1-1-2006

Review: Stability of Bases and Frames of Reproducing Kernels in Model Spaces

Stephan Ramon Garcia
Pomona College

Recommended Citation
MR2207388 (2006j:30064) Baranov, A., Stability of bases and frames of reproducing kernels in model spaces, Ann. Inst. Fourier (Grenoble) 55 (2005), no. 7, 2399–2422. (Reviewer: Stephan R. Garcia)

MR2207388 (2006j:30064) 30D55 (30F45 46E22 47A45)
Baranov, Anton [Baranov, Anton D.] (RS-STPT-MM)
Stability of bases and frames of reproducing kernels in model spaces. (English, French summaries)
Ann. Inst. Fourier (Grenoble) **55** (2005), *no. 7,* 2399–2422.1777-5310

This paper presents several results on the stability of bases and frames of reproducing kernels in model spaces based on the estimates of derivatives (Bernstein-type inequalities) recently obtained by the author [J. Funct. Anal. **223** (2005), no. 1, 116–146; MR2139883 (2007b:46041)].

To be more specific, let Θ be an inner function on the upper half-plane $\mathbb{C}^+$ and let $K^2_\Theta = H^2 \ominus \Theta H^2$ denote the corresponding model space. For $\lambda \in \mathbb{C}^+$, the function

$$k_\lambda(z) = \frac{i}{2\pi} \cdot \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{z - \overline{\lambda}}$$

is the reproducing kernel for the point evaluation functional at λ:

$$f(\lambda) = \int_{\mathbb{R}} f(t)\overline{k_\lambda(t)}\, dt$$

for $f \in K^2_\Theta$. A system of vectors $\{h_n\}$ in a Hilbert space H is a Riesz basis if $\{h_n\}$ is the image of an orthonormal basis for H under a bounded, invertible linear operator on H. Equivalently, each h in H can be written as an unconditionally convergent series $h = \sum_n c_n h_n$ and there exist positive constants A and B such that

$$A \sum_n |c_n|^2 \le \left\| \sum_n c_n h_n \right\|_H^2 \le B \sum_n |c_n|^2.$$

Let $\{k_{\lambda_n}\}$ be such a basis in K^2_Θ, let $\langle u, v\rangle$ denote the interval with endpoints u and v, and let $\delta_{\langle u,v\rangle}$ denote the Lebesgue measure on $\langle u, v\rangle$. Let $G = \bigcup_n G_n \subset \overline{\mathbb{C}^+}$ satisfy the following properties: (i) there exist positive constants c and C such that

$$c \le \|k_{z_n}\|_2 / \|k_{\lambda_n}\|_2 \le C, \quad z_n \in G_n,$$

and (ii) for any $z_n \in G_n$, the measure $\nu = \sum_n \delta_{\langle \lambda_n, z_n\rangle}$ is a Carleson measure whose Carleson constants M_ν are uniformly bounded with respect to z_n. The main theorem of this paper is the following:

Theorem 1.1. Let $\{k_{\lambda_n}\}$ be a basis in K^2_Θ, $p \in (1,2)$, $1/p + 1/q = 1$. Then for any set G satisfying (i)–(ii) there is $\varepsilon > 0$ such that the system of reproducing kernels $\{k_{\mu_n}\}$ is a basis whenever $\mu_n \in G_n$ and

$$\sup_n \frac{1}{\|k_{\lambda_n}\|_2^2} \int_{\langle \lambda_n, \mu_n\rangle} \|k_z^2\|_q^{\frac{2p}{p+1}} |dz| < \varepsilon.$$

Here ε depends on p, the constants involved in the definition of G, and on the constant A, but not on the inner function Θ.

For the case when Θ and λ_n satisfy $\sup_n |\Theta(\lambda_n)| < 1$, E. Fricain [J. Operator Theory **46** (2001), no. 3, suppl., 517–543; MR1897152 (2003b:46033)] showed that if $\{k_{\lambda_n}\}$ is a basis in K^2_Θ, then there is an $\varepsilon > 0$ such that $\{k_{\mu_n}\}$ is a basis whenever $\sup_n \rho(\lambda_n, \mu_n) < \varepsilon$. Here $\rho(z, w) = |(z - w)/(z - \overline{w})|$ is the pseudohyperbolic distance between z and w ($z, w \in \mathbb{C}^+$).

A corollary of Theorem 1.1 is as follows:

Corollary 1.2. Let $\{k_{\lambda_n}\}$ be a basis in K^2_Θ and $\gamma > 1/3$. Then there is $\varepsilon = \varepsilon(\gamma, A)$ such that the system $\{k_{\mu_n}\}$ is a basis whenever

$$\rho(\lambda_n, \mu_n) < \varepsilon(1 - |\Theta(\lambda_n)|)^\gamma.$$

As the author points out, the theorem of Fricain follows from the corollary above. An additional corollary (Corollary 1.3) pertains to the case where Θ is a “one-component inner function” (i.e. Θ satisfies the connected level set condition (CLS)).

Theorem 1.4 is an extension of Cohn’s theorem [W. S. Cohn, J. Operator Theory **15** (1986), no. 1, 181–202; MR0816238 (87g:47054)] on the stability of bases with real frequencies (in particular, the Clark bases) to general inner functions.

