*, Running Press Book Publishers, Philadelphia, 2005.

[8] Sir Thomas L. Heath, *Diophantus of Alexandria*, second edition, Cambridge University Press, second edition, Cambridge, 1910.

[9] Simon Hornblower, Antony Spawforth, and Esther Eidinow, editors, *The Oxford Classical Dictionary*, fourth edition, Oxford University Press, Washington, DC, 2012.

[10] George Gheverghese Joseph, *The Crest of the Peacock*, third edition, Princeton University Press, Princeton, 2011.

[11] Victor J. Katz and Karen Hunger Parshall, *Taming the Unknown*, Princeton University Press, Princeton NJ, 2014.

[12] Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, Massachusetts Institute of Technology Press, Cambridge MA, 1968.

[13] John Mason and David Pimm, "Generic Examples: Seeing the General in the Particular," *Educational Studies in Mathematics*, Volume **15** Issue 3 (1984), pages 277–289.

[14] Yuri V. Matiyasevich, *Hilbert's tenth problem*, Massachusetts Institute of Technology Press, Cambridge, MA, 1993.

[15] Ad Meskens, *Travelling Mathematics—The Fate of Diophantos' Arithmetic*, Birkheauser, Basel, 2010.

[16] Tim Rowland, "Generic Proofs in Number Theory," In *Learning and Teaching Number Theory: Research in Cognition and Instructions*, edited by Stephen R. Campbell and Rina Zazkis, pages 157–184. Ablex Publishing, New York NY, 2002.

[17] Jacques Sesiano, *An Introduction to the History of Algebra*, volume 27 of *Mathematical World*, American Mathematical Society, Providence RI, 2009.

[18] Maria Trigueros and Sally Jacobs, "On Developing a Rich Conception of Variable," Chapter 1 in *Making the Connection: Research and Teaching in Undergraduate Mathematics*, edited by Marilyn P. Carson and Chris Rasmussen, pages 3–13, Mathematical Association of America, Washington DC, 2008.

[19] Maria Trigueros and Sonia Ursini, "Starting College Students' Difficulties in Working with Different Kinds of Variable," *Research in Collegiate Mathematics Education*, Volume **5**, CBMS Issues in Mathematics Education, American Mathematical Society, Providence RI, 2003, pages 1–29.

[20] Keith Weber, Mary Porter, and David Housman, "Worked Examples and Concept Example Usage in Understanding Mathematical Concepts and Proofs," Chapter 19 in *Making the Connection: Research and Teaching in Undergraduate Mathematics*, edited by Marilyn P. Carson and Chris Rasmussen, pages 245–252, Mathematical Association of America, Washington DC, 2008.