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COMPLEX SYMMETRIC PARTIAL ISOMETRIES

STEPHAN RAMON GARCIA AND WARREN R. WOGEN

ABSTRACT. An operator $T \in B(\mathcal{H})$ is complex symmetric if there exists a conjugate-linear, isometric involution $C : \mathcal{H} \rightarrow \mathcal{H}$ so that $T = CT^*C$. We provide a concrete description of all complex symmetric partial isometries. In particular, we prove that any partial isometry on a Hilbert space of dimension ≤ 4 is complex symmetric.

1. INTRODUCTION

The aim of this note is to complete the classification of complex symmetric partial isometries which was started in [10]. In particular, we give a concrete necessary and sufficient condition for a partial isometry to be a complex symmetric operator.

Before proceeding any further, let us first recall a few definitions. In the following, \mathcal{H} denotes a separable, complex Hilbert space and $B(\mathcal{H})$ denotes the collection of all bounded linear operators on \mathcal{H} .

Definition. A *conjugation* is a conjugate-linear operator $C : \mathcal{H} \rightarrow \mathcal{H}$, which is both *involution* (i.e., $C^2 = I$) and *isometric* (i.e., $\langle Cx, Cy \rangle = \langle y, x \rangle$).

Definition. We say that $T \in B(\mathcal{H})$ is *C-symmetric* if $T = CT^*C$. We say that T is *complex symmetric* if there exists a conjugation C with respect to which T is *C-symmetric*.

It is straightforward to show that if $\dim \ker T \neq \dim \ker T^*$, then T is not a complex symmetric operator. For instance, the unilateral shift is perhaps the most ubiquitous example of a partial isometry which is not complex symmetric (see [7, Prop. 1], [9, Ex. 2.14], [6, Cor. 7]). On the other hand, we have the following theorem from [10]:

Theorem 1. *Let $T \in B(\mathcal{H})$ be a partial isometry.*

- (i) *If $\dim \ker T = \dim \ker T^* = 1$, then T is a complex symmetric operator,*
- (ii) *If $\dim \ker T \neq \dim \ker T^*$, then T is not a complex symmetric operator.*
- (iii) *If $2 \leq \dim \ker T = \dim \ker T^* \leq \infty$, then either possibility can (and does) occur.*

Although these results are the sharpest possible statements that can be made given only the data $(\dim \ker T, \dim \ker T^*)$, they are in some sense unsatisfactory. For instance, it is known that partial isometries on \mathcal{H} that are not complex symmetric exist if $\dim \mathcal{H} \geq 5$ and that every partial isometry on \mathcal{H} is complex symmetric if $\dim \mathcal{H} \leq 3$, the authors were unable to answer the corresponding question if

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$\dim \mathcal{H} = 4$. To be more specific, the techniques used in [10] were insufficient to resolve the question in the case where $\dim \mathcal{H} = 4$ and $\dim \ker T = 2$. Significant numerical evidence in favor of the assertion that all partial isometries on a four-dimensional Hilbert space are complex symmetric has recently been produced by J. Tener [13].

Suppose that T is a partial isometry on \mathcal{H} and let

$$\mathcal{H}_1 = (\ker T)^\perp = \text{ran } T^* \quad (1)$$

denote the *initial space* of T and $\mathcal{H}_2 = (\mathcal{H}_1)^\perp = \ker T$ denote its orthogonal complement (see [12, Pr. 127] or [2, Ch. VIII, Sect. 3] for terminology). With respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, we have

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \quad (2)$$

where $A : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ and $B : \mathcal{H}_1 \rightarrow \mathcal{H}_2$. Furthermore, the fact that T^*T is the orthogonal projection onto \mathcal{H}_1 yields the identity

$$A^*A + B^*B = I, \quad (3)$$

where I denotes the identity operator on \mathcal{H}_1 . Finally, observe that the operator $A \in B(\mathcal{H}_1)$ is simply the compression of the partial isometry T to its initial space.

