Abscissas and Ordinates

David Pierce

Mimar Sinan Güzel Sanatlar Üniversitesi

Follow this and additional works at: https://scholarship.claremont.edu/jhm

Part of the Geometry and Topology Commons

Recommended Citation


©2015 by the authors. This work is licensed under a Creative Commons License.

JHM is an open access bi-annual journal sponsored by the Claremont Center for the Mathematical Sciences and published by the Claremont Colleges Library | ISSN 2159-8118 | http://scholarship.claremont.edu/jhm/

The editorial staff of JHM works hard to make sure the scholarship disseminated in JHM is accurate and upholds professional ethical guidelines. However the views and opinions expressed in each published manuscript belong exclusively to the individual contributor(s). The publisher and the editors do not endorse or accept responsibility for them. See https://scholarship.claremont.edu/jhm/policies.html for more information.
Abscissas and Ordinates

David Pierce

Mathematics Department, Mimar Sinan Fine Arts University, Istanbul, TURKEY
http://mat.msgsu.edu.tr/~dpierce/
dpierce@msgsu.edu.tr

Synopsis

In the manner of Apollonius of Perga, but hardly any modern book, we investigate conic sections as such. We thus discover why Apollonius calls a conic section a parabola, an hyperbola, or an ellipse; and we discover the meanings of the terms abscissa and ordinate. In an education that is liberating and not simply indoctrinating, the student of mathematics will learn these things.

Contents

1 The liberation of mathematics 234
2 Lexica and registers 235
3 The gendered article 237
4 Applications 241
5 The cone 245
6 The parabola 250
7 The latus rectum 252
8 The hyperbola 256

Journal of Humanistic Mathematics  Vol 5, No 1, January 2015
In the third century before the Common Era, Apollonius of Perga (near Antalya in Turkey) wrote eight books on conic sections. The first four of these books survive in the original Greek; the next three survive in Arabic translation only. The last book is lost. Lucio Russo \[37\], page 8 uses this and other examples to show that we cannot expect all of the best ancient work to have come down to us.

In the first book of the *Conics* [4], Apollonius derives properties of the conic sections that can be used to write their equations in rectangular or oblique coordinates. The present article reviews these properties, because (1) they have intrinsic mathematical interest, (2) they are the reason why Apollonius gave to the three conic sections the names that they now have, and (3) the vocabulary of Apollonius is a source for a number of our other technical terms as well.

In a modern textbook of analytic geometry, the two coordinates of a point in the so-called Cartesian plane may be called the “abscissa” and “ordinate.”
Probably the book will not explain why. But the reader deserves an explanation. The student should not have to learn meaningless words, for the same reason that she should not be expected to memorize the quadratic formula without seeing a derivation of it. True education is not indoctrination, but liberation. I elaborate on this point, if only obliquely, in an article on my college [34]. Mathematics is liberating when it teaches us our own power to decide what is true. This power comes with a responsibility to justify our decisions to anybody who asks; but this is a responsibility that must be shared by all of us who do mathematics.

Mathematical terms can be assigned arbitrarily. This is permissible, but it is not desirable. The terms “abscissa” and “ordinate” arise quite naturally in Apollonius’s development of the conic sections. This development should be better known, especially by anybody who teaches analytic geometry. This is why I write.

2. Lexica and registers

Apollonius did not create his terms: they are just ordinary words, used to refer to mathematical objects. When we do not translate Apollonius, but simply transliterate his words, or use their Latin translations, then we put some distance between ourselves and the mathematics. When I read in high school that a conic section had a latus rectum, I had a sense that there was a whole theory of conic sections that was not being revealed, although its existence was hinted at by this peculiar Latin term. If we called the latus rectum by its English name of “upright side,” then the student could ask, “What is upright about it?” In turn, textbook writers might feel obliged to answer this question. In any case, I am going to answer it here. Briefly, it is called upright because, for good reason, it is to be conceived as having one endpoint on the vertex of the conic section, but as sticking out from the plane of the section.

English does borrow foreign words freely: this is a characteristic of the language. A large lexicon is not a bad thing. A choice from among two or more synonyms can help establish the register of a piece of speech. In the 1980s, a book called Color Me Beautiful was on the American bestseller lists. The New York Times blandly said the book provided “beauty tips for women”; the Washington Post described it as offering “the color-wheel approach to female pulchritude.” (The quotations are from my memory only.)
By using an obscure synonym for beauty, the *Post* mocked the book.

If distinctions between near-synonyms are maintained, then subtleties of expression are possible. “Circle” and “cycle” are Latin and Greek words for the same thing, but the Greek word is used more abstractly in English, and it would be bizarre to refer to a finite group of prime order as being circular rather than cyclic.

To propose or maintain distinctions between near-synonyms is a raison d’être of works like Fowler’s *Dictionary of Modern English Usage* [18]. Fowler laments, for example, the use of the Italian word *replica* to refer to any copy of an art-work, when the word properly refers to a copy *made by the same artist*. In his article on synonyms, Fowler sees in language the kind of liberation, coupled with responsibility, that I ascribed to mathematics:

> Synonym books in which differences are analysed, engrossing as they may have been to the active party, the analyst, offer to the passive party, the reader, nothing but boredom. Every reader must, for the most part, be his own analyst; & no-one who does not expend, whether expressly & systematically or as a half-conscious accompaniment of his reading & writing, a good deal of care upon points of synonymy is likely to write well.

Fowler’s own book is, in part, one of the synonym books that he denigrates here; so I suppose he is saying, “You will not be a good writer, just by reading me: you must read, and think, for yourself.” What other synonym books does he have in mind? Perhaps books like *Roget’s Thesaurus*, which I do find occasionally useful, but which is *not* the “indispensable guide to the English language” that the back cover of “my” edition [9] claims it to be. Fowler’s own book is closer to that description, for the example it sets of sound thinking. This is why I prefer to read his original work, rather than the posthumous second edition, edited and updated by Gowers [17].

