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Book Review: *Love and Math: The Heart of Hidden Reality* by Edward Frenkel

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Synopsis

This review traces Edward Frenkel's attempt to convey the excitement of mathematical research to a popular audience. In his expositions and explanations of his own research program, he shows how processes of mathematical discovery depend on the juxtaposition of various iconic and symbolic modes of representation as disparate fields of research (in this case algebraic number theory and complex analysis) are brought together in the service of problem solving. And he shows how crucial the encouragement of various older mathematicians was to his own development, as they guided his choice of problems, and served as inspiration. Along the way he gives an accessible description of the Langlands program.

Love and Math: The Heart of Hidden Reality. By Edward Frenkel, Basic Books, 2014. (ISBN-10: 0465050743. ISBN-13: 9780475050741. 304 pages.)

Nobody seemed to think much about the deeper correspondences between mathematics and the processes of nature before about 1600, when Kepler, Galileo and Descartes launched the balloon and Huygens, Leibniz, Newton, the Bernoullis and Euler attached a gondola to it so that we humans could be mathematical dreamers of all we survey, floating between earth and heaven. About fifteen years ago, I had to re-learn group theory and discover representation theory in order to write about molecules, and a bit later I read and taught Bas van Fraassen's *Laws and Symmetry* (Oxford University Press, 1990). So the best answer I now know to the question just posed is not only that discrete things (like apples) can be counted, and continua (like farmer's fields) can be measured. Such correspondences get arithmetic and geometry going, but that's only the beginning. Another key insight is also that things and systems–both natural and formal–have symmetries and (since periodicity is symmetry in time) so do natural processes! Carbon molecules as they throb, snowflakes as they form, and solar systems as they rotate exhibit symmetries and periodicities that are key to understanding what they are and how they work. Moreover, symmetry and periodicity are a kind of generalization of identity; they are the hallmark of stable existence.

What!?! You might rightly ask. Here is a short explanation. If you take a square, and rotate it 90 degrees without moving its center-point, there it is again: just the same. If you rotate it 180 degrees, or 270 degrees or back around 360 degrees, there it is again, just the same. These four rotational transformations of the square return it to a state indiscernible from the original state; and this mathematical "identity" is a kind of abstract schema (when scientists build the conceptual models that mediate between mathematics and the real world) for the stability and self-sameness and even the existence of things: carbon atoms, snowflakes and solar systems. Existence, in one important sense, is invariance under transformation. And, of course, the four rotations of the square also form a finite group.

The definition of an abstract group is deceptively simple, but it was the catalyst for a great number of 19^{th} and early 20^{th} century mathematical developments. Arithmetic is reorganized by the notion of congruence group, which then precipitates (with some help from topology) the theory of p-adic numbers; algebra is reorganized by Galois groups, geometry by Klein's symmetry groups, and analysis and topology by Lie groups. Some groups are finite, but some are infinite: the group of rotational symmetries of the circle, for example, is infinite. Fix a circle at its center: no matter what angle you rotate it through, there it is again, indiscernibly the same, and of course there are an infinite number of possible angles between 0 and 360 degrees. Some groups are discrete and some are continuous; in fact, you can treat a Lie group like a (smooth) differentiable manifold, the canonical object of algebraic topology. And with a bit of vector space magic, any group of symmetries can be nicely mapped to a group of matrices, which make calculation if not a breeze at least tractable. This is the central strategy of representation theory, and yet another example of the magic of mathematics.

Even if you already love mathematics, you should read the book I am reviewing here, *Love and Math.* Edward Frenkel, when he was a schoolboy in the old Russian city of Kolomna, two hours by train from Moscow, didn't love mathematics. He was however fascinated by the strange world of quantum mechanics. At one crucial juncture a friend of his parents (Evgeny Evgenievich, a mathematics professor in Kolomna's one small college), explained to him that what underlay the discovery of the quark was group theory, handed him three books about symmetry groups, p-adic numbers and topology, and invited him to come over once a week and ask questions. Frenkel writes, "I was instantly converted."

That observation is the last sentence of Chapter 1. The first page of Chapter 2, "The Essence of Symmetry," has attractive photographs of a snowflake and a butterfly; two pages later the square I was just talking about shows up, existent, "invariant under transformation," that is, under the group of four rotational symmetries, right there on page 17! One especially nice feature of this book is the illustrations that dramatize and personify the step-by-step, straightforward exposition of the mathematical ideas that Frenkel went on to fall in love with. We often think of mathematics as a collection of proofs, written out in sober prose, each result deduced from a set of already established principles. When you learn how to prove a theorem in geometry class in high school, or how to derive one proposition from a given set of propositions while studying predicate logic in college, it's mildly exciting. But the overall framework is discouraging: your "discovery" is just a matter of calculation and a bit of clever insight.

