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Review: Nontangential Limits in Pt(μ)-spaces and the Index of Invariant Subgroups

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Nontangential limits in $\mathcal{P}^t(\mu)$-spaces and the index of invariant subspaces. (English summary)
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This deep and interesting article answers a number of fundamental questions about boundary behavior in $\mathcal{P}^t(\mu)$ spaces and the relationship between the measure μ and the index of invariant subspaces of $\mathcal{P}^t(\mu)$. The introduction is well written and inviting and hence we freely paraphrase portions of it below.

In order the summarize the results of this article, a few preliminary definitions are required. First, let μ denote a finite positive measure on the closed unit disk $\overline{\mathbb{D}}$, let $1 \le t < \infty$, and let $\mathcal{P}^t(\mu)$ denote the closure of the analytic polynomials in $L^t(\mu)$. Multiplication by z is a bounded linear operator on $\mathcal{P}^t(\mu)$ and is denoted by S. An invariant subspace of $\mathcal{P}^t(\mu)$ is a closed linear subspace $\mathcal{M}$ of $\mathcal{P}^t(\mu)$ which satisfies $S\mathcal{M} \subseteq \mathcal{M}$.

The two most familiar examples of $\mathcal{P}^t(\mu)$ spaces are those corresponding to $\mu = \frac{1}{2\pi}m$ (normalized Lebesgue measure on $\partial\mathbb{D}$) and $\mu = \frac{1}{\pi}A$ (normalized Lebesgue measure on $\mathbb{D}$). These cases satisfy $\mu(\partial\mathbb{D}) > 0$ and $\mu(\partial\mathbb{D}) = 0$, respectively, and they illustrate the type of phenomenon that motivates the article. In particular, observe that:

- For $\mu = \frac{1}{2\pi}m$, one obtains the classical Hardy spaces H^t. It is well known that each $f \in H^t$ has a nontangential limit $f^*(z)$ at m-almost every $z \in \partial\mathbb{D}$ and that $f^* = f$ as elements on $L^t(\mu)$. Moreover, one has a strong uniqueness criterion, for $f^* = 0$ on a set of positive m-measure implies that $f = 0$. By Beurling's Theorem $(t = 2)$ and its extension to other values of t, it follows that every nonzero invariant subspace of H^t has index 1.
- For $\mu = \frac{1}{\pi}A$ one obtains the Bergman spaces L^t_a. Functions in L^t_a need not have nontangential limits at any point of $\partial\mathbb{D}$. Furthermore, for any $1 \le n \le \infty$ there exist invariant subspaces of L^t_a having index n [C. Apostol et al., J. Funct. Anal. **63** (1985), no. 3, 369–404; MR0808268 (87i:47004a); J. Eschmeier, Math. Ann. **298** (1994), no. 1, 167–186; MR1252824 (94k:47010); H. Hedenmalm, S. Richter and K. Seip, J. Reine Angew. Math. **477** (1996), 13–30; MR1405310 (97i:46044)].

In light of these results the authors are led to study the case where $\mathbb{D}$ is the set of analytic bounded point evaluations for $\mathcal{P}^t(\mu)$ and $\mathcal{P}^t(\mu)$ contains no nontrivial characteristic functions (i.e., $\mathcal{P}^t(\mu)$ is irreducible). It is known in this case that the restriction of μ to $\partial\mathbb{D}$ must be of the form $h\,|dz|$.

If $\mu(\partial\mathbb{D}) > 0$, then one has a boundary function $f|\partial\mathbb{D}$ for each $f \in \mathcal{P}^t(\mu)$. However, the precise relationship between $f|\partial\mathbb{D}$ and the limiting behavior of the analytic function $f|\mathbb{D}$ is not immediately clear. The following questions are suggested by the examples above:

1. Is $f|\partial\mathbb{D}$ the boundary value function of f in some suitable sense?
2. Is $f|\partial\mathbb{D} = f^*$ $\mu|\partial\mathbb{D}$-almost everywhere?

3. Is f determined by $f|\partial\mathbb{D}$? In other words, does $f|\partial\mathbb{D} = 0$ imply that $f = 0$?
4. Is the index of every nonzero invariant subspace equal to 1?

