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A prime strongly positive amphicheiral knot which is not slice

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We begin by giving several definitions. A knot K in S^3 is said to be *amphicheiral* if there is an orientation-reversing diffeomorphism h of S^3 which leaves K setwise invariant. Suppose, in addition, that K is given an orientation. Then K is said to be *positive amphicheiral* if h preserves the orientation of K. If, in addition, the diffeomorphism h is an involution then K is *strongly positive amphicheiral*. Finally, we say a knot is *slice* if it bounds a smooth disc in B^4. In this note we shall give a smooth example of a prime strongly positive amphicheiral knot which is not slice.

Levine[5] introduced an algebraic condition which is necessary for a knot to be slice. A knot satisfying this condition is said to be 'algebraically slice'. Recently, D. Long[8] showed that all strongly positive amphicheiral knots are algebraically slice. Long concludes his paper with the question of whether all prime strongly positive amphicheiral knots are, in fact, slice. So, our example answers Long's question in the negative.

To build our example we start with a knot L which is not concordant to its reverse (the 'reverse' of a knot is the knot with its string orientation reversed). We choose L to be the simplest of the collection of such knots constructed by Livingston[7]. Let L' denote the connected sum of our knot L with its mirror image L^*. Since L is not concordant to its reverse, the knot L' is not slice. Fig. 1 illustrates the knot L, and Fig. 2 illustrates this connected sum L'.
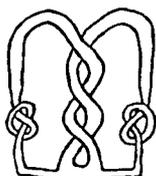

Fig. 1. L.
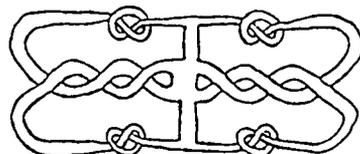

Fig. 2. L'.

Let h be the diffeomorphism of S^3 which is given by first reflecting through the plane of the page containing the projection of L', then rotating by π about an axis perpendicular to the page through the centre of the projection. This diffeomorphism is an orientation-reversing involution which preserves the orientation of L'. Thus L' is an example of a composite knot which is strongly positive amphicheiral but is not slice. We will now show that the knot K, which is shown in Fig. 3, is a prime such example.

Our example K is concordant to L' and hence also is not slice. This is because performing band moves along bands parallel to the top of each of the clasps would again yield the knot L'. The fact that K is strongly positive amphicheiral is clear by observing that again a reflection through the plane of the paper followed by a rotation
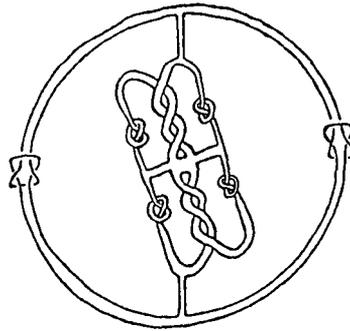
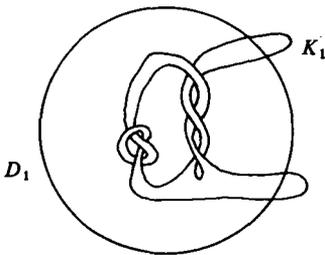Fig. 3. $K =$ a prime strongly positive amphicheiral knot which is not slice.
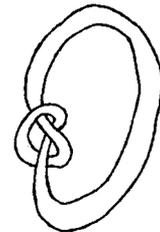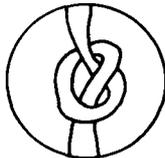

Fig. 4a. $(D_1, D_1 \cap K_1)$.
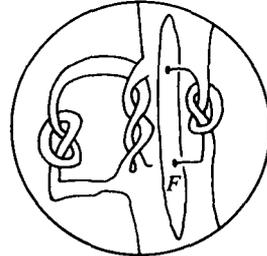

Fig. 4b. K_1.

Fig. 5. $(D_2, D_2 \cap K_2)$.

Fig. 6. $(A, A \cap L)$.

by π about a central axis provides the desired involution. We shall show that K is prime by reconstructing K out of prime tangles and using the methods of Lickorish [6].

We start with the link K_1 and the tangle $(D_1, D_1 \cap K_1)$ which are both defined by Fig. 4a. Fig. 4b shows K_1 after an isotopy.

It can be easily checked that K_1 is a prime link. So, by definition, any 2-sphere that meets K_1 transversely in two points, bounds a ball in S^3 which intersects K_1 in a single unknotted spanning arc. This observation allows us to show that $(D_1, D_1 \cap K_1)$ is a prime tangle. First, note that this tangle is not the untangle since each of the arcs of $D_1 \cap K_1$ is a trefoil knot. So, if $(D_1, D_1 \cap K_1)$ is not prime, then there is a 2-sphere in D_1 which meets K_1 transversely in two points and bounds a ball E meeting $D_1 \cap K_1$ in a knotted arc. Since K_1 has two components, the complementary ball $S^3 - E$ meets K_1 in an arc together with one complete component of K_1. Thus neither ball intersects K_1 in a single unknotted spanning arc. However, this contradicts the primeness of the link K_1. Hence, we conclude that the tangle $(D_1, D_1 \cap K_1)$ was, in fact, prime.

Let $(D_2, D_2 \cap K_2)$ be the double trefoil tangle which is illustrated in Fig. 5. This tangle is easily shown to be prime (see Lickorish[6], example 2b, or use the same argument as above).