The boredom, described by Fowler, of the reader of a *mere* book of synonyms is comparable to that of the reader of a mathematics textbook that begins with a bunch of strange words like “abscissa” and “ordinate.”

Mathematics can be done in any language. Greek does mathematics without a specialized vocabulary. It is worthwhile to consider what this is like.

I shall take Apollonius’s terminology from Heiberg’s edition [2]—actually
a printout of a pdf image downloaded from the Wilbour Hall website, wilbourhall.org. Meanings are checked with the big Liddell–Scott–Jones lexicon, the “LSJ” [25]. The articles of the LSJ are available from the Perseus Digital Library, perseus.tufts.edu, though I myself splurged on the print version of the whole book. In fact the great majority of works in the references, including all of the language books, are from the shelves of my personal library; some might not be the most up-to-date references, but they are the ones that I am pleased to have at hand.

I am going to write out Apollonius’s terms in Greek letters, using the NeoHellenic font made available by the Greek Font Society. I shall use the customary minuscule forms—today’s “lower case”—developed in the Middle Ages. Apollonius himself would have used only the letters that we now call capital; but modern mathematics uses minuscule Greek letters freely, and the reader ought to be able to make sense of them, even without studying the Greek language as such. I have heard a plausible rumor that it is actually beneficial for calculus students to memorize the Greek alphabet.

3. The gendered article

Apollonius’s word for cone is ὁ κῶνος, meaning originally “pine-cone.” Evidently our word “cone” comes ultimately from Apollonius’s word (and this is confirmed by such resources as the Concise Oxford Dictionary of English Etymology [20]). I write ὁ in front of κῶνος to indicate its gender: ὁ is the masculine definite article. The feminine definite article is ἡ. In each case, the sign over the vowel is the “rough breathing” mark, indicating the prefixed sound that is spelled in English with the letter H. The “smooth breathing” mark over an initial vowel, as in ἰ or ἲ, just means there is no “rough breathing.” The other accents (ἄ, Ὠ, ᾽) can be ignored; I reproduce them because they are in the modern texts (and are occasionally useful in parsing).

In the specific terminology of Apollonius, all of the nouns that we shall look at will be feminine or masculine. Greek does however have a neuter gender as well, and the neuter definite article is τὸ. English retains the threefold gender distinction in the pronouns “he, she, it.” English formerly had the distinction in its definite article as well [39, pages 28–9], with se, seo, δεητ corresponding to the Greek ὁ, ἡ, τὸ; the masculine se became δε, and then our “the,” while the neuter δεητ became our demonstrative “that”
Abscissas and Ordinates

[38, pages 550–1]. Thus residues of gender exist in English today. However, English no longer allows the economy of mathematical expression that is made possible by a gendered article as such.

In Greek mathematics, **point** is τὸ σημεῖον, neuter, while **line** is ἡ γραμμή, feminine. The feminine ἡ στιγμή can also be used for a mathematical point. However, in *The Shaping of Deduction in Greek Mathematics*, Netz argues [30, page 113] that mathematicians dropped στιγμή in favor of σημεῖον, so that an expression like ἡ Α could unambiguously designate a line labeled as Α in a diagram, while τὸ Α would designate a point.

Proposition 43 may be the most portentous of the 48 propositions of Book I of Euclid’s *Elements*. It makes possible the definition of an algebra of line segments such as Descartes would define, almost two millennia later. In particular, it makes possible the definition of the product of two line segments, once a unit segment is chosen. (In fact, for this definition, Descartes uses not Proposition 1.43, but the theory of proportion that Euclid will not develop until Book V. My article [33] features a contemporary theoretical development of Descartes’s idea.)

Euclid’s Proposition 1.43 establishes the *equality* of certain dissimilar parallelograms having the same angle. It bothers some of my students to say “equality” here; they apparently think equality must be congruence, if not identity, even though they will allow that the fractions \( \frac{2}{3} \) and \( \frac{4}{6} \) are equal. They want to say that the equality in Proposition 1.43 is an equality of *areas*. But they cannot say what area is, except that it is some real number of units. This is a doctrine that they have been taught without justification; the justification starts in Euclid, and this is a reason why my department asks students to read him.

Euclid’s Proposition is specifically that, if two straight lines parallel to sides of a given parallelogram intersect on a diagonal of that parallelogram, then the “complements” of the two smaller parallelograms formed about the diagonal are equal to one another. That is, parallelograms ΒΕΚΗ and ΚΖΔΘ in Figure 1 are equal to one another. The diagram is Euclid’s, at least as it is reproduced in Heiberg’s text [12, page 100]. But Euclid can express the equation of parallelograms more tersely. Heath’s translation [13, 14] of Euclid’s proof begins as follows, if we put Euclid’s Greek letters back in for Heath’s Latin replacements:

Let ΑΒΓΔ be a parallelogram, and ΑΓ its diameter; and about ΑΓ
let $\Theta, ZH$ be parallelograms, and $BK, KD$ the so-called complements; I say that the complement $BK$ is equal to the complement $KD$. For, since $ABGD$ is a parallelogram, and $AG$ is its diameter, the triangle $ABG$ is equal to the triangle $AGD$.

Euclid can refer to the parallelogram $\Delta \Theta$ simply as τὸ $\Theta$, using the neuter article τὸ, because παραλληλόγραμμον is neuter. The Greek reader cannot think that τὸ $\Theta$ is a line; the line would be ἑ $\Theta$, with the feminine article, just as the diagonal of the big parallelogram is Ἑ $\Gamma$. The parallelograms about this diagonal are τὰ $\Theta, ZH$, with the plural neuter article τὰ. Their complements, which are also parallelograms, are τὰ $BK, KD$, likewise with the plural neuter article, because παραπλήρωμα is neuter, as παραλληλόγραμμον is. Perhaps it is no coincidence that Euclid’s word for “complement” has the same gender as “parallelogram,” but Euclid imposed this as a necessary condition on his choice of the word. The LSJ lexicon cites only Euclid for the geometrical meaning of παραπλήρωμα; its other meanings are “expletive” and “sagina,” the latter being a Latin term for “meat to cram fowls” [44].

