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Melvin Henriksen
Harvey Mudd College

Meyer Jerison
Purdue University

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MINIMAL PROJECTIVE EXTENSIONS OF COMPACT SPACES

BY M. HENRIKSEN AND M. JERISON

A compact space E is called *projective* if for each mapping ψ of E into a compact space X, and each continuous mapping τ of a compact space Y *onto* X, there is a continuous mapping ϕ of E into Y such that $\psi = \tau \circ \phi$. Gleason proved in [1] that a compact space E is projective if and only if it is extremally disconnected. (A topological space E is *extremally disconnected* if the closure of each of its open sets is open. It is well known that E is extremally disconnected if and only if the Boolean algebra of open and closed subsets of E is complete.) Gleason showed, moreover, that for each compact space X, there is a unique compact extremally disconnected space $\mathcal{R}(X)$, and a continuous mapping π_X of $\mathcal{R}(X)$ onto X such that no proper closed subspace of $\mathcal{R}(X)$ is mapped by π_X onto X. (An alternate development of Gleason's results is given by Rainwater in [2].) We call $\mathcal{R}(X)$ the *minimal projective extension* of X; it can be described as follows.

Let $R(X)$ denote the family of regular closed subsets of X. (A closed subset of X is called *regular* if it is the closure of its interior.) Then $R(X)$ is a complete Boolean algebra if we define for α, β in $R(X)$

$$\alpha \vee \beta = \alpha \cup \beta; \alpha \wedge \beta = \text{cl int}(\alpha \cap \beta).$$

Note that the Boolean complement α^* of α is given by

$$\alpha^* = \text{cl}(X \sim \alpha).$$

The space $\mathcal{R}(X)$ is the Stone space of $R(X)$. That is, the points of $\mathcal{R}(X)$ are the prime ideals of $R(X)$, and a base for the topology of $\mathcal{R}(X)$ is the family of sets $\{P \in \mathcal{R}(X) : \alpha \notin P\}$, $\alpha \in R(X)$.

The mapping π_X is defined by letting $\pi_X(P) = \bigcap \{\alpha \in R(X) : \alpha \notin P\}$ for each $P \in \mathcal{R}(X)$.

1. LEMMA. *The mapping* $\alpha \to \pi_X^{-1}(\alpha)$ *is an isomorphism of* $R(X)$ *onto the Boolean algebra of open and closed subsets of* $\mathcal{R}(X)$.

From Gleason's theorems we deduce quickly the following induced mapping theorem which motivates this paper.

2. THEOREM. *Let* τ *be a continuous mapping of a compact space* Y *onto* X. *Then there exists a continuous mapping* $\bar{\tau}$ *of* $\mathcal{R}(Y)$ *onto* $\mathcal{R}(X)$ *such that* $\tau \circ \pi_Y = \pi_X \circ \bar{\tau}$. *Thus the following diagram is commutative.*

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$$\begin{array}{ccc} \mathcal{R}(Y) & \xrightarrow{\bar{\tau}} & \mathcal{R}(X) \\ \downarrow \pi_Y & & \downarrow \pi_X \\ Y & \xrightarrow{\tau} & X \end{array}$$

Proof. Since $\tau \circ \pi_Y$ maps $\mathcal{R}(Y)$ into X, and π_X maps $\mathcal{R}(X)$ onto X, the fact that $\mathcal{R}(Y)$ is projective implies the existence of a continuous mapping $\bar{\tau}$ of $\mathcal{R}(Y)$ into $\mathcal{R}(X)$ such that $\tau \circ \pi_Y = \pi_X \circ \bar{\tau}$. Moreover, $\bar{\tau}[R(Y)] = \tau[Y] = X$. But no proper closed subspace of $\mathcal{R}(X)$ is mapping by π_X onto all of X, so $\bar{\tau}$ maps $\mathcal{R}(Y)$ onto $\mathcal{R}(X)$.

This paper is devoted to answering the question: When is the mapping $\bar{\tau}$ unique?

In order to so do, we will make use of the well-known duality between Boolean algebras and their Stone spaces. In particular, we will use the following well known lemma.