Real mathematics, the program of mathematical research that Frenkel was swept away by–on waves that carried him to Moscow, Harvard, Princeton and Berkeley–isn't like that, as working mathematicians know. There are two important features of mathematics that he conveys very well in this book, which make it erotic in both of Plato's senses (go re-read the *Phaedrus*), and which twentieth century philosophy of mathematics as well as a great deal of classroom instruction fail to register. The first is the importance of iconic modes of representation (images, graphs, tables) to mathematical understanding, and more generally the rich superpositions and juxtapositions that occur when different modes of representation work in tandem. The second is the importance of narrative, the history of mathematics, stories about enlightenment: how problems are solved and new ideas born in individuals and clumps of mathematicians, sometimes located in the same city and sometimes widely dispersed. New solutions to problems typically go far beyond the mathematical context that gave rise to them, revising the meaning of earlier results and adding new concepts, methods and procedures. Mathematical problem-solving is ampliative, and for that very reason can't be fitted into the neat packaging of deduction from already available rules.

So if you look at Chapter 5, you will see strange but illuminating diagrams of braid groups, subjects of the first important mathematical puzzle that Frenkel solved while still at university in Moscow. He was studying at the Gubkin Institute of Oil and Gas, which received talented mathematics students who had been denied entry during the years that Moscow State University was implementing a strict anti-Semitic policy. (The story of how he "failed" the entrance exam is very dramatic.) One of his professors at the Gubkin Institute, noting his exceptional talent, put him in touch with the distinguished mathematician Dmitry Fuchs, who gave him the problem; after he solved it, it was published by Israel Gelfand in his mathematics journal. Gelfand was one of the greatest mathematicians of the twentieth century–he was awarded the Kyoto Prize–and was also sidelined by Moscow State University; like Fuchs, he emigrated a few years later to the United States.

The next problem that Frenkel worked on with Fuchs launched him into the Langlands Program. Andrew Wiles' 1995 proof of Fermat's Last Theorem is one of the harbingers of this important research program, but a clear, general, not-too-technical account of that program is not easy to find. Frenkel does an exceptional job of explaining some of the basic ideas in accessible terms in Chapters 7, 8 and 9. In those chapters and the copious, helpful notes that accompany his explanations, you will find a right triangle, various number series, generating functions, a lovely symmetry group on the disc, three famous Riemann surfaces (the sphere, the torus, and the Danish pastry–okay, that's not its technical name), a number line, a complex number represented as a point on the plane, and two more tori (donuts) with paths, as well as a chart of permutations, a table, lots more equations, and various "paths" to show what a fundamental group on a Riemann surface is. The mix of representations, so essential to Frenkel's explanations, is like Wallace Stevens' "Thirteen Ways of Looking at a Blackbird," which is in its own way explanatory though the truths are different. ("I do not know which to prefer.../ The blackbird whistling / Or just after.")

But what is the Langlands Program? One way to understand how and why the solution of problems in mathematics adds content is to recall the ancient method of analysis. (The philosopher Carlo Cellucci pursues this recollection especially well in his recent book, *Rethinking Logic* (Springer 2013).) In order to solve a problem, find another problem–so far unsolved–and show that if it were solved, it would guarantee the solution of the original problem. Reduce one problem to another, even more difficult, problem! This is just what happened with Fermat's Last Theorem, which states that the equation $x^n + y^n = z^n$, where $xyz \neq 0$, has no integer solutions when n is greater than or equal to 3. In 1990, Ken Ribet proved that its truth would follow from the truth of the Taniyama-Shimura-Weil Modularity Conjecture; or rather, if certain counter-examples to Fermat's Last Theorem existed, they would also contradict the Modularity Conjecture, so the truth of the latter would ensure the non-existence of those counter-examples. What Wiles actually proved in his famous hundred-page proof (*Annals of Mathematics* (1995), 443-551) was the truth of a version of the Modularity Conjecture: a "semi-stable" elliptic curve always corresponds in a precise way to a certain kind of modular form. And that conjecture, in turn, follows from the Langlands correspondence. But what is that? On even the simplest but still accurate account, it involves a fourfold correspondence: the Langlands program involves both the local and the global, as well as number theory and the study of automorphic representations. So there are two things to explain, the interplay of the local and the global, and the interplay of algebraic number theory (where elliptic curves live) and complex analysis (where modular forms live).