The concision that is possible for Euclid is confusing to the reader in English—and in Turkish, to name another language that is free of gender. This is the language that my students are reading Euclid in [15]. I said they could be uneasy that Proposition 1.43 was about equality of incongruent parallelograms; but in fact they have already had to get used to this kind of equality in Proposition 35:

Parallelograms on the same base and being in the same parallels are equal to one another.

When they get to Proposition 43, some students think that what is going
Abscissas and Ordinates

to be proved is the equality of the straight lines ΒΚ and ΚΔ. A sufficiently
careful reader in any language may avoid this mistake, especially because
those straight lines have not actually been drawn in Euclid’s diagram; but
no reader in Greek can make the mistake. This is one of the few cases where
gender in a language is actually useful.

Greek has no indefinite article. English and Turkish do have one, derived
from the word “one.” In Turkish it is still the same word, but positioned
differently: bir güzel gün “one fine day,” but güzel bir gün “a fine day.” Turkish
has no definite article.

Today, perhaps oddly, we may use the indefinite article where Greek uses
the definite. Heath translates the enunciation of Proposition 1.1 of Euclid as,

On a given finite straight line to construct an equilateral
triangle.

But the Greek says, “the given finite straight line.” One might read the
definite article here as the generic article, seen for example in the seventh
verse of Wordsworth’s little poem [45, page 85],

MY HEART LEAPS UP.

My heart leaps up when I behold
A rainbow in the sky:
So was it when my life began;
So is it now I am a man;
So be it when I shall grow old,
Or let me die!
The child is the father of the man;
And I could wish my days to be
Bound each to each by natural piety.

The old Descriptive English Grammar of Harman and House [22, page 76]
gives Wordsworth’s verse as an example, though without actually naming the
poet. Devoting little more than a page to “the,” the authors say of it only
that (1) it is derived from the masculine demonstrative pronoun ὁ and has
its meaning, weakened; (2) it also has a generic meaning, as in “The child
is the father of the man”; (3) it is used in names and titles, such as “The Soviet Union.” By contrast, the colossal Cambridge Grammar of the English Language [23], in its main section on the definite article as such (5.6.1, pages 368–71), does not mention the generic meaning of “the”: this comes later, in “Class uses of the definite article” (5.8.4(a), page 407), with examples like, “The African elephant will soon be extinct.”

We can say, “The isosceles triangle has its base angles equal” (Elements 1.5), meaning “Every isosceles triangle”; we can also say “An isosceles triangle,” with the same meaning. But the explanation for Euclid’s use of the definite article in Proposition 1.1 may be simpler. He is referring to the straight line that the reader or listener can already see in the diagram that Euclid has put on display. Netz points out [30, pages 14–16] that ancient schools would not have had our blackboards and whiteboards, on which diagrams could be drawn during the course of a lecture; diagrams were most likely prepared beforehand, perhaps by being graven in wax.

4. Applications

Ancient Greek uses no word for what we now label as propositions in Euclid and Apollonius (though modern Greek uses πρότασις, as for example at users.ntua.gr/dimour/euclid/). Writing hundreds of years later than those mathematicians, Pappus of Alexandria [43, pages 566–7] distinguishes between theorems and problems. In a theorem (τὸ θεώρημα), something is looked at (θεωρεῖται), while in a problem (τὸ πρόβλημα), a construction is proposed (προβάλλεται). Euclid’s Proposition 1.43 is a theorem, but it is used for Proposition 1.44, which (like Proposition 1.1) is a problem. The problem of 1.44 is to construct on a given straight line, and in a given angle, a parallelogram equal to a given triangle. (Euclid’s straight lines are our “line segments.”)

The verb προβάλλω from which τὸ πρόβλημα is derived has the root meaning of “throw forward.” Changing the prefix makes παραβάλλω, “throw alongside,” which is just what is proposed in Proposition 1.44: to throw a parallelogram alongside a given straight line. An English term (derived from Latin) for this activity is application. This is a translation of ἡ παραβολή, which will be Apollonius’s term for the parabola, because the parabola can be described in terms of applications in the sense just defined, as we shall see.
Euclid’s Proposition 1.44 is in turn a lemma for Proposition 45, which is another problem: given a figure with any number of straight sides, to construct a rectangle (or a parallelogram in any given angle) that is equal to this figure. This problem is the climax of Book I of the Elements. Implicitly, by 1.43 and 44, the rectangle can have a predetermined base, and then its height can be taken as a measure of the area of the original figure. Now we are a step on the way to justifying my students’ unexamined conviction that areas are numbers.

Euclid’s Proposition 1.45 would seem to corroborate the judgment of Herodotus: that the Greeks learned geometry from the Egyptians, who had to be able to measure land lost in the annual flooding of the Nile [19, ii.109]. It is probably better to say that the Greeks learned surveying from the Egyptians: this is the root meaning of ἡ γεωμετρία. It appears that Egyptian surveyors defined the area of a four-sided field as the product of the averages of the opposite sides. Thus they overestimated the area, unless the field was a perfect rectangle. In The Mathematics of Plato’s Academy [16], Fowler reports this, while doubting that the Greeks did “discover geometry from Egyptian land measurement” (pages 232, 281). It would seem to me that Euclid’s Proposition 1.45 could well be the result of attempts to improve the Egyptian formula for quadrilateral areas; but my interest now is in a passage that Fowler quotes (page 279) together with the passage of Herodotus mentioned above.

The passage is by Proclus in his Commentary on the First Book of Euclid’s Elements [35, 36, pages 64–5]:

...we say, as have most writers of history, that geometry (ἡ γεωμετρία) was first discovered among the Egyptians and originated in the remeasuring of their lands. This was necessary (ἀναγκαία) for them because the Nile overflows and obliterates the boundary lines between their properties. It is not surprising that the discovery of this and the other sciences had its origin in necessity (ἡ χρεία) ... Just as among the Phoenicians the necessities of trade and exchange gave the impetus to the accurate study of number, so also among the Egyptians the invention of geometry came from the cause (ἡ αἰτία) mentioned.