3. LEMMA. *There is a one-one correspondence between the continuous mappings of $\mathcal{R}(Y)$ onto $\mathcal{R}(X)$ and the isomorphisms of $R(X)$ into $R(Y)$ as follows: If ϕ is such a continuous mapping, the corresponding isomorphism f_ϕ is given by*

$$f_\phi(\alpha) = \pi_Y \phi^{-1} \pi_X^{-1}(\alpha) \quad \text{for all} \quad \alpha \in R(X).$$

This lemma enables us to replace the quest for a condition for uniqueness of $\bar{\tau}$ with one for uniqueness of the corresponding isomorphism. To accomplish this latter task, we must translate the condition that $\tau \circ \pi_Y = \pi_X \circ \bar{\tau}$ into one about the corresponding isomorphism. An immediate consequence of this commutativity condition and Lemma 3 is that

$$(1) \qquad \tau[f_\tau(\alpha)] = \alpha \quad \text{for all} \quad \alpha \in R(X),$$

so we examine those regular closed subsets of Y mapped onto α by τ.

First, we introduce some notation. For each $\alpha \in R(X)$, let $A(\alpha) = cl(\tau^{-1} \operatorname{int} \alpha)$, and $B(\alpha) = cl(\operatorname{int} \tau^{-1} \alpha)$. Clearly $A(\alpha) \subset B(\alpha)$ for all $\alpha \in R(X)$.

4. LEMMA. *For each $\alpha \in R(X)$, $\tau[A(\alpha)] = \tau[B(\alpha)] = \alpha$ and $B(\alpha)$ is the largest regular closed subset of Y mapped onto α by τ.*

By (1) and the lemma, any candidate for $f(\alpha)$ must be a subset of $B(\alpha)$. Unfortunately, there need be no smallest regular closed subset of Y that is mapped by τ onto α. Indeed, if τ denotes the projection mapping of the unit square Y onto the unit interval X, then unless the regular closed subset α of X is empty, there is *never* a smallest regular closed subset of X that is mapped by τ onto α.

Our next lemma will relate the sets $A(\alpha)$ and $B(\alpha)$ via the Boolean structure of $R(Y)$.

5. LEMMA. *For any $\alpha \in R(X)$, we have $(B(\alpha))^* = A(\alpha^*)$.*

Proof. Recall that $\alpha^* = cl(X \sim \alpha) = X \sim \operatorname{int} \alpha$. So, $\operatorname{int} \alpha^* = \operatorname{int}(X \sim \operatorname{int} \alpha) =$

$X \sim cl \operatorname{int} \alpha$. Since α is a regular closed set, we have

$$\operatorname{int} \alpha^* = X \sim \alpha, \tag{2}$$

and the analogous relation is also valid for members of $R(Y)$.

Now,

$$A(\alpha^*) = cl(\tau^{-1} \operatorname{int} \alpha^*) = cl(\tau^{-1}(X \sim \alpha)) = cl(Y \sim \tau^{-1}\alpha) = Y \sim \operatorname{int} \tau^{-1}\alpha.$$

And

$$(B(\alpha))^* = cl(Y \sim B(\alpha)) = cl(Y \sim cl \operatorname{int} \tau^{-1}\alpha) = cl \operatorname{int} (Y \sim \operatorname{int} \tau^{-1}\alpha)$$
$$= cl \operatorname{int} A(\alpha^*) = A(\alpha^*).$$

We can now translate our commutativity conditon on the mapping into a condition on the corresponding isomorphism.

6. LEMMA. *Let ϕ be a continuous mapping of $\mathcal{R}(Y)$ onto $\mathcal{R}(X)$, and let $f = f_\phi$ be the corresponding isomorphism of $R(X)$ into $R(Y)$. Then, the following are equivalent.*

(i) $\tau \circ \pi_Y = \pi_X \circ \phi$.

(ii) $f(\alpha) \subset B(\alpha)$ for all $\alpha \in R(X)$.

(iii) $A(\alpha) \subset f(\alpha) \subset B(\alpha)$ for all $\alpha \in R(X)$.

