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Kaleidoscopes, Chessboards, and Symmetry

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Abstract

This paper describes the n-queens problem on an $n \times n$ chessboard. We discuss the possible symmetries of n-queens solutions and show how solutions to this classical chess question can be used to create examples of colorful artwork.

Keywords: *n-queens; symmetry; art.*

1. Introduction

When I was a child, I grew up next door to a wonderful older couple who would open their home and their treasures to me and my siblings. My favorite object was a beautiful (and breakable) glass kaleidoscope. I can remember very carefully lifting up the triangular case, pointing toward the sunlight, and slowly rotating the glass knob. I was rewarded with an exciting array of motion and color and enchanted by the symmetry of the reflections. Eventually our neighbors moved away and the kaleidoscope was relegated to my memories, until I was reminded of it by patterns found in solution sets of the n-queens problem from recreational mathematics. To me, these patterns, as illustrated in Figure 1, have the same wonderful symmetry and feeling of motion as did the forgotten kaleidoscope of many years ago.

While pretty, the two images in Figure 1 do lack some of the depth of color that the kaleidoscope is able to showcase. In order to enhance these illustrations with more color and more motion while preserving the essential mathematical structure from the n-queens problem, I began to experiment by overlapping and coloring the designs from multiple solutions. The result was the creation of some more dramatic, but still symmetric designs as shown in Figures 2 and 3. The rest of this article will explore the symmetry and creation of such artwork.

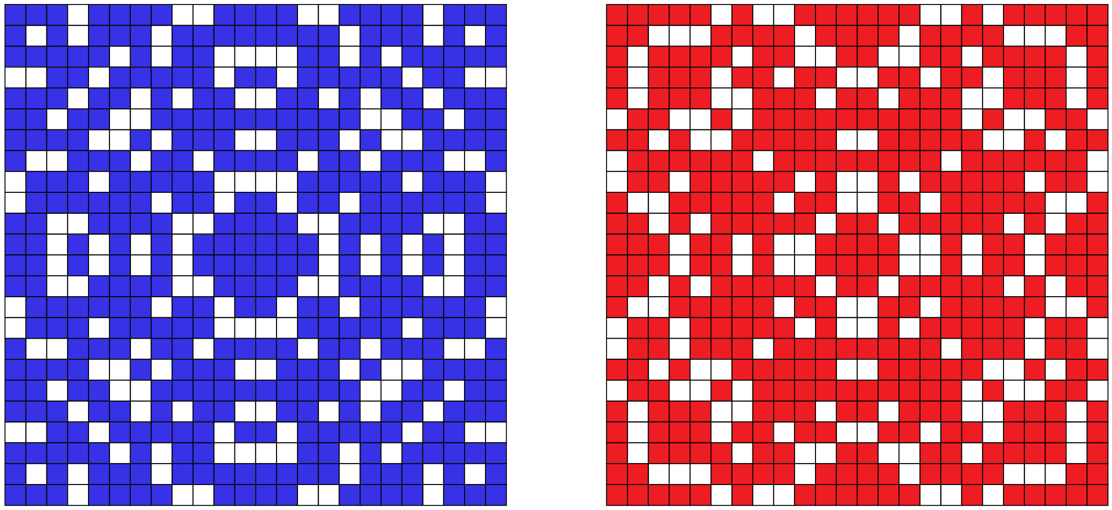


Figure 1: Two equivalence classes of n-queens solutions.

We will begin with a description of the n-queens problem on a chessboard in Section 2, and then in Section 3, we discuss the possible symmetries of solutions to this classical problem. Finally we conclude in Section 4 with more artwork created by n-queens solutions and their symmetries followed by a discussion on beauty and symmetry.

