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"Notation, Notation, Notation"
OR
Book Review: *Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers*, by Joseph Mazur

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Synopsis

This review describes Mazur's engaging popularization of an interesting and important topic, the history of mathematical symbols and notation. The reviewer only wishes that some of the history had been done better.

Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers. By Joseph Mazur, Princeton University Press, Princeton, NJ, 2014. (Hardcover US$29.95, ISBN: 978-0-691-15463-3. 285 pages.)

I like popularizations of mathematics and mathematical ideas, and I tend to read a lot of them. Anything that brings mathematics in an interesting way to a wider public is to be applauded. And Joseph Mazur's book focuses on what most people think of when they think of mathematics: its characteristic symbolism.

This book is not a history of mathematical notation though; let's get that straight right now. If you want that, there is still no better place to go than Florian Cajori's magisterial and encyclopedic two-volume *A History of Mathematical Notations* (first published in the 1920s but now available from Dover [1]), one of Mazur's sources. Mazur's book does treat some important innovations in mathematical notation. He rightly observes that the mathematical notation most high school students use today, like plus and minus and $\sqrt{\cdot}$, $ax^2 + bx + c = 0$ and π, did not exist before the Renaissance. Before that time, except for a few abbreviations, everything was a word problem. Those unfamiliar with the history of different ways of writing numbers, or with the history of notation in algebra and calculus, will discover many interesting pieces of information in this book.

Mazur also discusses the way notation helps (or hinders) thinking about mathematics, says a bit about what parts of the brain are involved in symbolic thinking as compared to processing language or vision, and briefly addresses some philosophical questions about mathematics related to notation. And he does all of this with a lively style and in a readable 220 pages of double-spaced text, including timelines of events he thinks important. The narrative is followed by endnotes, appendices, and an index.

However, *Enlightening Symbols* is not a reliable guide to the history it covers. It contains omissions, errors, and inconsistencies, some of which arise from the author's having assembled a range of sources that do not entirely agree, and some of which suggest that his enthusiasm for the subject outpaced his attention to detail. This is unfortunate, because that enthusiasm accompanied by Mazur's fluid prose could have made *Enlightening Symbols* a much better book.

Let me start with what I like, and then warn readers about the pitfalls. Mazur's Introduction has striking examples that make the reader think about notation, and striking statements about symbolic reasoning. Of the latter, I especially like his "Mathematical symbols have a definite initial purpose: to tidily package complex information in order to facilitate understanding," and "Symbolism ... helps the mind to transcend the ambiguities and misinterpretations dragged along by written words in natural language."

The rest of the narrative is divided into three parts. The first is about number symbols, the second about algebra, and the third about broader philosophical and psychological questions. In Part One, we find described the method of writing numbers among the ancient Babylonians and Egyptians, the Greeks, the ancient Hebrews, the Romans, the Mayans and the Chinese, before the numbers of India and the number symbols used in the Islamic world, which are the immediate ancestors of what we now call the Hindu-Arabic system. Illustrations show what the various symbols looked like, and give a feeling for how calculation was done. There is a discussion of various kinds of abacus computation, and also of the work of Leonardo of Pisa, better known as Fibonacci.

Part Two, about algebra, begins by saying a little bit about Euclid and al-Khwārizmī (mostly to observe that they do not have any kind of algebraic notation at all). Mazur then discusses Diophantus and the notation found in his texts, correctly noting that copyists may have been responsible for the notation. He presents examples of algebraic problems in the Islamic world. He tells the story of the solution of the cubic equation in the sixteenth century with great verve, converting the verbal arguments into modern notation so the reader can figure out what's going on. In the sixteenth century, the (mostly) verbally described solutions to cubics with real roots could give rise to expressions involving $\sqrt{-1}$, and Mazur describes how Rafael Bombelli in the 1570s dealt with this necessary but puzzling idea. Then Mazur moves on to the various Renaissance notations for square roots, for unknowns, for powers of an unknown, for equality, for plus and minus, and for multiplication and division.