The last clause but one would be more literally translated as,
Among the Phoenicians, through trade and exchange, the accurate study of numbers took a beginning (ἡ ἀρχή).

There is no specific mention of necessity, except that ἀρχή itself can be translated as “cause.”

While it may be inspired by the physical world, I want to propose that mathematics has no external cause. Any necessity in its development is internal. Apparently the Egyptians were not required by their physical conditions to find a formula for quadrilateral areas that we would recognize today as strictly correct. Euclid did happen to develop a precise theoretical understanding of areas, as presented in Book 1 of the Elements; but nothing made him do it but his own internal drive.

The Devil says to the Angel in Blake’s Marriage of Heaven and Hell [7, plate 23],

bray a fool in a mortar with wheat
yet shall not his folly be beaten out of him.

(Actually this is Proverbs 27:22.) Nothing external can separate the fool from his folly. This might be contradicted by the experience of George Orwell, as recounted in the essay called “Such, such were the joys . . .” [31]. The essay happens to take its ironic title from the middle stanza of “The Ecchoing Green,” one of Blake’s Songs of Innocence [6]:

Old John, with white hair,
Does laugh away care,
Sitting under the oak,
Among the old folk.
“Such, such were the joys
When we all, girls and boys,
In our youth time were seen
On the Ecchoing Green.”

At a boarding school at the age of eight, Orwell was not pounded by a pestle, but he was whipped with a riding crop until he stopped wetting his bed. Since he did in fact stop, “perhaps this barbarous remedy does work, though at a heavy price, I have no doubt.”
During a lesson, Orwell might be taken out for a beating, right in the middle of construing a Latin sentence; then he would be brought back in, "red-wealed and smarting," to continue.

It is a mistake to think that such methods do not work. They work very well for their special purpose. Indeed, I doubt whether classical education ever has been or can be successfully carried on without corporal punishment. The boys themselves believed in its efficacy.

Orwell mentions a boy who wished he had been beaten more before an examination that he failed. The student did fail though, and I doubt a whipping would have helped him. Orwell succeeded at examinations; but he was at school as a "scholarship boy" only because he had been thought likely to be successful. Moreover, this success was of doubtful value, at least given its price:

Over a period of two or three years the scholarship boys were crammed with learning as cynically as a goose is crammed for Christmas. And with what learning! This business of making a gifted boy’s career depend on a competitive examination, taken when he is only twelve or thirteen, is an evil thing at best, but there do appear to be preparatory schools which send scholars to Eton, Winchester, etc., without teaching them to see everything in terms of marks. At Crossgates the whole process was frankly a preparation for a sort of confidence trick. Your job was to learn exactly those things that would give the examiner the impression that you knew more than you did know, and as far as possible to avoid burdening your brain with anything else.

I quote all of this because it sounds as if American schools today are becoming like Orwell’s “Crossgates,” at least with their standardized examinations. There may be no corporal punishment, though the threat of school closure or reduced funding might be comparable to it.

I return to the Devil’s assertion that folly cannot be beaten out of you. Neither can wisdom—or true academic success—be beaten into you, be it by a stick, or the flooding of the Nile, or frequent examinations for that matter. If the weapons seem efficacious, it is only because their victims had
it in themselves to perform. Is there really no better way to encourage this performance? As one of Blake’s “Proverbs of Hell” runs [7, plate 7],

If the fool would persist in his folly he would become wise.

In the academic context, I interpret this as recommending attention, the avoidance of distractions: distractions such as certain portable electronic devices are designed to be. Another word for what is wanted is—application.

After the finding of the area of an arbitrary polygon by triangulation and application, the dénouement of Book 1 of Euclid is Propositions 47 and 48: the Pythagorean Theorem and its converse.

5. The cone

Again, the cone of Apollonius is ὁ κῶνος, a “pine-cone.” It is a solid figure determined by (1) a base (ἡ βάσις), which is a circle, and (2) a vertex (ἡ κορυφή “summit”), which is a point that is not in the plane of the base. The surface of the cone contains all of the straight lines drawn from the vertex to the circumference of the base. A conic surface consists of such straight lines, not bounded by the base or the vertex, but extended indefinitely in both directions. The Greek for “conic surface” is ἡ κωνικὴ ἐπιφάνεια, the last word meaning originally “appearance” and being the source of the English “epiphany.”

The straight line drawn from the vertex of a cone to the center of the base is the axis (ὁ ἀξίων “axle”) of the cone. If the axis is perpendicular to the base, then the cone is right (ὁρθὸς); otherwise it is scalene (σκαληνός “uneven”). Apollonius considers both kinds of cones indifferently.

A plane containing the axis intersects the cone in a triangle. Suppose a cone with vertex A has the axial triangle ABC shown in Figure 2. Then the base BC of this triangle is a diameter of the base of the cone. Let an arbitrary chord DE of the base of the cone cut the base BC of the axial triangle at right angles at a point F, again as in Figure 2. We are going to cut the cone itself with a plane that contains this chord DE, thus obtaining a conic section.

Figure 2 consists of a “cross section” and “floor plan” of a cone, as if in an architectural drawing. However, it is important to note that the cross section may not be strictly vertical. I shall not attempt an “axonometric projection,”
showing the cone itself, with a conic section drawn along its surface. I am reluctant to spend my own time figuring out how to draw such a projection, using \textsc{PSTricks} (as for the present figures) or another program.

I am not \textit{opposed} to making such projections. I should not mind being able to write a code for drawing them. I have a dream that a sculptor can be inspired to make three-dimensional diagrams for the propositions of Apollonius, using heavy gauge wires perhaps. I have made them myself, using cardboard, as in Figure 3. But there may be pedagogic value in having to construct, in one’s own mind, the vision of a cone and its sections.