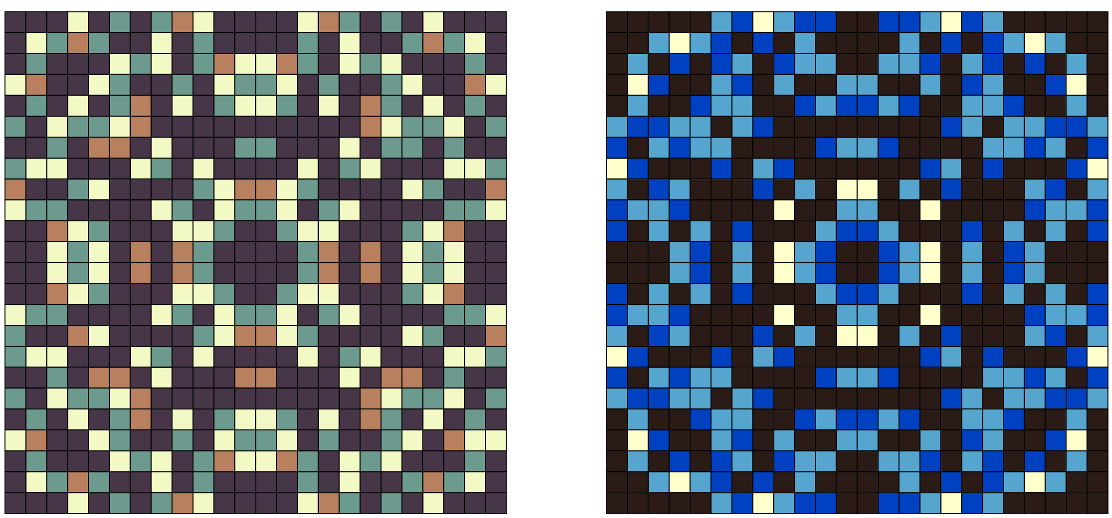


Figure 2: Artwork created by overlapping 24-queens solutions.

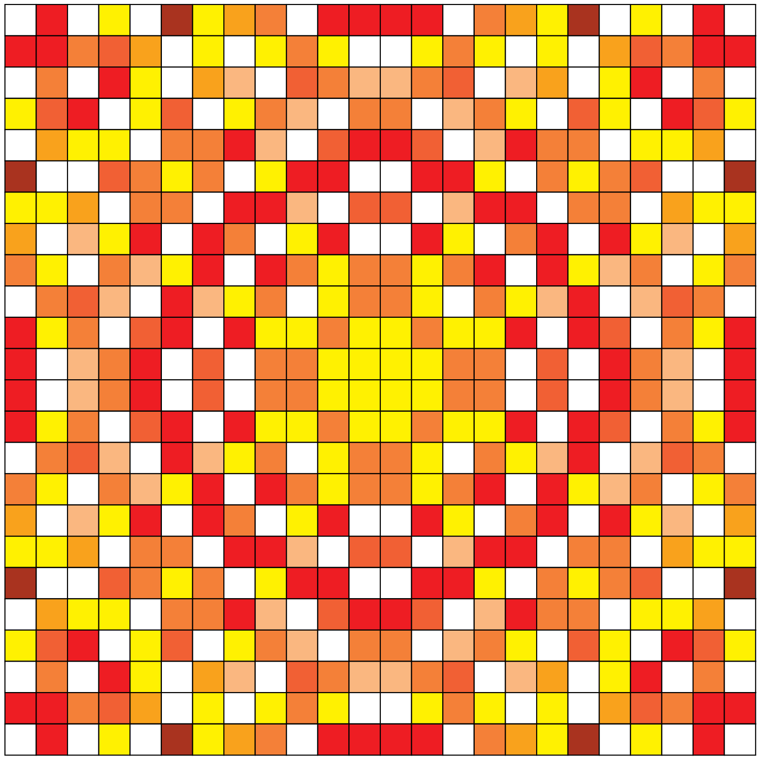


Figure 3: Noonday Sun.

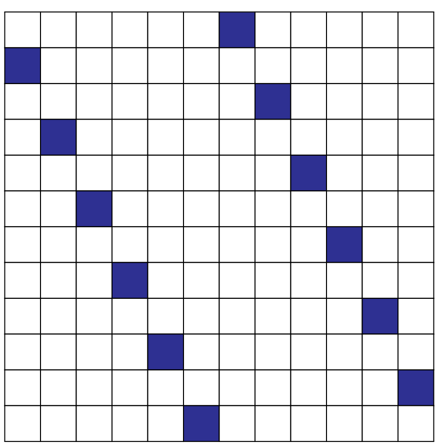
2. The n-queens problem

The n-queens problem is a classical chess problem that asks, "When can n queens be placed on an $n \times n$ chessboard so that no queen can attack another?" The origins of this problem go back to Max Bezzel [4] in 1841, who asked if eight non-attacking queens could be placed on a standard 8×8 chessboard. In 1850 all 92 solutions in the standard case were published by Nauck [14], and in 1874 Pauls, in two articles [16, 17], gave the first set of solutions for the general n-queens problem, that is, the problem of finding a specific solution set of n queens for every $n \geq 3$.

