**Humanistic Mathematics Network Journal**

Issue 14 Article 12

11-1-1996

# Fibonacci Melodies

Robert Lewand
*Goucher College*

# Fibonacci Melodies

*Robert Lewand*
*Goucher College*
*Baltimore, MD 21204*

### INTRODUCTION

One way in which mathematics has informed modern musical composition is through the use of algorithms. To compose algorithmically one begins with a sequence of numbers and maps the terms of the sequence into various musical parameters such as pitch, duration, dynamics and even timbre. Students of Arnold Schoenberg (1874-1951) and Anton von Webern (1883-1945) are credited with first using this composition technique whose incipient stage was known as *serialism* [4, p. 544]. More recently composers have been employing iterated function systems and chaos theory (e.g., [1], [3], [5]) to produce music.

Little is written, however, about the influence which music has had on mathematics. This article describes how the creation of a musical composition suggested a theorem concerning the Fibonacci sequence: $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$

### THE FIBONACCI COMPOSITION

One can create a very simple example of algorithmic music by associating the terms of the Fibonacci sequence with notes on a keyboard. In the example that follows, no attention is paid to timbre, dynamics, or duration; unless otherwise specified, each note is taken to be a quarter note. Of course since there are more terms in the Fibonacci sequence than there are keys on a keyboard, a more reasonable association would map the terms of the Fibonacci sequence modulo $m \in Z$, the set of integers, to the keyboard keys. Although any value of $m \geq 2$ would work, reasonable values include $m = 88$ (a piano has 88 keys) and $m = 8$ (an octave includes 8 notes). Selecting $m = 8$, one can establish the very straightforward correspondence appearing in Table 1 where x is any positive integer.

| x mod 8 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| Note | * | C | D | E | F | G | A | B |

**Table 1**
Correspondence between integers modulo 8 and notes on a keyboard

In this example the symbol * is a wildcard and may be interpreted in any number of ways. Let's agree that the effect of encountering a * in a string of notes is to change the duration of the previous note (if one exists) from a quarter note to a whole note. Another arbitrary decision concerns the octave in which the notes will be played. Again let's let that decision be idiosynchractic, entirely up to the discretion of the composer.

Table 2 reveals the sequence of notes generated in this manner by the first 24 terms of the Fibonacci sequence.

| Term | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Fibonacci Sequence mod 8 | 1 | 1 | 2 | 3 | 5 | 0 | 5 | 5 | 2 | 7 | 1 | 0 |
| Note | C | C | D | E | G | * | G | G | D | B | C | * |
| Term | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| Fibonacci Sequence mod 8 | 1 | 1 | 2 | 3 | 5 | 0 | 5 | 5 | 2 | 7 | 1 | 0 |
| Note | C | C | D | E | G | * | G | G | D | B | C | * |

**Table 2**
Notes Generated by the Fibonacci Sequence mod 8

With our conventions concerning octaves, duration, and the interpretation of the *, and assuming common (4/4) time, the first four measures of this song are depicted in Figure 1.

While Figure 1 reveals a surprisingly mellifluous sequence of notes, Table 2 invites us to examine the cyclic nature of the Fibonacci sequence mod 8. Specifi-

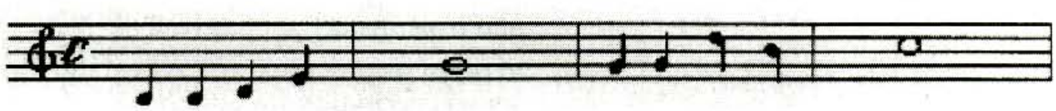


**Figure 1**
The First Four Measures of a Fibonacci Song

cally, with $f(n)$ representing the $n^{th}$ term of the Fibonacci sequence, Table 2 suggests the following conjecture:

(1) $$f(n) \bmod 8 = f(n+12) \bmod 8$$

Before attempting to prove this conjecture, it may be of interest to examine the effect of changing the modulus from 8 to some other numbers.

It is a simple matter to verify that the Fibonacci sequence mod 7 is:
$\{1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, \ldots\}$

and that the Fibonacci sequence mod 6 is:
$\{1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, 5, 5, 4, 3, 1, 4, 5, 3, 2, 5, 1, 0, 1, 1, 2, 3, 5, 2, \ldots\}$

These sequences suggest the following conjectures:

(2) $$f(n) \bmod 7 = f(n+16) \bmod 7$$

(3) $$f(n) \bmod 6 = f(n+24) \bmod 6$$

Since all three conjectures involve the terms $f(n)$ and $f(n+x)$, it may be of interest to examine the relationship between these two expressions.

### PROPERTIES OF THE FIBONACCI SEQUENCE

Recall that the Fibonacci sequence is defined recursively by the equations:

(4)
$$\begin{aligned} &f(1) = 1 \\ &f(2) = 1 \\ &f(k) = f(k-1) + f(k-2), \; for \; k > 2 \end{aligned}$$

Using (4) repeatedly, notice that:

(5)
$$\begin{aligned} f(n+x) &= f(n+x-1) + f(n+x-2) \\ &= 1f(n+x-1) + 1f(n+x-2) \\ &= 2f(n+x-2) + 1f(n+x-3) \\ &= 3f(n+x-3) + 2f(n+x-4) \\ &= 5f(n+x-4) + 3f(n+x-5) \\ &\vdots \end{aligned}$$

The coefficients of the terms on the right are all Fibonacci numbers and so (5) may be written as:

(6)
$$\begin{aligned} f(n+x) &= f(2)f(n+x-1) + f(1)f(n+x-2) \\ &= f(3)f(n+x-1) + f(2)f(n+x-2) \\ &= f(4)f(n+x-2) + f(3)f(n+x-3) \\ &= f(5)f(n+x-3) + f(4)f(n+x-4) \\ &\vdots \end{aligned}$$

Equations (6) suggest the following theorem:

**Theorem 1**:
For $n \geq 2$ and $x \geq 1$, $f(n+x) = f(n)f(x+1) + f(n-1)f(x)$.

This theorem can be readily proved by induction on x [2, p. 289].