The first major theorem of this paper answers all four of these questions in the affirmative:

Theorem A. Suppose that μ is supported in $\overline{\mathbb{D}}$ and is such that the set of analytic bounded point evaluations for $\mathcal{P}^t(\mu)$ is equal to $\mathbb{D}$ and $\mathcal{P}^t(\mu)$ is irreducible, and that $\mu(\partial\mathbb{D}) > 0$. Then:

(a) If $f \in \mathcal{P}^t(\mu)$, then the nontangential limit $f^*(z)$ of f exists for $\mu|\partial\mathbb{D}$-almost all z, and $f^* = f|\partial\mathbb{D}$ as elements of $L^t(\mu|\partial\mathbb{D})$.

(b) Every nonzero invariant subspace of $\mathcal{P}^t(\mu)$ has index 1.

An important consequence of this work is an affirmative answer to a conjecture of J. B. Conway and L. M. Yang [in *Holomorphic spaces (Berkeley, CA, 1995)*, 201–209, Cambridge Univ. Press, Cambridge, 1998; MR1630651 (99e:47027)]. In particular, the present authors show that for $1 < t < \infty$ one has $\dim \mathcal{M}/z\mathcal{M} = 1$ for every nonzero invariant subspace $\mathcal{M}$ of $P^t(\mu)$ if and only if $h \neq 0$.

On the other hand, away from the part of $\partial\mathbb{D}$ where μ has mass, the boundary behavior of $\mathcal{P}^t(\mu)$ functions can be wild. To be more specific, there is a natural notion of interpolating sequences for $\mathcal{P}^t(\mu)$ spaces and Theorem B of this article shows (under the hypotheses of Theorem A) that for $t \in (1, \infty)$ and $E \subseteq \partial\mathbb{D}$ with $\mu(E) = 0$, there is an interpolating sequence for $\mathcal{P}^t(\mu)$ which clusters nontangentially at m-almost every point of E. In particular, this implies that there are functions in $\mathcal{P}^t(\mu)$ which have nontangential limits at m-almost no points of E. In the case $\mu(\partial\mathbb{D}) = 0$ and $t \in (1, \infty)$, this proves the existence of an interpolating sequence for $\mathcal{P}^t(\mu)$ that clusters nontangentially at m-almost every point of $\partial\mathbb{D}$. The argument used to prove [A. Aleman, S. Richter and C. Sundberg, J. Anal. Math. **86** (2002), 139–182; MR1894480 (2003g:30058) (Proposition 7.3)] then yields invariant subspaces of $\mathcal{P}^t(\mu)$ of index greater than 1. Thus one has a new proof of the results of [C. Apostol et al., op. cit.] and [J. Eschmeier, op. cit.] on the index of invariant subspaces of the Bergman spaces L^t_a.

For bounded point evaluations λ for $\mathcal{P}^t(\mu)$, let e_λ denote the associated evaluation functional and let $M_\lambda = \|e_\lambda\|_{\mathcal{P}^t(\mu)^*}$. One of the key elements in proving Theorem A is obtaining the inequality

$$\limsup_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}} (1-|\lambda|^2)^{1/t} M_\lambda \le \frac{C}{h(z)^{1/t}},$$

for m-almost all $z \in \partial\mathbb{D}$ where C is some constant. Remarkably, for $t > 1$ the authors are even able to prove the following asymptotic result:

Theorem C. Under the hypotheses of Theorem A, if $t > 1$, then

$$\lim_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}} (1-|\lambda|^2)^{1/t} M_\lambda = \frac{1}{h(z)^{1/t}}$$

for m-almost all $z \in \partial\mathbb{D}$.

The proofs of these results utilize portions of J. E. Thomson's proof of the existence of bounded point evaluations [Ann. of Math. (2) **133** (1991), no. 3, 477–507; MR1109351 (93g:47026)] along with X. Tolsa's recent work on analytic capacity [Acta Math. **190** (2003), no. 1, 105–149; MR1982794 (2005c:30020)]. The methods used are similar to those developed independently by J. E. Brennan in his recent proof, based upon Tolsa's work, of Thomson's Theorem [Algebra i Analiz **17** (2005), no. 2, 1–32; MR2159582 (2006m:41009)]. The authors are even able to refine Thomson's result somewhat. For a compactly supported measure ν in $\mathbb{C}$, they describe the locations of bounded point evaluations for $P^t(\mu)$ in terms of the Cauchy transform of an annihilating measure (Corollary 2.2).

Reviewed by *Stephan R. Garcia*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.