Apropos the local and the global: late nineteenth and early twentieth century mathematics are transformed by topology. Let's go back to the sphere (and leave aside the pastry), a canonical Riemann surface that nicely illustrates the novel perspective that topology offers. The important insight is that locally a sphere (as every topologist knows) is very like the plane: if you choose any point on a sphere and look at a sufficiently small neighborhood around that point, you find that it is almost flat. But of course globally a sphere is very different from the plane. In topology we think of the sphere as a certain well-constructed collection of small neighborhoods, so that if you specify the neighborhoods and their mutual relations carefully enough, you can retrieve important global features of the sphere (or any other Riemann manifold) on the basis of the local information you've collected. This approach produces important insights. First, it gives rise to a novel and much more abstract concept of distance. The ordinary conception of distance leads to the "completion" of the rationals $\mathbb{Q}$ by the real numbers $\mathbb{R}$. But another conception, which is based on group theory and the concept of congruence, leads to a whole family of completions of $\mathbb{Q}$, the p-adic numbers, one such field for each prime number p. Second, topology introduces a subtle understanding of the interplay between the local and the global in mathematics. In general it is quite difficult to assemble local results and derive a global result, but very rewarding when you succeed. Mid-twentieth century mathematicians, trying to exploit this insight in number theory, introduced the notions of adèle rings and idèle groups to formulate "class field theory," in a way that clearly exhibits and exploits local-global relations. Third, it leads mathematicians to think of almost anything as a "space," even a collection of discrete things or a highly infinitary collection. The rational numbers (and its finite extensions, algebraic number fields) are called global fields, as are function fields in one variable over finite fields; the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, and the p-adic number systems are called local fields, because of their topological properties.

If you carefully adjoin $i = \sqrt{-1}$ to the integers $\mathbb{Z}$ you get the Gaussian integers, and if you carefully adjoin i to the rational numbers $\mathbb{Q}$, you get my favorite algebraic number field, $\mathbb{Q}[i]$. To construct an algebraic number field, you start with $\mathbb{Q}$ and then adjoin a finite set of elements not in $\mathbb{Q}$, like i or $\sqrt{2}$ or $\sqrt{-3}$, making sure that the new elements (like $n + im$, for example) obey the usual algebraic rules so that you can still call it a field; within this new system, you then try to find the analogue of units, integers, and prime numbers. There are a lot of these new fields, they are quite mysterious, and number theorists love to investigate them because they explain many of the odd properties of ordinary sheepish integers and the polynomials that organize them like benevolent sheepdogs. And Galois plays the role of shepherd, historically, because if you look at the automorphisms of an algebraic number field which map $\mathbb{Q}$ to itself but permute the adjoined elements, they form a Galois group! In Chapter 7, Frenkel explains that the Galois group is precisely the symmetry group of an algebraic number field–since formal systems, like snowflakes, have symmetries. You can see the possibilities for more problem reduction here, referring problems about the natural numbers to problems about these new fields.

Just as you can do calculus on the real line $\mathbb{R}$ (if you're studying the derivatives or integrals of curves) or the Euclidean plane $\mathbb{R}^2$ (if you're studying surfaces), you can do it on the complex numbers $\mathbb{C}$, which Gauss realized was also beautifully represented by the plane. The great thing about complex analysis is that, while it is rather easy for a function of one variable (a curve) to be differentiable at a point in real analysis, because you only have to come in from two directions, it is really hard in complex analysis, because you have to come in from every angle. This means that any analytic function rising to that standard is so well-behaved that it has angelic properties: complex analysis is the *Paradiso* of ordinary calculus.

Now you only need to keep in mind that elliptic curves live in algebraic number theory and modular forms live in complex analysis. In Chapter 8, Frenkel shows in remarkably precise and vivid terms one version of the correspondence between them, the version pertinent to Fermat's Last Theorem. First he explains how, if you begin with $\mathbb{N}$, the natural numbers $0, 1, 2, 3, 4, 5, 6, \ldots$ and impose an p-fold periodicity on it (where p is a prime number), you get a finite number field, $\mathbb{F}_p$. For example, if we say that any two natural numbers are equivalent if their difference is a multiple of five (that is, we mod out base 5), we get five infinite equivalence classes of number which we can label 0, 1, 2, 3 and 4. When addition and multiplication of these classes are suitably defined, with 0 as the additive identity and 1 as the multiplicative identity, because 5 is a prime number, the resulting structure is not just a ring but indeed a field.