One can find wooden models of cones cut by various planes; but I have seen only \textit{right} cones in this way. The beauty of Apollonius is that his cones can be scalene. He is not doing projective geometry though: there is still a distance between any two points, and any two distances have a ratio. Earlier mathematicians knew the ratios that arose from sections of \textit{right} cones; Apollonius understood that the restriction was not needed.

We do not know what Apollonius’s own diagrams looked like. This frees the editors of the Green Lion English edition \cite{4} to provide the best diagrams according to their own judgment. However, Netz recognizes the possibility and value of figuring out what the original diagrams of the Greek mathematicians looked like. They do not necessarily look like what we might draw. Netz himself has initiated the recovery of the diagrams of Archimedes \cite{5}. Given that Heiberg’s edition of Apollonius \cite{2} does \textit{not} give diagrams in the architectural style of Figure 2, we can probably assume that Apollonius

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{cone_diagram.png}
\caption{Axial triangle and base of a cone}
\end{figure}
did not draw them this way. Nonetheless, this manner of drawing may clarify some points.

We drew the chord $DE$ of the base of a cone. Apollonius uses no word for a chord as such, even though he proves in his Proposition 1.10 that the straight line joining any two points of a conic section is a chord, in the sense that it falls within the section. The English words “cord” and “chord” are derived from the Greek χορδή [20]; but this word means gut, hence anything made with gut, be it a lyre-string or a sausage [25].

In the axial triangle in our figure, let a straight line $FG$ be drawn from the base to the side $AC$. Some possibilities are shown in Figure 4. This straight line $FG$ may, but need not, be parallel to the side $BA$. It is not at right angles to the chord $DE$ of the base of the cone, unless the plane of the axial triangle is at right angles to the plane of the base of the cone. In any case, the two straight lines $FG$ and $DE$, meeting at $F$, are not in a straight line with one another, and so they determine a plane. This plane cuts the surface of the cone in such a curve $DGE$ as is shown in Figure 5. So now we have another cross section of our cone, but again, perhaps not a strictly vertical one. Apollonius refers to a curve like $DGE$ as a section (τομή) in the surface of the cone, in Proposition 1.7; later, in 1.10, it is just a section of a cone. All of the chords of this section that are parallel to $DE$ are bisected by the straight line $GF$. Therefore Apollonius calls this straight
line a **diameter** (ἡ διάμετρος [γραμμή]) of the section. The associated verb is διαμετρέω “measure through”; this is the verb used in Homer’s *Iliad* [21, π.315] for what is done to prepare for the single combat of Paris and Menelaus (the reference is in the LSJ lexicon):

But Hector, Priam’s son, and goodly Odysseus first *measured out* (διμέτρεων) a space, and thereafter took the lots and shook them in the bronze-wrought helmet, to know which of the twain should first let fly his spear of bronze.

The parallel chords bisected by the diameter of a conic section are said to be drawn to the diameter **in an orderly way**. Our prepositional phrase stands for the Greek adverb τεταγμένως, from the verb τάσσω, which has meanings like “to draw up in order of battle” or “marshall”—what Cyrus
the Great did in posting his troops outside the walls of Babylon, where the Euphrates entered and exited the city, so that when he had diverted enough of the river’s waters, the troops might enter the city and take it \[19, 1.191]\.

A Greek noun derived from \(\tau\alpha\sigma\sigma\omega\) is \(\tau\alpha\zeta\varsigma\), which is found in English technical terms like “taxonomy” and “syntax” \[26\]. The Latin adverb corresponding to the Greek \(\tau\epsilon\tau\alpha\gamma\mu\epsilon\nu\varsigma\) is \textit{ordinate} (four syllables), from the verb \textit{ordino}. From the Greek expression for “the straight line drawn in an orderly way,” Apollonius will elide the middle part, leaving “the in-an-orderly-way.” This term will refer to \textit{half} of a chord bisected by a diameter. Similar elision in the Latin leaves us with the word \textit{ordinate} for this half-chord \[28\]. In the \textit{Geometry}, Descartes refers to ordinates as “\[lignes\] qui s’appliquent par ordre [au] diametre” \[10, page 328\].

Heath \[1, page clxi\] translates \(\tau\epsilon\tau\alpha\gamma\mu\epsilon\nu\varsigma\) as “ordinate-wise”; Taliaferro \[4, page 3\], as “ordinatewise.” But this usage strikes me as anachronistic. The term “ordinatewise” seems to mean “in the manner of an ordinate”; but ordinates are just what we are trying to define when we translate \(\tau\epsilon\tau\alpha\gamma\mu\epsilon\nu\varsigma\).

I do not know whether the classical \textit{orders} of architecture—the Doric, Ionic, and Corinthian orders—are so called because of the mathematical use of the word “ordinate.” But we may compare the ordinates of a conic section as in Figure 6 with the row of columns of a Greek temple, as in Figure 7.

![Figure 6: Ordinates of a conic section](image)

Back in Figure 5, the point \(G\) at which the diameter \(GF\) cuts the conic section \(DGE\) is called a \textit{vertex} (\(\kappa\omicron\upsilon\rho\acute{f}\iota\varsigma\) as before). The segment of the diameter between the vertex and an ordinate has come to be called in English an \textit{abscissa}; but this just the Latin translation of Apollonius’s Greek for “being cut off”: \(\acute{\alpha} \pi\omicron\omicron\lambda\alpha\mu\beta\alpha\omicron\omicron\nu\omicron\epsilon\eta\). This participle is used in Proposition
1.11 [2, page 38], and its general usage for what we translate as *abscissa* is confirmed in the LSJ lexicon; however, the root sense of the verb is actually not of cutting, but of taking.

Apollonius will show that every point of a conic section is the vertex for some unique diameter. If the ordinates corresponding to a particular diameter are at right angles to it, then the diameter will be an *axis* of the section. Meanwhile, in describing the relation between the ordinates and the abscissas of conic section, there are three cases to consider.