The main result of this note is the following concrete description of complex symmetric partial isometries:

Theorem 2. *Let $T \in B(\mathcal{H})$ be a partial isometry. If A denotes the compression of T to its initial space, then T is a complex symmetric operator if and only if A is a complex symmetric operator.*

Due to its somewhat lengthy and computational proof, we defer the proof of the preceding theorem until Section 3. We remark that Theorem 2 remains true if one instead considers the final space of T . Indeed, simply apply the theorem with T^* in place of T and then take adjoints.

Corollary 1. *Every partial isometry of rank ≤ 2 is complex symmetric.*

Proof. Let $T \in B(\mathcal{H})$ be a partial isometry such that $\text{rank } T \leq 2$. If $\text{rank } T = 0$, then $T = 0$ and there is nothing to prove. If $\text{rank } T = 1$, then this is handled in [10]. In the case $\text{rank } T = 2$, we may write

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

where A is an operator on a two-dimensional space. Since every operator on a two-dimensional Hilbert space is complex symmetric (see [1, Cor. 3], [3, Cor. 3.3], [7, Ex. 6], [10, Cor. 1], [13, Cor. 3]), the desired conclusion follows from Theorem 2. \square

Corollary 2. *Every partial isometry on a Hilbert space of dimension ≤ 4 is complex symmetric.*

Proof. As mentioned earlier, the results of [10] indicate that only the case $\dim \mathcal{H} = 4$ and $\dim \ker T = 2$ requires resolution. The corollary is now immediate consequence of Theorem 2 and the fact that every operator on a two-dimensional Hilbert space is complex symmetric. \square

We conclude this section with the following theorem, which asserts that each C -symmetric partial isometry can be extended to a C -symmetric unitary operator on the whole space (the significance lies in the fact that the corresponding conjugations for these two operators are the same).

Theorem 3. *If T is a C -symmetric partial isometry, then there exists a C -symmetric unitary operator U and an orthogonal projection P such that $T = UP$.*

Proof. Since T is a C -symmetric partial isometry, it follows that $|T| = P$ is an orthogonal projection and that $T = CJP$ where J is a conjugation supported on $\text{ran } P$ which commutes with P [8, Sect. 2.2]. We may extend J to a conjugation \tilde{J} on all of \mathcal{H} by forming the internal direct sum $J \oplus J'$ where J' is a partial conjugation supported on $\ker P$. The operator $U = C\tilde{J}$ is a C -symmetric unitary operator. \square

2. PARTIAL ISOMETRIES AND THE NORM CLOSURE PROBLEM

Partial isometries on infinite-dimensional spaces often provide examples of note. For instance, one can give a simple example of a partial isometry T satisfying $\dim \ker T = \dim \ker T^* = \infty$ which is not a complex symmetric operator:

Example 1. Let S denote the unilateral shift on $l^2(\mathbb{N})$, Although S is certainly *not* a complex symmetric operator (by (ii) of Theorem 1, see also [9, Ex. 2.14], or [6, Cor. 7]), part (i) of Theorem 1 does ensure that the partial isometry $S \oplus S^*$ is complex symmetric. Indeed, simply take N to be the bilateral shift on $l^2(\mathbb{Z})$ and note that $S \oplus S^*$ is unitarily equivalent to $N - Ne_0 \otimes e_0$. That $S \oplus S^*$ is complex symmetric can also be verified by a direct computation [8, Ex. 5]. On the other hand, the partial isometry $T = S \oplus 0$ on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ is *not* a complex symmetric operator by Lemma 1.