Following Frenkel, we examine the behavior of an important kind of algebraic equation in two variables, our long-awaited elliptic curve. For instance, consider

$$y^2 + y = x^3 - x. \tag{1}$$

We can look for solutions to this equation mod p, that is, we can ask how many solutions it will have in a given finite number field $\mathbb{F}_p$. Call the numbers which (with a bit of legerdemain) record these totings up, a_p for each p. Just as, quite amazingly, there is an "effective formula" for generating the Fibonacci numbers, so there is a generating function (discovered by Martin Eichler in 1954) that produces an infinite sum

$$q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13} + \ldots$$

which is equally amazing. If you inspect the sequence of its coefficients b_n you discover that for each p, $b_p = a_p$. This is the kind of discovery that makes mathematicians very happy, but of course has to lead to further analysis in the sense of problem reduction: what in the world can account for this correspondence? You start with an elliptic curve and get the sequence of a_p's and then you examine a modular form (that's what the Eichler function is) and get the sequence of b_p's and they're just the same! Damn. We demand an explanation! This is why the problem reduction doesn't just stop there, but refers the Taniyama-Shimura-Weil conjecture upwards to the Langlands conjectures.

If you're interested, you can look at Frenkel's technical summary of this problem reduction on page 8 of his textbook *Langlands Correspondence for Loop Groups* (Cambridge University Press, 2007) and then follow it upwards. He writes, "Many interesting representations of Galois groups can be found in 'nature.' For example, the group Gal($\mathbb{Q}/\mathbb{Q}$) will act on the geometric invariants (such as the étale cohomologies) of an algebraic variety defined over $\mathbb{Q}$. Thus, if we take an elliptic curve E over $\mathbb{Q}$, then we will obtain a two-dimensional Galois representation on its first étale cohomology. This representation contains a lot of important information about the curve E, such as the number of points of E over $\mathbb{Z}/\mathbb{Z}_p[= \mathbb{F}_p]$ for various primes p. The Langlands correspondence is supposed to relate these Galois representations to automorphic representations of $GL_2(A_{\mathbb{F}})$ [a certain group of 2 by 2 matrices with entries from the ring of adèles over the field $\mathbb{F}$, a concept invented to simplify class field theory], in such a way that the data on the Galois side, like the number of points of $E(\mathbb{Z}/\mathbb{Z}_p)$, are translated into something more tractable on the automorphic side, such as the coefficients in the q-expansion of the modular forms that encapsulate automorphic representations of $GL_2(A_{\mathbb{Q}})$. This leads to some startling consequences, such as the Taniyama-Shimura conjecture." That's what he manages to exhibit, in plain English with lots of pictures, in Chapter 8, fittingly entitled "Magic Numbers."

This result, however, belongs to the "global" part of the Langlands program; Frenkel's most important work engages the "local" Langlands correspondence. In Chapter 9, Frenkel announces a very important idea; it takes him the whole last half of the book to explain its meaning. He writes, "A deep insight of [André] Weil was that the most fundamental object here is an algebraic equation... Depending on the choice of the domain where we look for solutions, the same equation gives rise to a surface, a curve, or a bunch of points. But these are nothing but the avatars of... the equation itself." And he continues: "The connection between Riemann surfaces and curves over finite fields should now be clear: both come from the same kind of equations, but we look for solutions in different domains, either finite fields or complex numbers. On the other hand, 'any argument or result in number theory can be translated, word for word,' to curves over finite fields, as Weil puts it in his letter..." (pp. 103-104) Curves on finite fields are the "middle terms" which mediate between number theory and Riemann surfaces. But these objects belong to the "local" side of the Langlands correspondences: there are two kinds of "non-Archimedean" local fields, finite extensions of a p-adic number field, and the field of formal Laurent series $\mathbb{F}_q((T))$ where $\mathbb{F}_q$ is a finite field.

Chapters 10 through 17 explore this research, which launched him from Moscow (where he worked with Fuchs and Boris Feigin) to Harvard and MIT (where he met Victor Kac and Vladimir Drinfeld, as well as Barry Mazur and John Tate) and Princeton (where he met Robert Langlands himself). Much of the material here is really too difficult to be conveyed in a popular exposition, but he does his best and includes helpful adumbrations in a series of long footnotes. Mathematically trained readers can refer to some of the articles and texts cited in those footnotes, as well as Frenkel's textbook mentioned above. His quest led him through the rugged but inspiring terrain of infinite dimensional Lie groups known as loop groups, Lie algebras, and Kac-Moody algebras (extensions of the Lie algebra of a loop group by a one-dimensional space), branes, fibrations, and sheaves, all in the context of more and more Riemann surfaces: see the pictures in Chapter 17, "Uncovering Hidden Connections." And Chapter 16, "Quantum Duality," explains how this research project led him back to his high school sweetheart, quantum mechanics, and the symmetries that link the natural world (at all levels) to mathematics. Right in the middle of that chapter is his mother's secret recipe for borscht, represented both iconically and symbolically: but why is it sitting there, that delicious bowl of soup, in the midst of an explanation of electromagnetic duality? You must buy the book and read it through, in order to find out. It makes perfect sense.