6. The parabola

Suppose the diameter of a conic section is parallel to a side of the corresponding axial triangle. For example, suppose in Figure 8 that $FG$ is parallel to $BA$. The square on the ordinate $DF$ is equal to the rectangle whose sides are $BF$ and $FC$ (by Euclid’s Proposition III.35). More briefly,

$$DF^2 = BF \cdot FC.$$  

But $BF$ is independent of the choice of the point $D$ on the conic section. That is, for any such choice (aside from the vertex of the section), a plane
containing the chosen point and parallel to the base of the cone cuts the cone in another circle, and the axial triangle cuts this circle along a diameter, and the plane of the section cuts this diameter at right angles into two pieces, one of which is equal to $BF$. The square on $DF$ thus varies as $FC$, which varies as $FG$. That is, the square on an ordinate varies as the abscissa (Apollonius 1.20). By means of Euclid's Proposition 1.43, used as in 1.44 as discussed earlier, we obtain a straight line $GH$ such that

$$DF^2 = FG \cdot GH;$$

and $GH$ is independent of the choice of $D$.

This straight line $GH$ can be conceived as being drawn at right angles to the plane of the conic section $DGE$. Therefore Apollonius calls $GH$ the **upright side** ($\delta \rho \theta \iota \alpha$ $[\pi \lambda \varepsilon \upsilon \rho \acute{\alpha}]$), and Descartes accordingly calls it *le costé droit* [10, page 329]. Apollonius calls the conic section itself Η $\pi \alpha \rho \alpha \beta \omega \lambda \eta$; we transliterate this as **parabola**. The Greek word is also the origin of the English “parable,” but can have various related meanings, like “juxtaposition, comparison, conjunction, application.” The word is self-descriptive, being itself a juxtaposition of the preposition $\pi \alpha \rho \acute{\alpha}$ “along, beside” and the noun Η $\beta \omega \lambda \eta$ “throw.” Alternatively, $\pi \alpha \rho \alpha \beta \omega \lambda \eta$ is a noun derived from the verb $\pi \alpha \rho \alpha \beta \omega \lambda \omega$, which as suggested earlier is $\pi \alpha \rho \acute{\alpha}$ plus $\beta \alpha \lambda \omega$ “throw.” In the parabola of Apollonius, the rectangle $FH$ bounded by the abscissa $FG$ and the upright side $GH$ is the result of *applying*, to the upright side, the square whose side is the ordinate $DF$. 
7. The latus rectum

The Latin term for the upright side is latus rectum. This term is also used in English. In the Oxford English Dictionary \([28]\), the earliest quotation illustrating the use of the term is from a mathematical dictionary published in 1702. Evidently the quotation refers to Apollonius and gives his meaning:

App. Conic Sections 11 In a Parabola the Rectangle of the Diameter, and Latus Rectum, is equal to the rectangle of the Segments of the double Ordinate.

Apparently “diameter” was used here, rather than “abscissa.” The “segments of the double ordinate” are presumably the two halves of a chord, so that each of them is what we are calling an ordinate, and the rectangle contained by them is equal to the square on one of them.

The possibility of defining the conic sections in terms of a directrix and focus is shown by Pappus \([32, \text{vii.312–8, pages 1004–15}]\) and was presumably known to Apollonius. Pappus does not use such technical terms though; there is just a straight line and a point, as in the following, a slight modification of Thomas’s translation \([42, \text{pages 492–503}]\):

If \(AB\) be a straight line given in position [Figure 9], and the point \(\Gamma\) be given in the same plane, and \(\Delta\Gamma\) be drawn, and \(\Delta E\) be drawn perpendicular [to \(AB\)], and if the ratio of \(\Gamma\Delta\) to \(\Delta E\) be given, then the point \(\Delta\) will lie on a conic section.

As Heath explains \([1, \text{pages xxxvi–xl}]\), Pappus proves this theorem because Euclid did not supply a proof in his treatise on surface loci. (This treatise
itself is lost to us.) Euclid must have omitted the proof because it was already well known; and Euclid predates Apollonius. Kline summarizes all of this by saying that the focus-directrix property “was known to Euclid and is stated and proved by Pappus” [24, page 96]. Later (on his page 128), Kline gives a precise reference to Pappus: it is Proposition 238, in Hultsch’s numbering, of Book vii. Actually this proposition is a recapitulation, which is incomplete in the extant manuscripts; one must read a few pages earlier in Pappus for more details, as in the selection in Thomas’s anthology. In any case, Kline says, “As noted in the preceding chapter, Euclid probably knew” the proposition. According to Boyer however, “It appears that Apollonius knew of the focal properties for central conics, but it is possible that the focus-directrix property for the parabola was not known before Pappus” [8, §xi.12, page 211].

I modified Thomas’s translation of Pappus by putting “the ratio of $\Gamma\Delta$ to $\Delta\varepsilon$” where Thomas has “the ratio $\Gamma\Delta : \Delta\varepsilon$.” Pappus uses no special notation for a ratio as such, but refers merely to $\lambda\gamma\varsigma\cdot\cdot\cdot \tau\iota\varsigma \Gamma\Delta \pi\rho\varsigma \Delta\varepsilon$. The recognition of ratios as individual mathematical objects (namely positive real numbers) distinguishes modern from ancient mathematics, although the beginnings of this recognition can be seen in Pappus, as Descartes observed.

A modern textbook may define the parabola in terms of a directrix and focus, explicitly so called. An example is Nelson, Folley, and Borgman, Analytic Geometry [29]. I have this book because my mother used it in college. I perused it at the age of 12 when I wanted to understand how curves could be encoded in equations. Dissatisfaction with such textbooks leads me back to the Ancients. According to Nelson et al.,

The chord of the parabola which contains the focus and is perpendicular to the axis is called the *latus rectum*. Its length is of value in estimating the amount of “spread” of the parabola.