Let $\mathcal{S}(\mathcal{H})$ denote the subset of $B(\mathcal{H})$ consisting of all bounded complex symmetric operators on \mathcal{H} . There are several ways to think about $\mathcal{S}(\mathcal{H})$. By definition, we have

$$\mathcal{S}(\mathcal{H}) = \{T \in B(\mathcal{H}) : \exists \text{ a conjugation } C \text{ s.t. } T = CT^*C\}.$$

If C is a fixed conjugation on \mathcal{H} , then we also have

$$\mathcal{S}(\mathcal{H}) = \{UTU^* : T = CT^*C, U \text{ unitary}\}.$$

Thus if we identify \mathcal{H} with $l^2(\mathbb{N})$ and C denotes the canonical conjugation on $l^2(\mathbb{N})$ (i.e., entry-by-entry complex conjugation), we can think of $\mathcal{S}(\mathcal{H})$ as being the *unitary orbit* of the set of all bounded (infinite) complex symmetric matrices.

The following example shows that the set $\mathcal{S}(\mathcal{H})$ is not closed in the strong operator topology (SOT):

Example 2. We maintain the notation of Example 1. For $n \in \mathbb{N}$, let P_n denote the orthogonal projection onto the span of the basis vectors $\{e_i : i \geq n\}$ of $l^2(\mathbb{N})$. Now observe that each operator $T_n = P_n S \oplus S^*$ is unitarily equivalent to $S \oplus 0_n \oplus S^*$ where 0_n denotes the zero operator on an n -dimensional Hilbert space. Each T_n is complex symmetric since $S \oplus S^*$ is complex symmetric (by Lemma 1). On the other hand, since $P_n S$ is SOT-convergent to 0, it follows that the SOT-limit of the sequence T_n is $0 \oplus S^*$, which is not a complex symmetric operator (by Lemma 1).

The preceding example demonstrates that the set of all complex symmetric operators (on a fixed, infinite-dimensional Hilbert space \mathcal{H}) is not SOT-closed. We also remark that the conjugations corresponding to the operators T_n from Example 2 depend on n . In contrast, if we fix a conjugation C , then it is elementary to see that the set of C -symmetric operators is a SOT-closed subspace of $B(\mathcal{H})$.

We conclude with a related question, which we have been unable to resolve:

Question. Is $\mathcal{S}(\mathcal{H})$ norm closed?

3. PROOF OF THEOREM 2

This entire section is devoted to the proof of Theorem 2. We first require the following lemma:

Lemma 1. *If \mathcal{H}, \mathcal{K} are separable complex Hilbert spaces, then $T \in B(\mathcal{H})$ is a complex symmetric operator if and only if $T \oplus 0 \in B(\mathcal{H} \oplus \mathcal{K})$ is a complex symmetric operator.*

Proof. If T is a C -symmetric operator on \mathcal{H} , then it is easily verified that $T \oplus 0$ is $(C \oplus J)$ -symmetric on $\mathcal{H} \oplus \mathcal{K}$ for any conjugation J on \mathcal{K} . The other direction is slightly more difficult to prove.

Suppose that $S = T \oplus 0$ is a complex symmetric operator on $\mathcal{H} \oplus \mathcal{K}$. Before proceeding any further, let us remark that it suffices to consider the case where

$$\mathcal{H} = \overline{\text{ran } T + \text{ran } T^*}. \quad (4)$$

Otherwise let $\mathcal{H}_1 = \overline{\text{ran } T + \text{ran } T^*}$ and note that \mathcal{H}_1 is a reducing subspace of \mathcal{H} . If \mathcal{H}_2 denotes the orthogonal complement of \mathcal{H}_1 in \mathcal{H} , then with respect to the orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{K}$, the operator S has the form $T' \oplus 0 \oplus 0$, where T' denotes the restriction of T to \mathcal{H}_1 . By now considering S with respect to the orthogonal decomposition $\mathcal{H} \oplus \mathcal{K} = \mathcal{H}_1 \oplus (\mathcal{H}_2 \oplus \mathcal{K})$, it follows that we need only consider the case where (4) holds.