Proof. From (i), we have $\tau[f(\alpha)] = \alpha$, which implies (ii) by Lemma 4. Suppose that (ii) holds. Then $f(\alpha) = f(\alpha^{**}) = f(\alpha^*)^* \supset B(\alpha^*)^* = A(\alpha^{**}) = A(\alpha)$ by Lemma 5, so (iii) holds.

Obviously, (iii) implies (ii).

If (i) does not hold, there is $p \in \mathcal{R}(Y)$ such that $x = (\pi_X \circ \phi)(p) \neq (\tau \circ \pi_Y)(p) = x'$. Let $\alpha \in R(X)$ contain x but not x'. Since $p \in \phi^{-1}\pi_X^{-1}(\alpha)$, we have $\pi_Y(p) \in f(\alpha)$ by Lemma 3. Then $\tau[f(\alpha)]$ contains $\tau[\pi_Y(p)] = x'$, which does not belong to α. Thus, $f(\alpha)$ is not contained in $B(\alpha)$. Thus (ii) implies (i).

7. THEOREM. *Given a continuous mapping τ of a compact space Y onto a compact space X, there is a unique continuous mapping $\bar{\tau}$ of $\mathcal{R}(Y)$ onto $\mathcal{R}(X)$ satisfying $\tau \circ \pi_Y = \pi_X \circ \bar{\tau}$ if and only if $A(\alpha) = B(\alpha)$ for all $\alpha \in R(X)$.*

Proof. Sufficiency follows immediately from Lemma 6. Suppose, conversely, that there is $\alpha \in R(X)$ such that $A(\alpha) \neq B(\alpha)$. We will use this to construct distinct mappings $\bar{\tau}'$ and $\bar{\tau}''$ satisfying the condition of the theorem. Since $A(\alpha)$ and $B(\alpha)$ are in $R(Y)$, by Lemma 1, the sets $\pi_Y^{-1}[A(\alpha)]$ and $\pi_Y^{-1}[B(\alpha)]$ are open and closed sets, and the first is properly contained in the second. Let $G = \pi_Y^{-1}[B(\alpha)] \sim \pi_Y^{-1}[A(\alpha)]$ and note that this is a nonempty open and closed subset of $\mathcal{R}(Y)$. Moreover, by (2),

$$\tau[B(\alpha) \sim A(\alpha)] \subset \tau[\tau^{-1}\alpha \sim \tau^{-1}(\operatorname{int} \alpha)] \tag{3}$$
$$= \alpha \sim \operatorname{int} \alpha \subset \alpha \sim (X \sim \alpha^*) = \alpha \cap \alpha^* .$$

So,

$$(\tau \circ \pi_Y)[G] = [B(\alpha) \sim A(\alpha)] \subset \alpha \cap \alpha^* .$$

Consider the mapping $\sigma' : G \to \alpha$ defined by letting $\sigma'(p) = (\tau \circ \pi_Y)(p)$ for all $p \in G$, and the mapping $\pi_X \mid (\pi_X^{-1}\alpha)$. Since the latter maps $\pi_X^{-1}\alpha$ onto α and since G is extremally disconnected, there exists, by Gleason's theorem, a continuous mapping $\bar{\sigma}' : G \to \pi_X^{-1}\alpha$ such that $\pi_X \circ \bar{\sigma}' = \sigma'$. Likewise, the mappings $\sigma'' : G \to \underset{\sim}{\alpha}$ defined by $\sigma''(p) = (\tau \circ \pi_Y)(p)$ and $\pi_X \mid (\pi_X^{-1}\underset{\sim}{\alpha}^*)$ yield a mapping $\sigma'' \cdot G \to \pi_X^{-1}\underset{\sim}{\alpha}$ such that $\pi_X \circ \bar{\sigma}'' = \sigma''$. Now by Theorem 2, there exists a mapping $\bar{\tau}$ that satisfies the commutativity condition. We define mappings $\bar{\tau}'$ and $\bar{\tau}''$ of $\mathfrak{R}(Y)$ onto $\mathfrak{R}(X)$ to agree with $\bar{\tau}$ on $\mathfrak{R}(Y) \sim G$ and to agree with $\bar{\sigma}'$ and $\bar{\sigma}''$, respectively, on G.