Mazur clearly explains the importance of one of Vieta's innovations of the 1590s, which, considering how important it is, deserves to be much better known. Vieta explicitly distinguished between knowns and unknowns by alphabetical convention—not as we do following Descartes' later choice to use letters from the end of the alphabet for unknowns and earlier for knowns, but by using vowels for unknowns and consonants for knowns. The use of letters to stand for known quantities allows mathematicians, for the first time, to write the general quadratic equation and give a general solution. Though Vieta's notational conventions are not entirely modern—for instance, he wouldn't write the expression $AB+C$ because it seems to add an area to a line—his work changed forever the focus of algebra from looking for solutions of individual problems to studying general mathematical structures.

Mazur goes on to describe how Descartes used the new general symbolic algebra to solve geometric problems. Descartes, like his contemporary Thomas Harriot, recognized that a polynomial with roots a, b, and $-c$ would have $(x-a)$, $(x-b)$, and $(x+c)$ as factors. Among other innovations, Descartes introduced the raised number notation for exponents. Mazur goes on to describe Leibniz's notation for the differential calculus and the heuristic power of that notation. He provides more details on this in his Appendix A, including how Leibniz's notation produces the chain rule, but the prospective audience for a popular work apparently discouraged Mazur from showing how Leibniz's notation also lets one discover integration by parts or how to change variables under the integral sign. Mazur also has a brief discussion of Newton and his calculus, and briefly and enthusiastically lists the many scientific and technological achievements that followed.

Part Three is rather wide-ranging, but again has good examples and some striking statements, like "With the right and proper symbols, we focus on the patterns, the symmetries, the similarities, the differences that might appear rather dim and blurred through the lens of natural language." As an illustration of this statement that would certainly enlighten a nonmathematical reader, Mazur explains why, if x^n is the way to write the product of n x's, $x^{1/n}$ should be the notation for the n^{th} root. And more philosophically, the nonmathematical reader needs to hear that "patterns that appear to us symbolically ... open the gates to logical worlds that are external to visible nature". Again, in a chapter titled The Good Symbol, Mazur says that a good symbol "must function as a revealer of patterns, a pointer to generalizations. It must have an intelligence of its own, or at least it must support our own intelligence and help us think for ourselves." Leibniz would surely applaud.

In a chapter entitled Invisible Gorillas (the title refers to a famous psychological experiment), Mazur addresses philosophical and psychological matters involving symbolic thinking in an interesting if episodic way. He discusses visualization, the way symbols are assigned meaning, the way different notational conventions cause mathematicians to see different things, ways in which external events and emotions can affect the way people reason, and experiments about which part of the brain "lights up" when a person reasons about words, numerals, pictures, or strings of algebraic symbols. This is all very interesting, and the research questions that are just beginning to be addressed are important, but I agree with Mazur's statement that not very much is yet known about how people come to understand mathematical expressions. He has a chapter about the mental pictures mathematicians use, especially about his own. His conclusion continues to address such issues, and I especially like his statement, "Although $\sqrt{x^2+y^2}$ is not generally found in the classical catalogue of archetype symbols with subconscious powers coming from folklore, it nevertheless encourages connections between the unknown and the familiar."

So what's not to like? Omissions, errors, inconsistencies, which make the historical account hard to trust. Not getting the history complete isn't a matter of missing a few names and dates; there are conceptual consequences of errors and omissions. Let me address the two instances that trouble me the most.