Suppose now that (4) holds and that S is C -symmetric where C denotes a conjugation on $\mathcal{H} \oplus \mathcal{K}$. Writing the equations $CS = S^*C$ and $CS^* = SC$ in terms of the 2×2 block matrices

$$S = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \quad (5)$$

(the entries C_{ij} of C are conjugate-linear operators), we find that

$$C_{11}T = T^*C_{11}, \quad (6)$$

$$C_{21}T = C_{21}T^* = 0, \quad (7)$$

$$T^*C_{12} = TC_{12} = 0. \quad (8)$$

Since $C_{21}T = C_{21}T^* = 0$, it follows that C_{21} vanishes on $\text{ran } T + \text{ran } T^*$ and hence on \mathcal{H} itself by (4). On the other hand, (8) implies that C_{12} vanishes on the orthogonal complements of $\ker T$ and $\ker T^*$ in \mathcal{H} . By (4), this implies that C_{12} vanishes identically.

It follows immediately from (5) that C_{11} and C_{22} must be conjugations on \mathcal{H} and \mathcal{K} , respectively, whence T is C_{11} -symmetric by (6). This concludes the proof of the lemma. \square

Now let us suppose that T is a partial isometry on \mathcal{H} and let

$$\mathcal{H}_1 = (\ker T)^\perp = \text{ran } T^*.$$

and $\mathcal{H}_2 = \ker T$. With respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, it follows that

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

where $A : \mathcal{H}_1 \rightarrow \mathcal{H}_1$, $B : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, and

$$A^*A + B^*B = I. \quad (9)$$

(\Rightarrow) Suppose that T is a complex symmetric operator. For an operator with polar decomposition $T = U|T|$ (i.e., U is the unique partial isometry satisfying $\ker U = \ker T$ and $|T|$ denotes the positive operator $\sqrt{T^*T}$), the *Aluthge transform* of T is defined to be the operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Noting that

$$T^*T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

we find that

$$\tilde{T} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

By [5, Thm. 1], we know that the Aluthge transform of a complex symmetric operator is complex symmetric. Applying Lemma 1 to \tilde{T} , we conclude that A is complex symmetric, as desired.

(\Leftarrow) Let us now consider the more difficult implication of Theorem 2, namely that if A is a complex symmetric operator, then T is as well. We claim that it suffices to consider the case where $\overline{\text{ran } B} = \mathcal{H}_2$. In other words, we argue that if

$$\mathcal{K} = \overline{\text{ran } T + \text{ran } T^*},$$

then we may suppose that $\mathcal{K} = \mathcal{H}$. Indeed, \mathcal{K} is a reducing subspace for T and $T = 0$ on \mathcal{K}^\perp . By Lemma 1, if $T|_{\mathcal{K}}$ is a complex symmetric operator, then so is T .

Write $B = V|B|$ where $V : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a partial isometry with initial space $(\ker B)^\perp \subseteq \mathcal{H}_1$ and final space \mathcal{H}_2 (since $\overline{\text{ran } B} = \mathcal{H}_2$). In particular, we have the relations

$$V^*B = |B| = B^*V, \quad |B| = \sqrt{I - A^*A}. \quad (10)$$

By hypothesis, the operator $A \in B(\mathcal{H}_1)$ is complex symmetric. Therefore suppose that K is a conjugation on \mathcal{H}_1 such that $KA = A^*K$ and observe that the equations

$$\begin{aligned} A\sqrt{I - A^*A} &= \sqrt{I - AA^*}A, \\ A^*\sqrt{I - AA^*} &= \sqrt{I - A^*AA^*}, \\ K\sqrt{I - A^*A} &= \sqrt{I - AA^*}K, \\ K\sqrt{I - AA^*} &= \sqrt{I - A^*AK}, \end{aligned}$$

follow from a standard polynomial approximation argument (i.e., if $p(x) \in \mathbb{R}[x]$, then $Ap(A^*A) = p(AA^*)A$ and $Kp(A^*A) = p(AA^*)K$ hold whence the desired identities follow upon passage to the strong operator limit). In particular, it follows from the preceding that