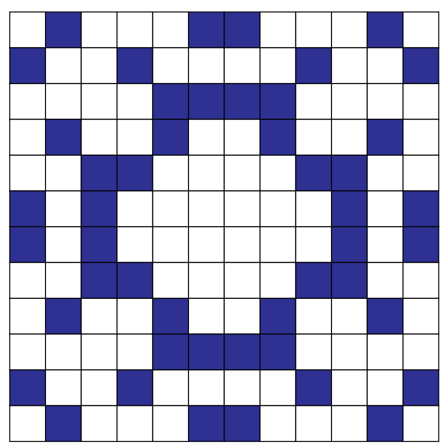
In the following years, researchers began to look for other sets of general solutions to this problem. Computationally given a small enough n (or fast enough computer) all possible solutions can be listed out, but this is not practical for large n. A general solution must allow a user to generate an infinite set of sets of non-attacking queens of various sizes. Techniques for finding these infinite sets include congruency classes, d-circulants, magic or semi-magic squares, Latin squares, and quasi-groups. Generalizations of this problem also abound, including investigating n-queens solutions on boards of different shapes and topological structures as well as boards with other pieces, such as pawns, blocking queens' attacks. A reader could discover more about techniques used to solve and generalizations of the n-queens problem in a survey paper by Bell and Stevens [3].

For this paper, we restrict the size of our chessboards to n congruent to 0 modulo 6 and first consider classical n-queens solutions. One historical technique for finding an n-queens solution is to choose a starting square on the board and place queens at regular intervals of $(1, k)$ for some integer k. Figure 4a illustrates this strategy, showing an example of the 12-queens solution originally due to Pauls. Here one can place the first queen on the first square of the second row and place further queens on the left half of the board by moving right one and down two, that is, steps of $(1, -2)$. When the bottom of the chessboard is reached we circle back to the top, starting with the seventh square of the first row and continuing is steps of $(1, -2)$. This approach finds what are called regular solutions on each half of the chessboard.

Other solutions can be found by transformations of Pauls' solution. The refection of Pauls' solution across a diagonal was originally due to Scheid [18] and the vertical reflection was due originally to Yaglom and Yaglom [21].



(a) 12-queens solutions due to Pauls.



(b) 12-queens solutions due to Pauls, Scheid, Yaglom and Yaglom with fourth reflective solution.

Figure 4: Classical 12-queens solutions.

In subsequent work, other authors have found different ways of representing general solutions, but in the case where n is congruent to 0 modulo 6, these new solution were always equal to one of the solutions found by Pauls, Scheid, or Yaglom and Yaglom. Figure 4b shows these three solutions along with a fourth solution found by composing the reflections of Scheid and Yaglom and Yaglom or equivalently rotating Pauls solutions 90° in the case $n = 12$. Given the lack of general solutions in this case historically, it is an interesting challenge to find more such solutions. (See [5] for a more complete discussion on n-queens solutions in this congruency class.)

Now, let us consider the types of symmetry we can find in n-queens solutions by exploring an algebraic group action.

3. Exploring symmetry in n-queens solutions

A traditional chessboard is a square and hence any symmetry of a solution must transpose the square onto the square. There are four rotations which are given by rotating the board by 0°, 90°, 180°, and 270°, and there are four reflections, namely the reflections across the center of the board vertically or horizontally and reflections across each diagonal. This group of eight actions is know as the dihedral group on eight elements or D_8. (More generally for any regular polygon with n sides the group of $2n$ reflections and rotations is the dihedral group D_{2n}.) On an $n \times n$ chessboard the rotations and reflections in D_8 will act on any n-queens solution by reflecting or rotating the positions of each individual queen on the board. In particular any solution created by rotating or reflecting an n-queens solution is also a viable n-queens solution. We ask, given an n-queens solutions how many distinct solutions are produced by actions of these eight group elements?