First, Mazur's account of ancient Greek computation is lacking. He describes the everyday notation for numbers among the Greeks that used the Greek alphabet, which starts with alpha, beta, and gamma, which stand for one, two, and three. Continuing, the tenth letter stands for 10, the next stands for 20, and so on up to 90. The subsequent letters stand for 100, 200, and so on. This convention was also, as Mazur says, used by ancient Hebrews, although I'm not convinced by his unsupported assertion that the Greeks got it from them. With alphabetic numbers, he observes, one runs out of symbols to use for large numbers. Even addition requires memorizing—the notation doesn't help you at all—and multiplication is even harder. Furthermore, Mazur rightly says that Euclid uses virtually no numerical values at all in his *Elements*. But Mazur generalizes from this fact to say that the lack of appropriate notation barred the Greeks from complicated computation. This contradicts what we know about the sophistication of Greek astronomy, with its careful determination of parameters in the mathematical models for planetary orbits. Even accomplished theoretical mathematicians like Eudoxus and Apollonius developed mathematical models for astronomy. How could they have done this if Mazur's account is complete?

In fact, computations and observations in a place-value system using base 60 came to the Greeks from Babylonian astronomy. As early as the 3$^{\text{rd}}$ century BCE, Archimedes' contemporary Eratosthenes, who used trigonometric ideas to measure the circumference of the earth, used the base-60 system in his geographical work. Ptolemy's 2$^{\text{nd}}$ century CE system of mathematical astronomy, which reigned supreme until the time of Copernicus, did its computations using base-60 place value. Indeed, it is from Greek astronomy that later scientists got the Babylonian base-60 fractions of a degree that we now call minutes and seconds. Nor were Greek mathematicians hostile to computation in other arenas. Archimedes, using 96-sided regular polygons and good approximations to square roots, calculated that the ratio of the circumference of a circle to its diameter lies between $3\frac{10}{71}$ and $3\frac{1}{7}$. The book attributed to Heron of Alexandria entitled *Metrica* gives approximations to square roots and calculates areas and volumes. The fact that the Greeks are most notable for using self-evident axioms to prove geometric truths should not make us think that they did not do applied mathematics and use numerical methods.

The second point that troubles me is Mazur’s incomplete treatment of the Islamic mathematician Muḥammad ibn Mūsā al-Khwārizmī. Mazur mentions three different works by al-Khwārizmī: *al-Kitāb al-mukhtasar fī hisāb al-jabr wa’l-muqābala* (“The Compendious Book on Calculation by Completion and Balancing”) from whose title we get the word “algebra”; the *Kitāb hisāb al-’adad al-hindi* (“Treatise on Calculation With the Hindu Numerals”); and an astronomical work, the *Zij al-sindhind*. At various points in Mazur’s exposition, results from one are conflated with results from the others. At one point he calls the treatise on calculation “al-Khwārizmī’s *Algorism*”, which, as we’ll see, cannot possibly be what its author called it. Let us concentrate on the content of the first two of these works.

The work in whose title “al-jabr” appears, which in the modern world is called “al-Khwārizmī’s *Algebra*”, gave a systematic classification of the different classes of what we call quadratic equations (if one does not have general symbolic notation, the quadratics of the form $x^2 + 10x = 39$ and those of the form $x^2 = 10x + 11$ must be treated separately). Al-Khwārizmī’s *Algebra* was translated into Latin in the twelfth century as part of the highly influential wave of translations of scientific works from Arabic into Latin that jump-started the revival of learning in Europe. It would have been valuable to point out that when al-Khwārizmī solves quadratic equations by completing the square, he justifies doing so by a geometric diagram similar to that in a theorem in Euclid’s *Elements* (Book II, Theorem 4), whose statement is a verbally stated geometric equivalent to the identity $(a + b)^2 = a^2 + 2ab + b^2$. Al-Khwārizmīs diagram for completing the square is one of many examples in the mathematics of the medieval Islamic world of the close relationship between geometry and algebra.