$$(KA)\sqrt{I - A^*A} = \sqrt{I - A^*A}(KA),$$

that is

$$KA|B| = |B|KA, \quad A^*K|B| = |B|A^*K. \quad (11)$$

Let us now define a conjugate-linear operator C on \mathcal{H} by the formula

$$C = \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix}. \quad (12)$$

Assuming for the moment that C is a conjugation on \mathcal{H} , we observe that

$$\underbrace{\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}}_T = \underbrace{\begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix}}_C \underbrace{\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}}_J \underbrace{\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}}_{|T|}.$$

Since it is clear that J is a partial conjugation which is supported on the range of $|T|$ and which commutes with $|T|$, it follows immediately that T is a C -symmetric operator (see [8, Thm. 2]).

To complete the proof of Theorem 2, we must therefore show that C is a conjugation on \mathcal{H} . In other words, we must check that C^2 is the identity operator on \mathcal{H} and that C is isometric. Since these computations are somewhat lengthy, we perform them separately:

Claim: $C^2 = I$.

Pf. of Claim. We first expand out C^2 as a 2×2 block matrix:

$$\begin{aligned} C^2 &= \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \\ &= \begin{pmatrix} AKAK + KB^*BK & AKKB^* - KB^*VA^*KV^* \\ BKAK - VA^*KV^*BK & BKKB^* + VA^*KV^*VA^*KV^* \end{pmatrix} \\ &= \begin{pmatrix} AA^* + KB^*BK & AB^* - KB^*VA^*KV^* \\ BA^* - VA^*KV^*BK & BB^* + VA^*KV^*VA^*KV^* \end{pmatrix}. \end{aligned}$$

To obtain the preceding line, we used the fact that K is a conjugation and A is K -symmetric. Letting E_{ij} denote the entries of the preceding block matrix we find that

$$\begin{aligned} E_{11} &= AA^* + KB^*BK \\ &= AA^* + K(I - A^*A)K \\ &= AA^* + (I - AA^*) \\ &= I. \end{aligned}$$

$$\begin{aligned} E_{12} &= AB^* - KB^*VA^*KV^* \\ &= AB^* - K|B|A^*KV^* && \text{by (10)} \\ &= AB^* - KA^*K|B|V^* && \text{by (11)} \\ &= AB^* - A|B|V^* \\ &= AB^* - AB^* && \text{since } B^* = |B|V \\ &= 0. \end{aligned}$$

$$E_{21} = BA^* - VA^*KV^*BK$$

$$\begin{aligned}
 &= BA^* - VA^*K|B|K && \text{since } V^*B = |B| \\
 &= BA^* - V|B|A^*KK && \text{by (11)} \\
 &= BA^* - V|B|A^* \\
 &= BA^* - BA^* && \text{since } B = V|B| \\
 &= 0.
 \end{aligned}$$

As for E_{22} , it suffices to show that E_{22} agrees with I (the identity operator on \mathcal{H}_2) on the range of B , which is dense in \mathcal{H}_2 . In other words, we wish to show that $E_{22}Bx = Bx$ for all $x \in \mathcal{H}_2$, which is equivalent to showing that

$$E_{22}Bx = BB^*Bx + VA^*KV^*VA^*KV^*Bx = Bx \quad (13)$$

for all $x \in \mathcal{H}_2$. Let us investigate the second term of (13):

$$\begin{aligned}
 VA^*KV^*VA^*KV^*Bx &= VA^*KV^*VA^*K|B|x && \text{by (10)} \\
 &= VA^*KV^*V|B|A^*Kx && \text{by (11)} \\
 &= VA^*K|B|A^*Kx && \text{since } V^*V = P_{\overline{\text{ran}}|B|} \\
 &= V|B|A^*KA^*Kx && \text{by (11)} \\
 &= BA^*KA^*Kx && \text{since } B = V|B| \\
 &= BA^*Ax \\
 &= B(I - B^*B)x && \text{since } A^*A + B^*B = I \\
 &= Bx - BB^*Bx.
 \end{aligned}$$

Putting this together with (13), we find that $E_{22}Bx = Bx$ for all $x \in \mathcal{H}_2$ whence $E_{22} = I$, as claimed. \square

Claim: C is isometric.