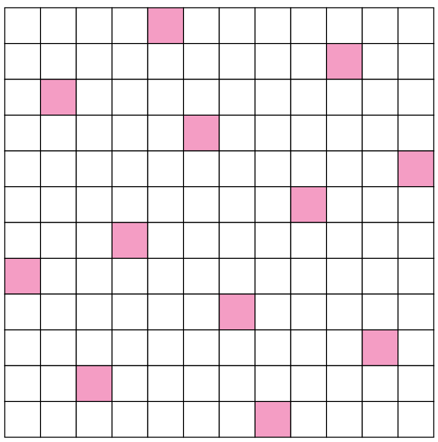
From abstract algebra, we know the number of distinct solutions generated by a group action must divide the order of the group. Thus on our chessboard, the dihedral group has order eight, so the possible sets of distinct solutions given an n-queens solution have cardinality one, two, four, or eight. Because an n-queens solutions does not have two queens in the same row, column, or diagonal, observe that applying any of the four reflections to an n-queens solution must produce a new distinct solution. Therefore the set of distinct solutions cannot have cardinality one. In fact each of the other three types of symmetry are possible, as we will see in the examples below. First, we need a few definitions.

Definition 3.1. *Given an n-queens solution, if the group action of the dihedral group D_8 produces four distinct solutions, we say each of these solutions is* ***symmetric****.*

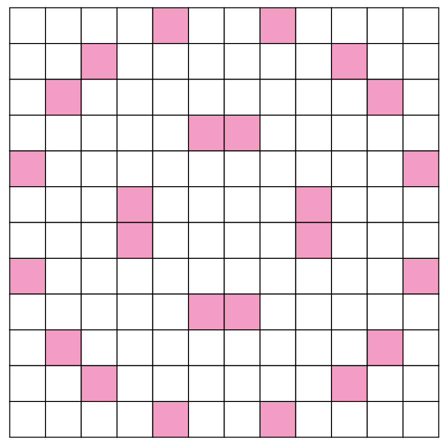
The solution found in Figure 4a is an example of a symmetric 12-queens solution. We see all four n-queens solution generated by Pauls' solution in Figure 4b.

Definition 3.2. *Given an n-queens solution, if the group action by the dihedral group produces only two distinct solutions, each of the solutions is called **doubly symmetric**.*

Figures 5a and 5b, respectively, display a doubly symmetric 12-queens solution and the pair of distinct solutions generated by the group action, respectively.



(a) A doubly symmetric 12-queens solution.



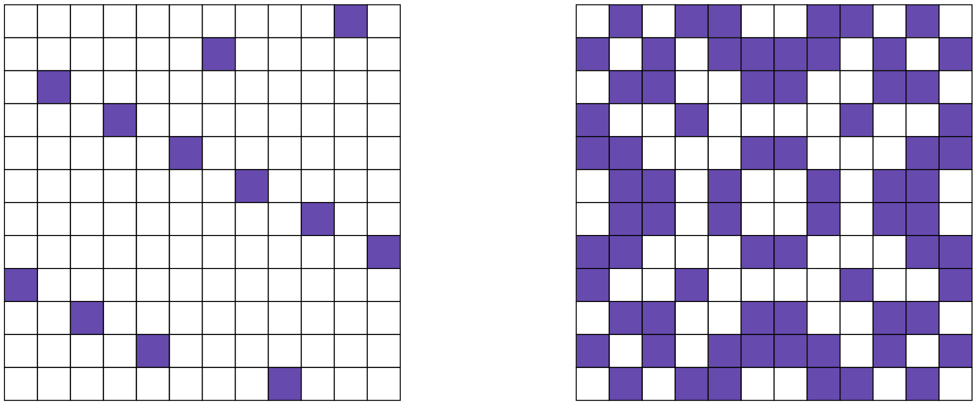
(b) Both solutions generated by the doubly symmetric 12-queens solution.

Figure 5: Doubly symmetric 12-queens solutions.

A 12-queens solution which is not symmetric or doubly symmetric is given in Figure 6a and we display all eight solutions generated by the group action in Figure 6b, see the following page.

Thus far we have seen examples for symmetric, doubly symmetric, and non-symmetric solutions in the case $n = 12$. It is interesting to note that it is not true in general that for all n there even exist symmetric and doubly symmetric solutions, especially when n is small. See Kraitchik's book [12] for some of the known counts for symmetric, doubly symmetric, and non-symmetric solutions or reference OEIS sequences A032522 and A033148 from [15] for totals.

In this article we wish to present artwork that can be created from an n-queens solution or combinations of n-queens solutions along with their reflections and rotations from the group action. Figure 7 (also on the next page) shows the picture created by overlapping the three sets of solutions found in Figures 4b, 5b, and 6b.



(a) A non-symmetric 12-queens solution.

(b) All eight solutions generated by the non-symmetric 12-queens solution.

Figure 6: Non-symmetric 12-queens solutions.