Al-Khwārizmī’s influential book on the Hindu numbers explained the base-10 place value system, and, through its Latin translation, introduced the system to Europe. Even though the book is elementary, somebody had to be the first to fully explain the system to the Europeans, and this is the book that did so. He tells his readers how to carry out addition, subtraction, multiplication, division, and finding square roots using the Hindu numerals. Latin works based on it often included al-Khwārizmī’s name in the title, Latinized as "algorismus" or something similar. For instance, as Mazur notes, John of Seville produced a *Liber alghoarismi* based on al-Khwārizmī's book as early as the 1130s, and Johannes de Sacrobosco in 1240 produced a textbook on the subject called *Algorismus.* Why was this term used? Because the name of al-Khwārizmī, in various Latin spellings, became so closely associated with the method of computation using the Hindu numerals that the method was referred to as the "method of algorismus". This phrase first meant only the "algorithm" of computing with the Hindu-Arabic number system, but later became applied more generally.

The origin of the term "algorithm", which Mazur does not explain, is not just an obscure historical fact, and the introduction of base-10 methods into Europe is not just one more story about notation. Here's the key point I wish Mazur had made. The Hindu-Arabic numbers were the first example in Europe of what we now call an *algorithm*, that is, a powerful symbolic method of computation that proceeds more or less mechanically according to fixed rules and which does some of our thinking or us. The concept of algorithm broadened when another example was introduced in the late sixteenth century: general symbolic algebra. General symbolic notation was an instrument of discovery in algebra, as the modern quadratic formula illustrates. Such notation also was an instrument of discovery in geometry, as Descartes and Fermat so creatively used it in their independent inventions of analytic geometry—a fact that especially impressed Leibniz. Using his chosen term "calculus" as we use the term "algorithm", Leibniz wrote in 1686 of his "calculus" for using differentials, "From it flow all the admirable theorems and problems of this kind with such ease that there is no more need to teach and retain them *than for him who knows our present algebra to memorize many theorems of ordinary geometry*" (excerpted in [3, page 281]; emphasis added). So the idea of the rule-based computations in base-10 arithmetic that immortalized the name of al-Khwārizmī in the term "algorithm" served first as a model and inspiration for symbolic algebra, and then the algorithmic character of symbolic algebra helped inspire Leibniz's notation for the calculus of differentials and integrals. All the examples in this paragraph could have been used to illustrate Mazur's point that good notation has "an intelligence of its own".

Something else I wish had been explained is the conceptual framework Vieta provided for the already existing practice of designating an unknown quantity by a single letter. When we say "let $x =$ the unknown" in a problem presented in words, what we are effectively doing, Vieta pointed out, is *treating that unknown as if it were known*. Even though we don't know what x is, we can still calculate with it. This is an amazingly powerful tool. The notation allows us to square the unknown quantity, add it to itself, or write its cosine, just as if we knew what it was. We can then manipulate the resulting equations to allow us to write the unknown in terms of known quantities. Vieta describes naming the unknown and then treating it as if it were known as an example of what the Greek geometers called “analysis”—which means “solution backwards”. As an example of analysis, let us suppose that a Greek geometer wants to discover how to construct a line. “Analysis” or “solution backwards” says, “Assume that the line is already constructed, and then work backwards from that assumption until you find something you already know how to construct.” Because Vieta recognized that designating an unknown with the letter A and then treating it as if it were known was an example of what the Greeks called “analysis”, he called algebra “the analytic art”. This name reflects the Greek sense of the term “analysis” as a method of finding the solutions to problems. Indeed, Vieta ended his book by saying “There is no problem that cannot be solved.”

I wish also that Mazur had recognized the influence of Vieta's notational innovations on Thomas Harriot. Harriot followed Vieta's convention of using vowels for unknowns, although Harriot used lower-case letters. Harriot, as Mazur points out, was the first to decompose polynomials into linear factors that look like what we see in high-school algebra today, thus making visible the root-coefficient relations. (A good and easily accessible account is in [2, pages 33-44].) But Mazur's presenting Harriot's work in Descartes' notation with x as the unknown detracts from understanding Harriot's chief predecessor and his own originality.