It is routine to check that $\pi_X \circ \bar{\tau}' = \tau \circ \pi_Y = \pi_X \circ \bar{\tau}''$. Moreover $\bar{\tau}'$ and $\bar{\tau}''$ are distinct. For, $\bar{\tau}'[G] \subset \pi_X^{-1}\alpha$ while $\bar{\tau}''[G] \subset \pi_X^{-1}\alpha^*$ and, since $\alpha \wedge \alpha^* = \phi$, $(\pi_X^{-1}\alpha) \cap (\pi_X^{-1}\alpha^*) = \phi$.

It seems natural to seek conditions on the mapping which insure that $A(\alpha) = B(\alpha)$ for all $\alpha \in R(X)$. One such is given by

8. PROPOSITION. *If, for every nonempty open subset U of Y,* int $\tau[U]$ *is nonempty, then $A(\alpha) = B(\alpha)$ for all $\alpha \in R(X)$.*

Proof. If $A(\alpha)$ is properly contained in $B(\alpha)$ for some $\alpha \in R(X)$, then, by (3) above, $B(\alpha) \sim A(\alpha)$ is a nonempty open set whose image has empty interior.

It follows that $\bar{\tau}$ is unique in case τ is open or τ maps no proper closed subspace of Y onto X. For, as is noted in [3], this latter condition is equivalent to the condition that every nonempty open subset of Y contains the inverse image of a nonempty open subset of X. Indeed, $\bar{\tau}$ is then a homeomorphism. For the commutativity condition implies that $\bar{\tau}$ maps no proper closed subspace of $\mathfrak{R}(Y)$ onto $\mathfrak{R}(X)$, and any such mapping onto an extremally disconnected space is a homeomorphism [1].

The condition $A(\alpha) = B(\alpha)$ can hold independently of τ. In particular, if α is open and closed, $A(\alpha) = cl \operatorname{int} \tau^{-1}\alpha = cl \tau^{-1}\alpha = \tau^{-1}\alpha = B(\alpha)$, irrespective of the mapping τ. Thus, if every member of $R(X)$ is open as well as closed, precisely, if X is extremally disconnected, then $A(\alpha) = B(\alpha)$ for all $\alpha \in R(X)$. The statement of Theorem 7 that the mapping $\bar{\tau}$ is uniquely determined by τ in this case is hardly surprising, since π_X is then a homeomorphism.

This remark enables us to show that the converse of Proposition 8 is false. Let X be any compact extremally disconnected space (e.g., $X = \beta N$, the Stone-Čech compactification of the countable discrete space) and let Y be the topological sum of X and any compact space T. Let τ map X identically onto itself, and let τ map T onto any nonisolated point of X. The image of the open set T has empty interior, but $A(\alpha) = B(\alpha)$ for all $\alpha \in R(X)$ since X is extremally disconnected.

As we noted above, $A(\alpha) = B(\alpha)$ for any open and closed set α. Indeed $\{\alpha \in R(X) : A(\alpha) = B(\alpha)\}$ is a subalgebra of $R(X)$, which will be substantial in size in case X is totally disconnected. But, for any compact space X, it is possible to find a space Y and a mapping τ of Y onto X such that $A(\alpha) \neq B(\alpha)$ unless α is open. (In particular, if X is connected, $A(\alpha) = B(\alpha)$ will imply that $\alpha = \phi$ or $\alpha = X$.) Simply let D denote the discrete space whose points are those of X, let $Y = \beta D$, and let τ denote the Stone extension of the identity map of D onto X. Then, if $x \in (\alpha \sim \text{int } \alpha)$, the point x, regarded as a point of D, is isolated in βD and belongs to $B(\alpha) \sim A(\alpha)$.

In case the conditions of Theorem 7 are satisfied, the isomorphism $f_{\bar{\tau}}$ is given by the formula $f_{\bar{\tau}}(\alpha) = B(\alpha) = A(\alpha)$ for all $\alpha \in R(X)$. If $\bar{\tau}$ is not unique, however, neither the mapping $\alpha \rightarrow A(\alpha)$ nor the mapping $\alpha \rightarrow B(\alpha)$ is an isomorphism.

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Purdue University