The solutions are colored similarly to their original colors with the intersection between the solution set in Figure 6b and each of the other two sets of solutions being a mix of the two original colors. For example, the third square in the second row is part of the equivalence class of solutions in Figures 6b and 5b, so is colored an orchid color which is a mix of the lavender and purple. Similarly, the sixth square in the third row contains a queen both solution sets in Figures 6b and 5b and hence is colored royal purple color as a mix of the blue and purple.

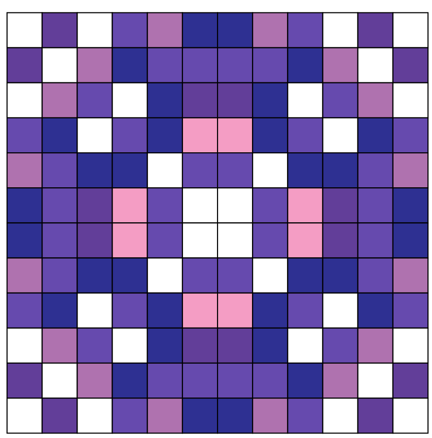


Figure 7: Overlap of the 12-queens solutions in 4b, 5b, 6b.

Some sets of n-queens solutions do not overlap.

Definition 3.3. *Two solutions or two solutions sets, respectively, are called* ***superimposable*** *if no square contains a queen in both solutions or solution sets, respectively.*

We note that the solutions in Figures 4a and 5a as well of the sets of solutions in Figures 4b and 5b are superimposable. Figure 8 displays a design with these two superimposable solutions. Finding and counting sets of superimposable solutions is an interesting topic in its own right, but when used to create a design, superimposable solutions use a minimal number of colors. In many cases it is the overlap and the utilization of more colors which makes the design more complex and attractive.

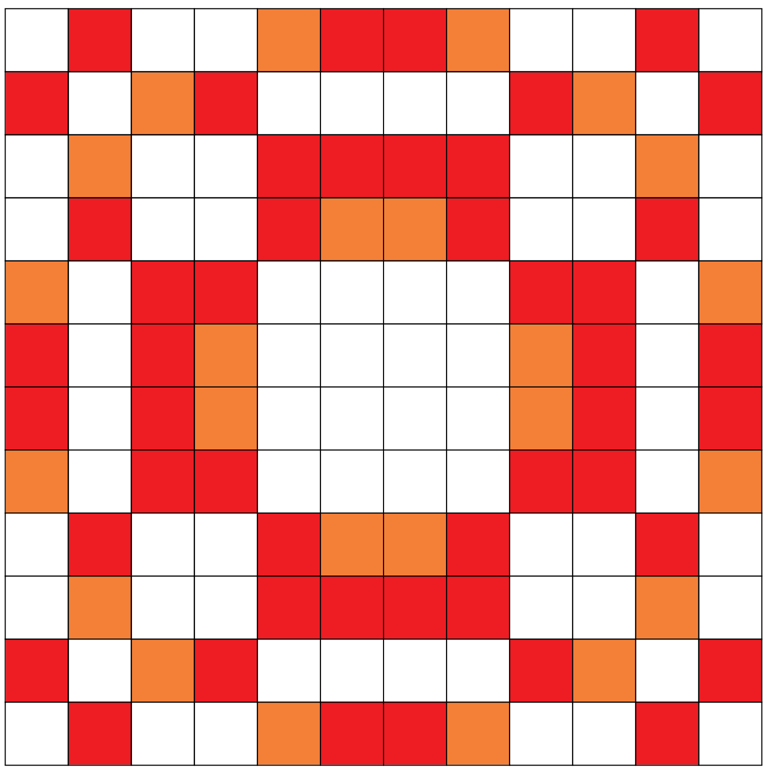
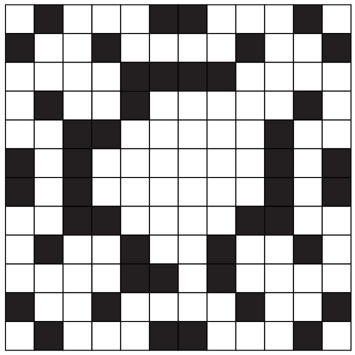
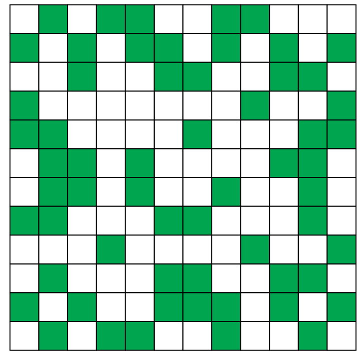


Figure 8: Two superimposable 12-queens solution sets

Finally we examine some designs lacking rotational or reflectional symmetry to contrast with the chessboards we have seen thus far. Consider the following four designs in Figures 9 and 10.