Pf. of Claim. The proof requires three steps:

- (i) Show that C is isometric on \mathcal{H}_1 ,
- (ii) Show that C is isometric on $B\mathcal{H}_1$, which is dense in \mathcal{H}_2 ,
- (iii) Show that $C\mathcal{H}_1 \perp C(B\mathcal{H}_1)$.

For the first portion, observe that

$$\begin{aligned}
 \left\| C \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^2 \\
 &= \left\| \begin{pmatrix} AKx \\ BKx \end{pmatrix} \right\|^2 \\
 &= \langle AKx, AKx \rangle + \langle BKx, BKx \rangle \\
 &= \langle A^*AKx, Kx \rangle + \langle B^*BKx, Kx \rangle \\
 &= \langle (A^*A + B^*B)Kx, Kx \rangle \\
 &= \langle Kx, Kx \rangle \\
 &= \|Kx\|^2 \\
 &= \|x\|^2.
 \end{aligned}$$

Thus (i) holds.

Now for (ii):

$$\begin{aligned}
\left\| C \begin{pmatrix} 0 \\ Bx \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} 0 \\ Bx \end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix} KB^*Bx \\ -VA^*KV^*Bx \end{pmatrix} \right\|^2 \\
&= \|KB^*Bx\|^2 + \|VA^*KV^*Bx\|^2 \\
&= \|B^*Bx\|^2 + \|VA^*K|B|x\|^2 \\
&= \|B^*Bx\|^2 + \|V|B|A^*Kx\|^2 \\
&= \|B^*Bx\|^2 + \|BA^*Kx\|^2 \\
&= \|B^*Bx\|^2 + \langle BA^*Kx, BA^*Kx \rangle \\
&= \|B^*Bx\|^2 + \langle B^*BA^*Kx, A^*Kx \rangle \\
&= \|B^*Bx\|^2 + \langle (I - A^*A)A^*Kx, A^*Kx \rangle \\
&= \|B^*Bx\|^2 + \langle A^*K(I - A^*A)x, A^*Kx \rangle \\
&= \|B^*Bx\|^2 + \langle K(I - A^*A)x, AA^*Kx \rangle \\
&= \langle B^*Bx, B^*Bx \rangle + \langle KAA^*Kx, (I - A^*A)x \rangle \\
&= \langle (I - A^*A)x, (I - A^*A)x \rangle + \langle A^*Ax, (I - A^*A)x \rangle \\
&= \langle x, (I - A^*A)x \rangle - \langle A^*Ax, (I - A^*A)x \rangle + \langle A^*Ax, (I - A^*A)x \rangle \\
&= \langle x, (I - A^*A)x \rangle \\
&= \langle x, B^*Bx \rangle \\
&= \langle Bx, Bx \rangle \\
&= \|Bx\|^2.
\end{aligned}$$

Thus (ii) holds.