And about π. It is true, as Mazur says, that William Jones in 1715 introduced the notation π for the ratio of the circumference to the diameter of a circle. Mazur even remarks that this letter can stand for the Greek word “periphery” but makes nothing else of this observation. In fact this is precisely why we use the symbol π. It became the standard when it was taken up by Leonhard Euler. To make clear that this reason for choosing the letter π is no accident, observe that a competing symbol in the eighteenth century was the letter c, the first letter in “circumference”. However, the original choice of π is due not to Jones but to William Oughtred in the 1630s, who wrote the ratio as π/δ (the denominator delta also comes from a Greek word, "diameter"). Mazur discusses other notations introduced by Oughtred, including the $\times$ for multiplication and the colon for a ratio; it's a shame that he missed this one. It would also have been worth pointing out that, until Vieta had introduced general symbolic algebra, there was no need for a notation for π. But once one starts writing formulas about circles and spheres using symbols for the various parameters, one needs a symbol for the crucial ratio of periphery to diameter as well.

There are also a few places that are not entirely clear. For instance, Newton spoke of curves as generated by the motion of a point. He used the term "fluxion" for what we now call a derivative, usually meaning a time derivative, and he used the term "fluent" for its inverse. Mazur states (in two separate places) that fluents are "quantities flowing along a curve", a rather obscure way to put it.

Mazur seems to appreciate the multicultural nature of mathematics, which is admirable, but then it is best to get the cultural references right. In his timeline, Mazur rightly identifies al-Khwārizmī as Persian, but elsewhere refers to him as "the greatest Arab mathematician of his day". And it is stylistically awkward as well as inaccurate to refer to all mathematicians from the Islamic world, as Mazur does, as "the Arabians" (page 109). Finally, in a footnote (page 244) that correctly explains the name al-Khwārizmī as "from the birthplace of Khwarazm" in what is now Uzbekistan, Mazur adds that "the *al* in an Arabic name means from the birthplace of"'. In fact *al* in Arabic is the definite article. Although the definite article can precede the designation of a birthplace, as it does in English when we say "Hamlet the Dane", it can also be a characterization, like "Harun al-Rashid" which means "Harun the Righteous", as in English when we say "John the Baptist".

Popular works in mathematics often draw on history, which provides a humanizing context, and which also has great explanatory power. It's important, therefore, to get the history right. Certainly we want historians and scholars of literature to get the mathematics right even if they deal with it only tangentially. We'd object if a literary scholar writing about Newton's influence on eighteenth-century poetry misstated the law of gravity, or if an economic historian writing about the Industrial Revolution misstated the laws of thermodynamics.

We'd also object if humanistic scholars omitted the mathematics and science relevant to their topic altogether. Bertrand Russell complained in his *History of Western Philosophy* that many Plato scholars are ignorant of and neglect the subject Plato thought most important, namely mathematics. When I was in graduate school, looking up Isaac Newton in Langer's respected *Encyclopedia of World History* revealed only that Newton had been Master of the Mint in the reign of William III. The Harvard History of Science Department's response to such deficiencies was to enlist us graduate students to update Langer's reference work to include major influential achievements from mathematics and science.

For the same reasons, popular writers about mathematics should handle the history involved with great care, and strive mightily to do justice to the richness of the historical context of mathematical ideas and their applications. Not only will this enhance the writer's credibility with a nonmathematical audience—certainly important for a popular work—but it will make a real difference for the audience's understanding of the causes and nature of mathematical progress.

I have gone on at such length about this book not just because I was asked to review it, but because I enjoyed reading it and liked what the author was trying to do—I only wish that he had done it better. Perhaps the kinds of improvements I suggest can serve as examples to future writers of popular works in mathematics, and help a wider audience understand our subject as a creative human endeavor.

References

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[3] Dirk J. Struik, *A Source Book in Mathematics, 1200-1800*, Harvard University Press, Cambridge, 1969.