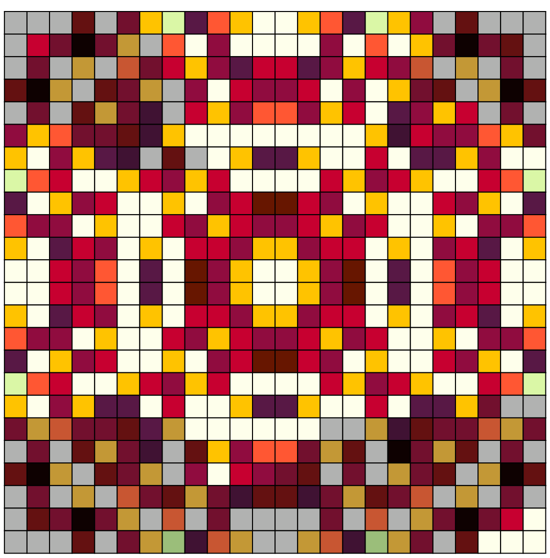
(a) Reflections and rotations of Pauls' 12-queens solution with queens removed.



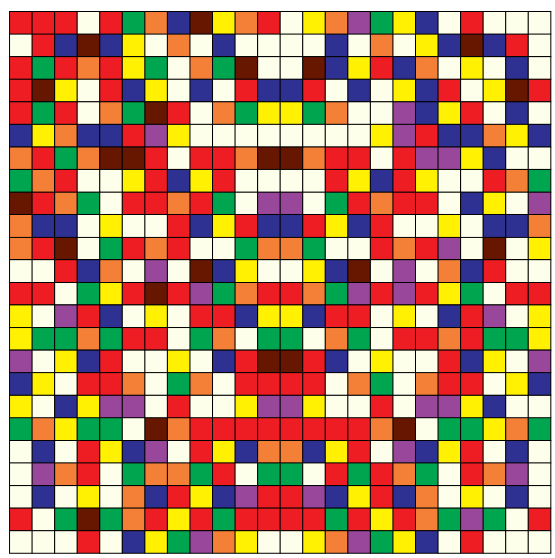
(b) Seven of eight solutions generated by a non-symmetric 12-queens solution.

Figure 9: Chessboards which are asymmetric with respect to reflection

Figures 9a and 9b illustrate designs which are not invariant under the group action of the dihedral group. These chessboards are not as predictable as and lack the symmetry of the previous designs. Two more complex examples of asymmetrical compositions are found in Figure 10. These pictures were created by switching the colors of a subset of squares in a set of three overlapping 24-queens solutions.



(a) Bi-toned 24-queens solutions.



(b) Chaotic 24-queens solution.

Figure 10: Asymmetrical 24-queens solutions.

In the final section, we explore why the symmetry in these designs may or may not appeal to an observer. We also display further examples of chessboard compositions for 24×24 boards, mainly focusing on the situation where a set of 24 queens produces eight distinct solutions through its symmetries.

4. The beauty of n-queens solutions

We begin our final section with illustrations of the general n-queens solutions in the case $n = 24$ mentioned above and used to create the overlapping designs. Starting with Pauls's solution, as is Figure 4b, the picture in Figure 11 shows the set of four solutions obtained by the symmetries of the square on the same 24×24 board.