**Corollary 1**:
For $x \geq 1$, if *m divides* $f(x)$ and *m divides* $f(x+1)-1$, then $f(n) \bmod m = f(n+x) \bmod m$.

**Proof**:
Given any positive integer $n$, suppose

$f(n+x) \bmod m = k$. Then there is some $c \in Z$ with

(7) $$f(n+x) = cm + k$$

We wish to show that $f(n) \bmod m = k$. Since, by hypothesis, $m$ *divides* $f(x+1)-1$ and $m$ *divides* $f(x)$, there exist $r, s \in Z$ with

(8) $$f(x+1)-1 = rm \quad (i.e., f(x+1) \equiv 1 \bmod m)$$

and

(9) $$f(x) = sm \quad (i.e., f(x) \equiv 0 \bmod m)$$

Case 1:
Assume $n = 1$. Then

$$\begin{aligned} f(x+n) \bmod m &= f(x+1) \bmod m \\ &= 1 \bmod m \quad (\textit{from } (8)) \\ &= f(1) \bmod m \\ &= f(n) \bmod m \end{aligned}$$

Case 2:
Assume $n \geq 2$. Then from Theorem 1

(10) $$\begin{aligned} f(n+x) &= f(n)f(x+1) + f(n-1)f(x) \\ &= f(n) + (f(x+1)-1)f(n) + f(n-1)f(x) \end{aligned}$$

from (7), (8), and (9) we have

$$cm + k = f(n) + (rm)f(n) + f(n-1)(sm)$$

so

$$\begin{aligned} f(n) &= (c - rf(n) - sf(n-1))m + k \\ f(n) &\equiv k \bmod m \end{aligned}$$

Corollary 1 establishes a condition that is sufficient to assure that $f(n) \bmod m = f(n+x) \bmod m$. Expressed in the musical context in which this investigation originated, this corollary asserts that if a song is created using the algorithm described in this paper, and if $m$ *divides* $f(x)$ as well as $f(x+1)-1$, a string of notes so generated will repeat indefinitely, i.e., the song is periodic with period $x$.

A natural question to raise is whether the sufficient condition of Corollary 1 is also necessary. The following corollary answers this question affirmatively.

**Corollary 2**:
If $f(n+x) \bmod m = f(n)$ for fixed positive integers $x$ and $m$ and for all positive integers $m$, then $m$ *divides* $f(x+1)-1$ and $m$ *divides* $f(x)$.

**Proof**:
Choosing $n=1$, the hypothesis implies that $f(1+x) \bmod m = f(1) \bmod m$ for fixed positive integers $x$ and $m$. So, $f(1+x) - f(1) \equiv 0 \bmod m$, or $f(1+x) - 1 \equiv 0 \bmod m$, i.e., $m$ *divides* $f(1+x)-1$.

To prove that $m$ *divides* $f(x)$, begin with (10):

$$f(n+x) = f(n) + (f(x+1)-1)f(n) + f(n-1)f(x)$$

Rewriting, we get:

$$f(n-1)f(x) = f(n+x) - f(n) - f(n)(f(x+1)-1)$$

In particular:

$$f(x) = f(1)f(x) = (f(2+x) - f(2)) - f(2)(f(x+1)-1)$$

Since by hypothesis $f(2+x) - f(2) \equiv 0 \bmod m$, and since $m$ *divides* $f(x+1)-1$, $m$ *divides* $f(x)$.

**CONCLUSIONS**
This article establishes necessary and sufficient conditions for $f(n+x) \bmod m = f(n) \bmod m$ where $f(n)$ is the $n^{th}$ term of the Fibonacci sequence. Musically, the result can be interpreted in terms of when a sequence of notes generated by the Fibonacci sequence is periodic. The novelty of the article lies in its demonstration that the relationship between mathematics and music is a two-way street. Beginning with a

mathematical algorithm involving the Fibonacci sequence and the concept of modularity to compose a musical piece, we were rather unexpectedly led to a result in number theory.

**REFERENCES**

[1] R. Bidlack, Chaotic Systems as Simple (but Complex) Compositional Algorithms, *Computer Music Journal*, 16, (1992) 33-47.

[2] D. Burton, *Elementary Number Theory*, Allyn and Bacon Publishing Company, Boston (1976).

[3] M. Gogins, Iterated Functions Systems Music, *Computer Music Journal*, 15, (1991) 40-48.

[4] G. Hindley (ed.), *Larousse Encyclopedia of Music*, Hamlyn Publishing Group, London, 1971.

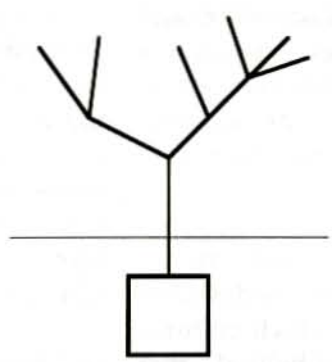
[5] J. Pressing, Non-linear Maps as Generators of Musical Design, *Computer Music Journal*, 12, (1988) 35-46.

# Mathematical Rebuses

*Florentin Smarandache*
*Pima Community College*
*Tucson, AZ 85709*

*A*
*R*
*I*
*T*
*H*
*M*
*E*
*T*
*I*
*C*

= Higher Arithmetic



= Square Root

*S P A*
*C E*

= Compact Space