The first sentence here defines the *latus rectum* as a certain straight line that is indeed equal to Apollonius’s upright side. The second sentence correctly describes the significance of the *latus rectum*. However, the juxtaposition of the two sentences may mislead somebody, like me, who knows just a little Latin. I suppose Latin was more commonly studied when Nelson et al. were writing, and I have the idea that they put the word “spread” in quotation marks to suggest that it is a translation of the Latin *latus*. The Latin adjective *latus*, -a, -um does mean “broad, wide; spacious, extensive”
it is the root of the English noun “latitude.” An extensive \textit{latus rectum} does mean a broad or spreading parabola. However, the Latin adjective \textit{latus} is unrelated to the noun \textit{latus}, -eris “side; flank,” which is found in English in the adjective “lateral”; and the noun \textit{latus} is what is used in the phrase \textit{latus rectum}.

It is possible that Nelson et al. put the word “spread” in quotation marks simply because, writing in 1949, they felt its use as a noun to be colloquial. It is not a noun in the \textit{OED}: this has only two entries for “spread,” as a verb and as a participial adjective. The section “Speech–Spring” of the dictionary was first published in September, 1914. The 1976 sixth edition of the \textit{Concise Oxford Dictionary} does list “spread” as a noun, without the tag of being colloquial or being otherwise restricted. Somewhere in between then, the nominal use of “spread” must have occurred only colloquially, and this may have been when Nelson et al. were writing.

Again, it is also possible that they mistakenly connected \textit{latus rectum} with “latitude.” Finally, it is possible that they knowingly and deliberately made the connection, being happy to suggest a false etymology to their students if it would help them. The students would be studying calculus, after all, not classics. In that case though, the students would still be better served with the correct mathematical etymology of \textit{latus rectum}.

Perhaps it was felt that introducing a third dimension to the students’ thoughts at this stage would be confusing. The last four of the fifteen chapters of Nelson et al. are on solid analytic geometry; but on the blank page before they begin, I see that my mother has written “THE END!” in her copy. Analytic geometry was the last mathematics course that she took at her liberal arts college, and apparently her course covered only plane geometry. Perhaps this was because, indeed, it was thought too hard to teach students to connect an equation

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}
\]

with a certain surface called a hyperbolic paraboloid. In that case though, why not teach the students to visualize cones, even with the aid of solid models, and to understand what a \textit{latus rectum} really is, rather than to manipulate equations?

There is even more that can be said about the Latin words spelled \textit{latus}; I give it here for completeness. In the noun phrase \textit{latus rectum}, the adjective
rectus, -a, -um “straight, upright” is given the neuter form, because the noun latus is (despite appearances) neuter. The plural of latus rectum is latera recta. The neuter plural of the adjective latus would be lata. The dictionary writes the adjective as lātus, with a long A; but the A in the noun is unmarked and therefore short. As far as I can tell, the adjective is to be distinguished from another Latin adjective with the same spelling (and the same long A), but with the meaning of “carried, borne.” This adjective is used for the past participle of the verb fero, ferre, tulī, lātum. The past participle appears in English in words like “translate,” while fer- appears in “transfer.” According to the American Heritage Dictionary [26],

lātus “broad” comes from the Indo-European root stel- and has the English derivatives “latitude” and “dilate”;

lātus “carried” comes from an Indo-European root tel- and is found in English words like “translate” and “relate,” but also “dilatory.”

Thus “dilatory” is not to be considered as a derivative of “dilate.” A French etymological dictionary [11] implicitly confirms this under the adjacent entries dilater and dilatoire. The older English etymological dictionary of Skeat [38] does give “dilatory” as a derivative of “dilate.” However, under “latitude,” Skeat traces lātus “broad” to the Old Latin stlātus, while under “tolerate” he traces lātum “borne” to tlātum. In his introduction, Skeat says he has collated his dictionary “with the New English Dictionary [as the Oxford English Dictionary was originally called] from A to H (excepting a small portion of G).” So Skeat should know what the OED has to say about “dilate.” In fact the OED distinguishes two English verbs “dilate,” one for each of the Latin adjectives lātus. But concerning the meanings of these verbs, the dictionary notes,

The sense ‘prolong’ comes so near ‘enlarge’, ‘expand’, or ‘set forth at length’…that the two verbs were probably not thought of as distinct words.

Thus it is possible for an etymological purist to get carried away, making distinctions without a difference. Still, the adjective or adjectives lātus have nothing to do with the noun latus.

So much for etymology then. Denoting abscissa by \(x\), and ordinate by \(y\), and latus rectum by \(ℓ\), we have for the parabola the modern equation

\[
y^2 = ℓx. \tag{1}\]
The letters here can be considered as numbers in the modern sense, or just as line segments, or congruence-classes of segments.

8. The hyperbola

The second possibility for a conic section is that the diameter meets the other side of the axial triangle when this side is extended beyond the vertex of the cone. In Figure 10, the diameter $FG$, crossing one side of the axial triangle $ABC$ at $G$, crosses the other side, extended, at $K$. Again $DF^2 = BF \cdot FC$; but the latter product now varies as $KF \cdot FG$. The upright side $GH$ can now be defined so that

$$BF \cdot FC : KF \cdot FG :: GH : GK.$$ 

We draw $KH$ and extend to $L$ so that $FL$ is parallel to $GH$, and we extend $GH$ to $M$ so that $LM$ is parallel to $FG$. Then

$$FL : KF :: FL \cdot FG : KF \cdot FG,$$

but also

$$FL : KF :: GH : GK.$$ 

Eliminating common ratios from the three proportions yields

$$FL \cdot FG : KF \cdot FG :: BF \cdot FC : KF \cdot FG,$$
and so $FL \cdot FG = BF \cdot FC$. Thus

$$DF^2 = FG \cdot FL.$$  

Apollonius calls the conic section here an **hyperbola** ($\eta \upsilon\pi\epsilon\rho\beta\omicron\lambda\eta$), that is, an **excess**, an **overshooting**, a **throw** ($\beta\omicron\lambda\eta$) **beyond** ($\upsilon\pi\epsilon\rho\rho\omicron\lambda\omega$), because the square on the ordinate $DF$ is equal to a rectangle $FM$ whose one side is the abscissa $FG$, and whose other side $GM$ is applied along the upright side. But the applied rectangle $FM$ actually **exceeds** ($\upsilon\pi\epsilon\rho\beta\omicron\omicron\lambda\lambda\omega$) the rectangle contained by the abscissa $FG$ and the upright side $GH$ itself. The **excess** rectangle—$MN$ in the figure—is similar to the rectangle contained by the upright side $GH$ and $GK$.