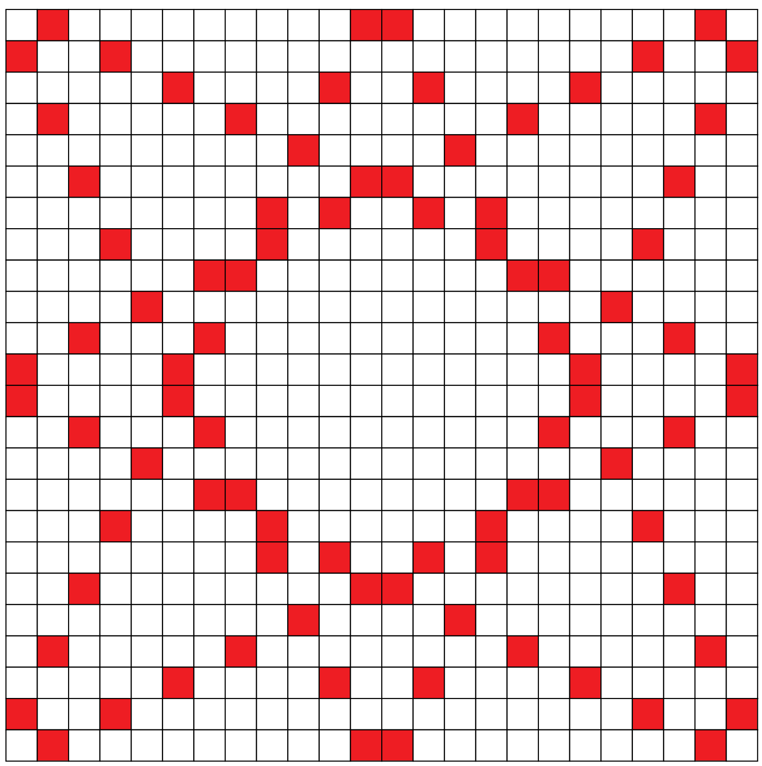


Figure 11: Symmetric 24-queens solutions generated from Pauls' solution.

Next, the pictures in Figures 12 and 13 illustrate designs generated by the group acting on different non-symmetric solutions.

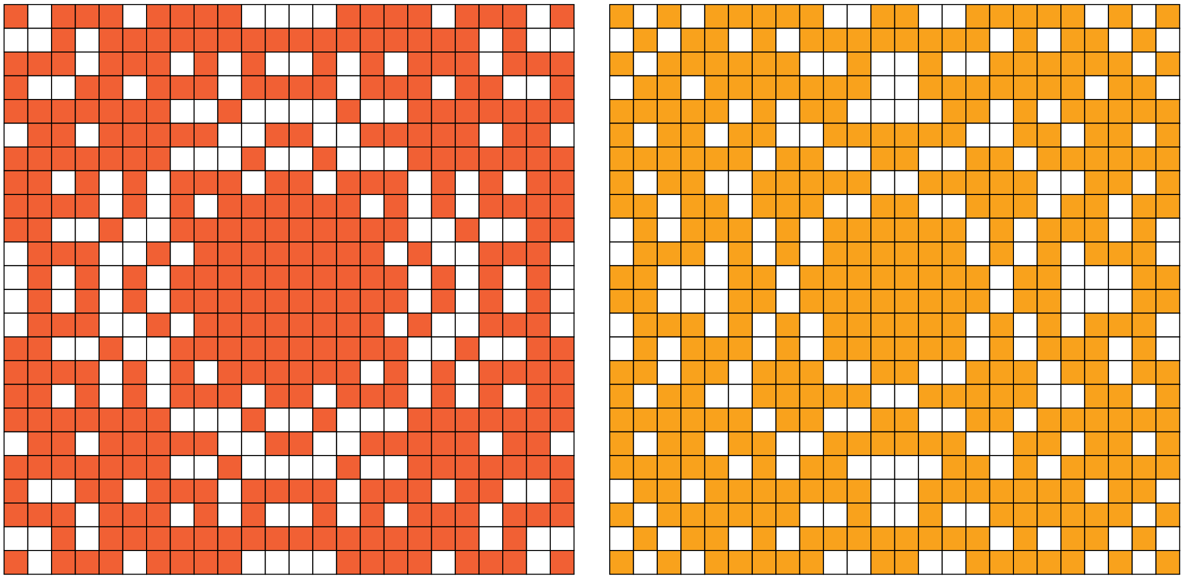


Figure 12: 24-queens solutions and their symmetries.