Recall that L was the Livingston knot shown in Fig. 1. Let Fig. 6 define the tangle $(A, A \cap L)$, together with a properly embedded disc F which separates it into the two prime tangles $(D_1, D_1 \cap K_1)$ and $(D_2, D_2 \cap K_2)$. Thus, it follows from theorem 3 of [6] that the tangle $(A, A \cap L)$ is prime.

We shall let the tangle $(A^*, A^* \cap L^*)$ denote the mirror image of the tangle $(A, A \cap L)$. Then it is clear that $(A^*, A^* \cap L^*)$ is also a prime tangle. So, again by theorem 3 of [6], we conclude that the tangle $(B, B \cap L')$, of Fig. 7, is prime. Furthermore, as Kirby and Lickorish[3] noted, the clasp tangle $(C, C \cap J)$, of Fig. 7, is prime as well.
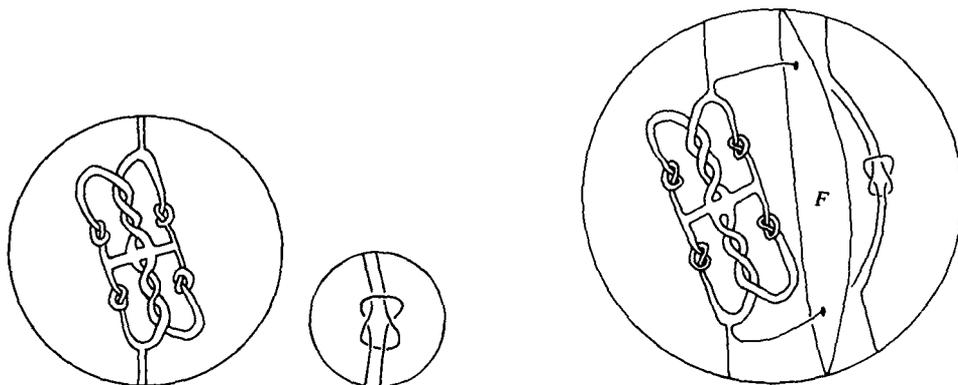

Fig. 7. $(B, B \cap L')$ $(C, C \cap J)$.

Fig. 8. $(D, D \cap L'')$ together with the disc F.

Figure 8 illustrates the tangle $(D, D \cap L'')$ together with a properly embedded disc F which separates $(D, D \cap L'')$ into the two prime tangles $(B, B \cap L')$ and $(C, C \cap J)$. So, once again we apply theorem 3 of [6] to see that the tangle $(D, D \cap L'')$ is prime.

Finally, our example K is the sum of the tangle $(D, D \cap L'')$ with another copy of the clasp tangle $(C, C \cap J)$. So, by theorem 1 of [6], the knot K is prime.

Using a similar method we can also answer a question of Hartley and Kawauchi[2]. In [2], they show that the Alexander polynomial $A(t)$ of any strongly positive amphicheiral knot is a square, $A(t) = f(t)^2$. Conversely, if $f(t)$ is the Alexander polynomial of a knot L, then the knot L', which is the connected sum of L with its mirror image L^*, is a strongly positive amphicheiral knot with Alexander polynomial $f(t)^2$. Hartley and Kawauchi ask the question of whether for a given $f(t)$ there is a prime knot which is
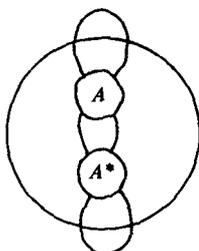

Fig. 9. $(B, B \cap L')$.

strongly positive amphicheiral and has Alexander polynomial $f(t)^2$. We answer this question in the affirmative as follows.

Given the knot L, by theorem 4 of [6], we can write L as the sum of a prime tangle $(A, A \cap L)$ together with the untangle. Also, the mirror image tangle $(A^*, A^* \cap L^*)$ is prime. So again, as in our example above, by theorem 3 of [6] the tangle $(B, B \cap L')$ shown in Fig. 9, is a prime tangle.

Now let J be the Kinoshita–Terasaka knot [4]; then it follows from Bleiler [1] that the tangle $(C, C \cap J)$, shown in Fig. 10, is also a prime tangle.

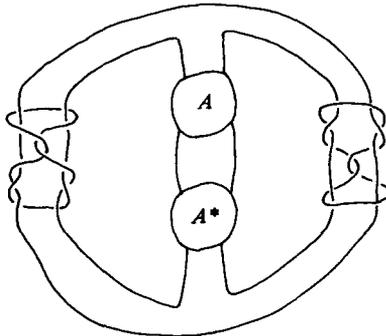Fig. 10. J = the Kinoshita–Terasaka knot $(C, C \cap J)$.

Fig. 11. K = a prime strongly positive amphicheiral knot
with Alexander polynomial $A(t) = f(t)^2$.

Here the desired knot K is obtained by taking the sum of the tangle $(B, B \cap L')$ with one copy of the tangle $(C, C \cap J)$ and one copy of its mirror image, as indicated in Fig. 11.

The knot K is prime and strongly positive amphicheiral for the same reasons as our first example. By using the fact that the Kinoshita–Terasaka knot has trivial Alexander polynomial, Bleiler [1] has shown that adding the Kinoshita–Terasaka tangle to a given knot has no effect on its Alexander polynomial. Thus K is, in fact, a prime strongly positive amphicheiral knot which has Alexander polynomial $f(t)^2$.

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