Now for (iii):

$$\begin{aligned}
\left\langle C \begin{pmatrix} x \\ 0 \end{pmatrix}, C \begin{pmatrix} 0 \\ By \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} 0 \\ By \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} AKx \\ BKx \end{pmatrix}, \begin{pmatrix} KB^*By \\ -VA^*KV^*By \end{pmatrix} \right\rangle \\
&= \langle AKx, KB^*By \rangle - \langle BKx, VA^*KV^*By \rangle \\
&= \langle B^*By, KAKx \rangle - \langle BKx, VA^*K|B|y \rangle \\
&= \langle B^*By, A^*x \rangle - \langle BKx, V|B|A^*Ky \rangle \\
&= \langle AB^*By, x \rangle - \langle BKx, BA^*Ky \rangle \\
&= \langle AB^*By, x \rangle - \langle B^*BKx, A^*Ky \rangle \\
&= \langle AB^*By, x \rangle - \langle (I - A^*A)Kx, A^*Ky \rangle \\
&= \langle AB^*By, x \rangle - \langle K(I - AA^*)x, A^*Ky \rangle \\
&= \langle AB^*By, x \rangle - \langle KA^*Ky, (I - AA^*)x \rangle \\
&= \langle AB^*By, x \rangle - \langle Ay, (I - AA^*)x \rangle
\end{aligned}$$

$$\begin{aligned}
&= \langle AB^*By, x \rangle - \langle (I - AA^*)Ay, x \rangle \\
&= \langle AB^*By, x \rangle - \langle A(I - A^*A)y, x \rangle \\
&= \langle AB^*By, x \rangle - \langle AB^*By, x \rangle \\
&= 0.
\end{aligned}$$

By the polarization identity, it follows that

$$\left\langle C \begin{pmatrix} x_1 \\ Bx_2 \end{pmatrix}, C \begin{pmatrix} y_1 \\ By_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x_2 \\ By_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ By_1 \end{pmatrix} \right\rangle$$

holds for all $x_1, x_2, y_1, y_2 \in \mathcal{H}_1$ whence C is isometric on \mathcal{H} . \square

REFERENCES

- [1] Balayan, L., Garcia, S.R., *Unitary equivalence to a complex symmetric matrix: geometric criteria*, (preprint).
- [2] Conway, J.B., *A Course in Functional Analysis* (second edition), Graduate Texts in Mathematics, **96**, Springer-Verlag, New York, 1990.
- [3] Chevrot, N., Fricain, E., Timotin, D., *The characteristic function of a complex symmetric contraction*, Proc. Amer. Math. Soc. **135** (2007), 2877–2886. MR2317964 (2008c:47025)
- [4] Garcia, S.R., *Approximate antilinear eigenvalue problems and related inequalities*, Proc. Amer. Math. Soc. **136** (2008), no. 1, 171–179. MR2350402
- [5] Garcia, S.R., *Aluthge transforms of complex symmetric operators*, Integral Equations Operator Theory **60** (2008), no. 3, 357–367. MR2392831
- [6] Garcia, S.R., *Means of unitaries, conjugations, and the Friedrichs operator*, J. Math. Anal. Appl. **335** (2007), 941–947. MR2345511 (2008i:47070)
- [7] Garcia, S.R., Putinar, M., *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), 1285–1315. MR2187654 (2006j:47036)
- [8] Garcia, S.R., Putinar, M., *Complex symmetric operators and applications II*, Trans. Amer. Math. Soc. **359** (2007), 3913–3931. MR2302518 (2008b:47005)
- [9] Garcia, S.R., *Conjugation and Clark Operators*, Contemp. Math. **393** (2006), 67–112. MR2198373 (2007b:47073)
- [10] Garcia, S.R., Wogen, W.R., *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. (to appear).
- [11] Gilbreath, T.M., Wogen, W.R., *Remarks on the structure of complex symmetric operators*, Integral Equations Operator Theory **59** (2007), no. 4, 585–590. MR2370050
- [12] Halmos, P.R., *A Hilbert Space Problem Book* (Second Edition), Springer-Verlag, New York, 1982.
- [13] Tener, J.E., *Unitary equivalence to a complex symmetric matrix: an algorithm*, J. Math. Anal. Appl. **341** (2008), no. 1, 640–648. MR2394112 (2008m:15062)
- [14] Sarason, D., *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices **1** (2007), no. 4, 491–526. MR2363975 (2008i:47060)

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