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Review: Hankel and Toeplitz Transforms on H^1: Continuity, Compactness and Fredholm Properties

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Hankel and Toeplitz transforms on H^1: continuity, compactness and Fredholm properties. (English summary)

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The authors discuss boundedness and compactness of Hankel and Toeplitz operators acting on the Hardy space H^1. Among other things, they provide a new proof (Theorem 1.5) of the fact that a Hankel operator H_a having symbol a is bounded on H^1 if and only if a has bounded logarithmic mean oscillation (recall that a function f belongs to $\mathrm{BMO}_{\log}$ if $f \in L^1$ and

$$\|f\|_{**} = \sup_I \frac{\log\left(\frac{4\pi}{|I|}\right)}{|I|} \int_I |f(\zeta) - f_I||d\zeta| < \infty$$

where the supremum runs over all arcs I of the unit circle $\mathbb{T}$). The original result, and a number of special cases, are due to several authors [J. A. Cima and D. A. Stegenga, in *Analysis at Urbana, Vol. I (Urbana, IL, 1986–1987)*, 133–150, Cambridge Univ. Press, Cambridge, 1989; MR1009172 (90g:47049); S. Janson, J. Peetre and S. W. Semmes, Duke Math. J. **51** (1984), no. 4, 937–958; MR0771389 (86m:47033); D. A. Stegenga, Amer. J. Math. **98** (1976), no. 3, 573–589; MR0420326 (54 #8340); V. A. Tolokonnikov, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **141** (1985), 165–175, 191; MR0788897 (86j:47038)].

Moreover, the authors also prove (Theorem 1.6) that the operator norm of H_a acting on H^1 is comparable to $\|P_1 a\|_{\mathrm{BMO}_{\log}}$. Here P_1 denotes the (unbounded) truncation operator

$$P_1 f(\zeta) \sim \sum_{n=1}^{\infty} \widehat{f}(n)\zeta^n.$$

Concluding their treatment of Hankel operators on H^1, the authors show that H_a is compact on H^1 if and only if $P_1 a \in \mathrm{VMO}_{\log}$ (Theorem 1.7). Here the subspace $\mathrm{VMO}_{\log}$ of $\mathrm{BMO}_{\log}$ is defined to be the set of all $f \in L^1$ such that

$$\lim_{\delta\to 0^+} \sup_{|I|<\delta} \frac{\log\left(\frac{4\pi}{|I|}\right)}{|I|} \int_I |f(\zeta) - f_I||d\zeta| = 0.$$

The second part of the article deals with the spectral properties of Toeplitz operators. In particular, the authors prove a theorem (Theorem 1.8) for $p = 1$ motivated by the well-known result of R. G. Douglas [Bull. Amer. Math. Soc. **74** (1968), 895–899; MR0229070 (37 #4648)] concerning the Fredholm properties of Toeplitz operators on H^2 with symbols $a \in C + \overline{H^\infty}$. Since the criteria for boundedness and compactness are different in the cases $p = 1$ and $1 < p < \infty$, it turns out that C must be replaced by $C \cap \mathrm{VMO}_{\log}$ and H^∞ must be replaced by $H^\infty \cap \mathrm{BMO}_{\log}$ in order for the analogous statements to go through.

Reviewed by *Stephan R. Garcia*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.