PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 2, February 2013, Page 549
S 0002-9939(2012)11347-4
Article electronically published on June 14, 2012

ON THE NORM CLOSURE PROBLEM FOR COMPLEX SYMMETRIC OPERATORS

STEPHAN RAMON GARCIA AND DANIEL E. POORE

(Communicated by Marius Junge)

Abstract. We prove that the set of all complex symmetric operators on a separable, infinite-dimensional Hilbert space is not norm closed.

In [2, Sect. 3], it is asked whether the set of all complex symmetric operators on a separable, infinite-dimensional Hilbert space is norm closed. We answer this question in the negative. Let $S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, \ldots)$ denote the unilateral shift on $\ell^2(\mathbb{N})$ and let $\cong$ denote unitary equivalence. Note that

$$T_n \quad = \quad \tfrac{n}{n+1} S \oplus \Big(\bigoplus_{\substack{j=1 \\ j\neq n}}^{\infty} \tfrac{j}{j+1} S\Big) \oplus \Big(\bigoplus_{j=1}^{\infty} \tfrac{j}{j+1} S^*\Big) \quad \cong \quad \bigoplus_{j=1}^{\infty} \tfrac{j}{j+1} (S \oplus S^*)$$

is complex symmetric by [1, Ex. 5]. On the other hand, T_n converges in norm to

$$T \quad = \quad S \oplus \Big(\bigoplus_{j=1}^{\infty} \tfrac{j}{j+1} S\Big) \oplus \Big(\bigoplus_{j=1}^{\infty} \tfrac{j}{j+1} S^*\Big) \quad \cong \quad S \oplus \bigoplus_{j=1}^{\infty} \tfrac{j}{j+1} (S \oplus S^*).$$

Since $\|S^k(1, 0, 0, \ldots)\| = 1$, there is an x so that $\|T^k x\| = 1$ for $k \geq 0$. However,

$$T^* = S^* \oplus \bigoplus_{j=1}^{\infty} \tfrac{j}{j+1} (S^* \oplus S) = S^* \oplus (\text{a strict contraction})$$

possesses no such vector since $(S^*)^k$ tends strongly to zero. This precludes the existence of a conjugation C (i.e., an isometric, conjugate-linear involution) such that $T = CT^*C$. Thus T is not complex symmetric. □

Acknowledgements

We thank D. Sherman for his helpful suggestions. We also note that S. Zhu, C.G. Li, and Y.Q. Ji discovered a different approach [3] shortly before we did.

References

[1] Stephan Ramon Garcia and Mihai Putinar. Complex symmetric operators and applications. II. *Trans. Amer. Math. Soc.*, 359(8):3913–3931 (electronic), 2007. MR2302518 (2008b:47005)
[2] Stephan Ramon Garcia and Warren R. Wogen. Complex symmetric partial isometries. *J. Funct. Anal.*, 257(4):1251–1260, 2009. MR2535469 (2011g:47005)
[3] Sen Zhu, Chun Guang Li, and You Qing Ji. The class of complex symmetric operators is not norm closed. *Proc. Amer. Math. Soc.*, 140(5):1705–1708, 2012. MR2869154

Department of Mathematics, Pomona College, Claremont, California 91711
E-mail address: Stephan.Garcia@pomona.edu
URL: http://pages.pomona.edu/~sg064747

Received by the editors March 27, 2011 and, in revised form, June 30, 2011.
2010 *Mathematics Subject Classification.* Primary 47A05, 47B35, 47B99.
Key words and phrases. Complex symmetric operator, norm closure, Hilbert space.
This work partially supported by National Science Foundation Grant DMS-1001614.