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Review: Embeddings of Model Subspaces of the Hardy Class: Compactness and Schatten–von Neumann Ideals

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For each inner function Θ and $p \in [1, \infty)$, let $K^p_\Theta = H^p \cap \Theta \overline{H^p_0}$ denote a typical $*$-*invariant subspace* of the classical Hardy space H^p. The paper under review concerns the general question of determining all Borel measures μ on the closed unit disk $\overline{\mathbb{D}}$ such that K^p_Θ can be embedded in $L^p(\mu)$, a question which dates back to the work of W. S. Cohn [Pacific J. Math. **103** (1982), no. 2, 347–364; MR0705235 (84m:30054)]. The embedding $K^p_\Theta \subset L^p(\mu)$ is equivalent to an estimate of the form $\|f\|_{L^p(\mu)} \leq C\|f\|_p$ for $f \in K^p_\Theta$ and for some constant $C > 0$. Currently, a complete description of such measures is known only in certain special cases. Of particular interest are *one-component inner functions*, those inner functions Θ such that the level set

$$\Omega(\Theta, \varepsilon) = \{z \in \mathbb{D} : |\Theta(z)| < \varepsilon\}$$

is connected for some $\varepsilon \in (0, 1)$. A noteworthy theorem in the area is due to A. L. Vol′berg and S. Treil [Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **149** (1986), Issled. Lineĭn. Teor. Funktsiĭ. XV, 38–51, 186–187; MR0849293 (88f:47028)], who proved that K^p_Θ embeds in $L^p(\mu)$ if there exists an $\varepsilon \in (0, 1)$ such that $\mu(S(I)) \leq C|I|$ for all Carleson squares $S(I)$ satisfying $S(I) \cap \Omega(\Theta, \varepsilon) \neq \varnothing$.

A new approach to this family of problems was suggested by the author in previous work [Algebra i Analiz **15** (2003), no. 5, 138–168; MR2068792 (2005b:30037); J. Funct. Anal. **223** (2005), no. 1, 116–146; MR2139883 (2007b:46041)]. This approach relies upon the use of *Bernstein inequalities* for the spaces K^p_Θ, which are weighted norm inequalities relating f' to the standard L^p-norm of $f \in K^p_\Theta$ in the space $L^p(\mathbb{T}, \mu)$. Using these techniques, the author proves that the embedding of K^p_Θ in $L^p(\mu)$ is compact whenever a certain natural vanishing condition on μ holds. In particular, this yields a positive answer to a question of J. A. Cima and A. L. Matheson [Quaest. Math. **26** (2003), no. 3, 279–288; MR2018910 (2004k:30083)]. This paper also contains a number of other results concerning the question of when the embedding operator belongs to a Schatten-von Neumann ideal $\mathfrak{S}_r$. For one-component inner functions, many of the author's results are also reversible and hence they yield definitive answers.

Reviewed by *Stephan R. Garcia*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.