Apollonius calls $GK$ the **transverse side** ($\eta \pi\lambda\alpha\gamma\acute{i} \pi\lambda\epsilon\upsilon\rho\acute{k}$) of the hyperbola. Denoting it by $a$, and the other segments as before, we have the modern equation

$$y^2 = \ell x + \frac{\ell}{a} x^2. \quad (2)$$

9. **The ellipse**

The last possibility is that the diameter meets the other side of the axial triangle when this side is extended below the base. All of the computations will be as for the hyperbola, except that now, if it is considered as a **directed** segment, the transverse side is negative, and so the modern equation is

$$y^2 = \ell x - \frac{\ell}{a} x^2. \quad (3)$$

In this case Apollonius calls the conic section an **ellipse** ($\eta \xi\lambda\lambda\epsilon\upsilon\rho\omicron\lambda$), that is, a **falling short**, because again the square on the ordinate is equal to a rectangle whose one side is the abscissa, and whose other side is applied along the upright side: but this rectangle now **falls short** ($\xi\lambda\lambda\epsilon\omicron\pi\omega$) of the rectangle contained by the abscissa and the upright side by another rectangle. Again this last rectangle is similar to the rectangle contained by the upright and transverse sides.

Apollonius concludes Book I concludes by showing that every curve given by an equation of one of the forms (1), (2), and (3) is indeed a conic section. This may be true for us today, by definition; but Apollonius finds a cone of which the given curve is indeed a section.
10. Descartes

We have seen that the terms “abscissa” and “ordinate” are ultimately translations of Greek words that describe certain line segments determined by points on conic sections. For Apollonius, an ordinate and its corresponding abscissa are not required to be at right angles to one another.

Descartes generalizes the use of the terms slightly. In one example [10, page 339], he considers a curve derived from a given conic section in such a way that, if a point of the conic section is given by an equation of the form

$$y^2 = \ldots x \ldots,$$

then a point on the new curve is given by

$$y^2 = \ldots x' \ldots,$$

where $xx'$ is constant. But Descartes just describes the new curve in words:

"toutes les lignes droites appliquées par ordre a son diametre estant esgales a celles d’une section conique, les segmens de ce diametre, qui sont entre le sommet & ces lignes, ont mesme proportion a une certaine ligne donnée, que cete ligne donnée a aux segmens du diametre de la section conique, auquels les pareilles lignes sont appliquées par ordre."

All of the straight lines drawn in an orderly way to its diameter being equal to those of a conic section, the segments of this diameter that are between the vertex and these lines have the same ratio to a given line that this given line has to the segments of the diameter of the conic section to which the parallel lines are drawn in an orderly way.

In particular, the new curve has ordinates, namely les lignes droites appliquées par ordre a son diametre. These ordinates have corresponding abscissas, which are les segmens de ce diametre, qui sont entre le sommet & ces lignes. There is still no notion that an arbitrary point might have two coordinates, called abscissa and ordinate respectively. A point determines an ordinate and abscissa only insofar as the point belongs to a given curve with a designated diameter.
The Wikipedia (en.wikipedia.org) articles “ordinate” and “abscissa” do not explain the origins of these terms (at least as of October 4, 2014). This is unfortunately true of many Wikipedia articles on mathematics. Of course, anybody who cares may work to change this (as I did, for example, in adding the section “Origins” to the article “Pappus’s hexagon theorem”).

The Wikipedia “ordinate” article does have a reference to the website Earliest Known Uses of Some of the Words of Mathematics (maintained by Jeff Miller, jeff560.tripod.com/mathword.html); but this site currently provides only a modern history of “ordinate” and “abscissa.” It is said that Descartes did not use the latter term, but did use the former; however, the quotation above suggests that he did not even use the term “ordinate” as such, but used only the longer phrase, derived from Apollonius, that was presumably the precursor of this term.

It appears that the use of “ordinate” and “abscissa” as technical terms is due to Leibniz. The articles on Miller’s website refer to Struik in A Source Book in Mathematics, 1200–1800, where a footnote [40, page 272, note 1] to an article of Leibniz reads,

Note the Latin term abscissa. This term, which was not new in Leibniz’s day, was made by him into a standard term, as were so many other technical terms. In the article “De linea ex lineis numero infinitis ordinatim ductis inter se concurrentibus formata . . . ,” Acta Eruditorum 11 (1692), 168–171 (Leibniz, Mathematische Schriften, Abth. 2, Band I (1858), 266–269), in which Leibniz discusses evolutes, he presents a collection of technical terms. Here we find ordinata, evolutio, differentiare, parameter, differentiabilis, functio, and ordinata and abscissa together designated as coordinatae. Here he also points out that ordinates may be given not only along straight but also along curved lines. The term ordinate is derived from rectae ordinatim applicatae, “straight lines designated in order,” such as parallel lines. The term functio appears in the sentence: “the tangent and some other functions depending on it, such as perpendiculars from the axis conducted to the tangent.”

Presumably Leibniz (1646–1716) knew Books I–IV of Apollonius’s Conics from Commandino’s 1566 Latin translation of them. Arabic manuscripts of the later books were brought to Europe, starting later in the 16th century;
but the first proper Latin translation did not appear until Halley’s 1710 edition [3; xxi–xxv]. In any case, all we have needed for the present article is Book I.

It was noted in the beginning that the original Greek of Books V–VII of Apollonius has been lost, and Book VIII exists in no language at all. Terminology based on Apollonius survives, thanks apparently to Leibniz; but a proper understanding of the terminology has not generally survived. This is unfortunate, but can change.

References