Reviewed by *Stephan R. Garcia*

References

1. P. R. Ahern, D. N. Clark, Radial limits and invariant subspaces, Amer. J. Math., 92, 2 (1970), 332–342. MR0262511 (41 #7117)
2. A. B. Aleksandrov, Invariant subspaces of shift operators. An axiomatic approach, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 113 (1981), 7–26; English transl.: J. Soviet Math. 22 (1983), 1695–1708. MR0629832 (83g:47031)
3. A. B. Aleksandrov, A simple proof of the Volberg–Treil theorem on the embedding of coinvariant subspaces of the shift operator, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 217 (1994), 26–35; English transl.: J. Math. Sci., 85 (1997), 2, 1773–1778. MR1327512 (96f:30031)
4. A. B. Aleksandrov, Embedding theorems for coinvariant subspaces of the shift operator. II, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 262 (1999), 5–48; English transl.: J. Math. Sci., 110, (2002), 5, 2907–2929. MR1734326 (2001g:46047)
5. A. D. Baranov, The Bernstein inequality in the de Branges spaces and embedding theorems, Proc. St. Petersburg Math. Soc., 9 (2001), 23–53; English transl.: Amer. Math. Soc. Transl. Ser. 2, 209 (2003), 21–49. MR2018371 (2004m:30043)
6. A. D. Baranov, Weighted Bernstein-type inequalities and embedding theorems for shift-coinvariant subspaces, Algebra i Analiz, 15, 5 (2003), 138–168; English transl.: St. Petersburg Math. J., 15 (2004), 5, 733–752. MR2068792 (2005b:30037)
7. A. D. Baranov, Bernstein-type inequalities for shift-coinvariant subspaces and their applications to Carleson embeddings, J. Funct. Anal., 223 (2005), 1, 116–146. MR2139883 (2007b:46041)
8. I. Boricheva, Geometric properties of projections of reproducing kernels on z^*-invariant subspaces of H^2, J. Funct. Anal., 161 (1999), 2, 397–417. MR1674647 (2000f:46031)
9. P. Borwein, T. Erdelyi, Sharp extensions of Bernstein’s inequality to rational spaces, Mathematika, 43 (1996), 2, 413–423. MR1433285 (97k:26014)

10. L. De Branges, Hilbert spaces of entire functions, Prentice Hall, Englewood Cliffs (NJ), 1968. MR0229011 (37 #4590)
11. D. N. Clark, One-dimensional perturbations of restricted shifts, J. Anal. Math., 25 (1972), 169–191. MR0301534 (46 #692)
12. W. S. Cohn, Radial limits and star invariant subspaces of bounded mean oscillation, Amer. J. Math., 108 (1986), 3, 719–749. MR0844637 (87j:30076)
13. W. S. Cohn, Carleson measures and operators on star-invariant subspaces, J. Oper. Theory, 15 (1986), 1, 181–202. MR0816238 (87g:47054)
14. W. S. Cohn, On fractional derivatives and star invariant subspaces, Michigan Math. J., 34 (1987), 3, 391–406. MR0911813 (89a:30029)
15. K. M. Dyakonov, Entire functions of exponential type and model subspaces in H^p, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 190 (1991), 81–100; English transl.: J. Math. Sci., 71 (1994), 1, 2222–2233. MR1111913 (92h:30072)
16. K. M. Dyakonov, Smooth functions in the range of a Hankel operator, Indiana Univ. Math. J., 43 (1994), 805–838. MR1305948 (96f:47047)
17. K. M. Dyakonov, Differentiation in star-invariant subspaces I, II, J. Funct. Anal., 192 (2002), 2, 364–409. MR1923406 (2003g:30060)
18. E. Fricain, Bases of reproducing kernels in model spaces, J. Oper. Theory, 46 (2001), 3 (suppl.), 517–543. MR1897152 (2003b:46033)
19. E. Fricain, Complétude des noyaux reproduisants dans les espaces modèles, Ann. Inst. Fourier (Grenoble), 52 (2002), 2, 661–686. MR1906486 (2003g:46029)
20. S. V. Hruscev, N. K. Nikolskii, B. S. Pavlov, Unconditional bases of exponentials and of reproducing kernels, Lecture Notes in Math., 864 (1981), 214–335. MR0643384 (84k:46019)
21. M. I. Kadec, The exact value of the Paley–Wiener constant, Dokl. Akad. Nauk SSSR, 155 (1964), 1253–1254; English transl.: Sov. Math. Dokl., 5 (1964), 559–561. MR0162088 (28 #5289)
22. M. B. Levin, An estimate of the derivative of a meromorphic function on the boundary of domain, Sov. Math. Dokl., 15 (1974), 3, 831–834. MR0352468 (50 #4955)
23. Yu. I. Lyubarskii, K. Seip, Complete interpolating sequences for Paley–Wiener spaces and Muckenhoupt's (A_p) condition, Rev. Mat. Iberoamericana, 13 (1997), 2, 361–376. MR1617649 (99e:42004)
24. N. K. Nikolski, Treatise on the shift operator, Springer-Verlag, Berlin-Heidelberg, 1986. MR0827223 (87i:47042)
25. N. K. Nikolski, Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz, Mathematical Surveys and Monographs, 92, AMS, Providence, RI, 2002. MR1864396 (2003i:47001a)
26. N. K. Nikolski, Operators, functions, and systems: an easy reading. Vol. 2. Model operators and systems, Mathematical Surveys and Monographs, 93, AMS, Providence, RI, 2002. MR1892647 (2003i:47001b)
27. J. Ortega-Cerda, K. Seip, Fourier frames, Ann. of Math. (2), 155 (2002), 3, 789–806. MR1923965 (2003k:42055)
28. K. Seip, On the connection between exponential bases and certain related sequences in

$L^2(-\pi, \pi)$, J. Funct. Anal., 130 (1995), 1, 131–160. MR1331980 (96d:46030)
29. A. L. Volberg, S. R. Treil, Embedding theorems for invariant subspaces of the inverse shift operator, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 149 (1986), 38–51; English transl.: J. Soviet Math., 42 (1988), 2, 1562–1572. MR0849293 (88f:47028)
30. R. M. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, New-York, 1980. MR0591684 (81m:42027)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.