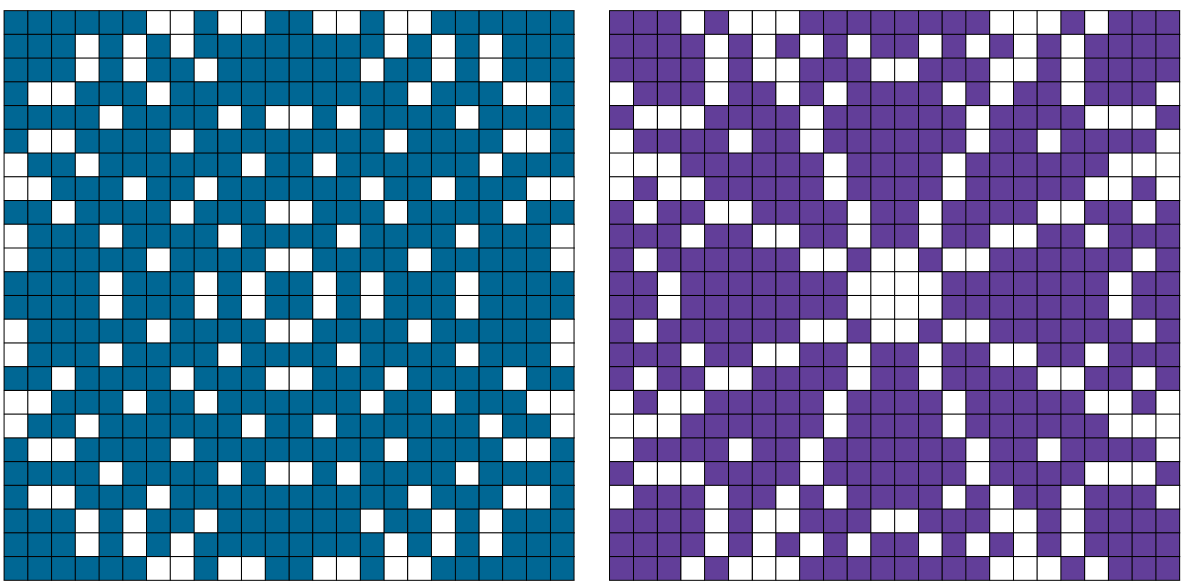


Figure 13: 24-queens solutions and their symmetries.

These solution sets along with those used to generate the artwork in Figure 1 are from general n-queens solutions in the case "n congruent to 0 modulo 6" found in [5, 6]. In each of these designs the white space shows the placement of the queens. We note that the patterns are not all alike; some solutions lead to flower-like pictures while others are more circular.

We can create even more complex designs by overlapping some of these 24-queens solutions. The general method to complete this process is as follows. Begin with a set, $Q = \{s_1, s_2, \ldots, s_k\}$, of k n-queens solutions for some natural number k. For each solution s_i in Q, apply the group action of the dihedral group and let S_i be the set of squares which contain a queen in at least on solution generated by the group action. Partition the squares of a chessboard into distinct sets as one would a Venn diagram; the sets being all non-empty intersections of exactly k sets from the set of equivalence classes and their complements $\{S_1, S_2, \ldots, S_k, S_1^c, S_2^c, \ldots, S_k^c\}$. For example, in the case $k = 3$, if no pair of solutions is superimposable, the nonempty sets are $S_1 \cap S_2 \cap S_3$, $S_1 \cap S_2 \cap S_3^c$, $S_1 \cap S_2^c \cap S_3$, $S_1^c \cap S_2 \cap S_3$, $S_1 \cap S_2^c \cap S_3^c$, $S_1^c \cap S_2 \cap S_3^c$, $S_1^c \cap S_2^c \cap S_3$, and $S_1^c \cap S_2^c \cap S_3^c$. Each of these sets is assigned a color and the chessboard is colored accordingly.

In many cases, as in Figure 7, the base colors assigned to the solutions can be combined and blended to choose colors for sets containing squares from more than one solution. Examples of overlapping solutions are displayed in Figure 14 and Figure 15 on the following pages. To me, each of these pictures is a snapshot of a single moment looking through the kaleidoscope just before the brightly colored glass falls into a new position. The compositions from these n-queens solutions demonstrate the symmetry and beauty of mathematics while providing a surprising connection between a classical chess problem and the memories from my childhood.

The final question which then arises is, "What is the connection between symmetry and beauty, especially in the context of the chessboard and n-queens artwork?"

The word symmetry can take on several connotations. Colloquially, symmetry implies a sense of balance and proportion and, according to Weyl in his influential book *Symmetry* [20], a "concordance of several parts by which they integrate into a whole." As a principle of art, symmetry is, by definition, a desirable and attractive property. To mathematicians the definition is more exacting. As discussed in Section 3, mathematical symmetry in a plane requires a transformation of an object onto itself as it is reflected across a line in the plane. Thus we ask, how does the more vague concept of artistic symmetry relate to precise mathematical definition and how is the beauty of an object revealed by both definitions?

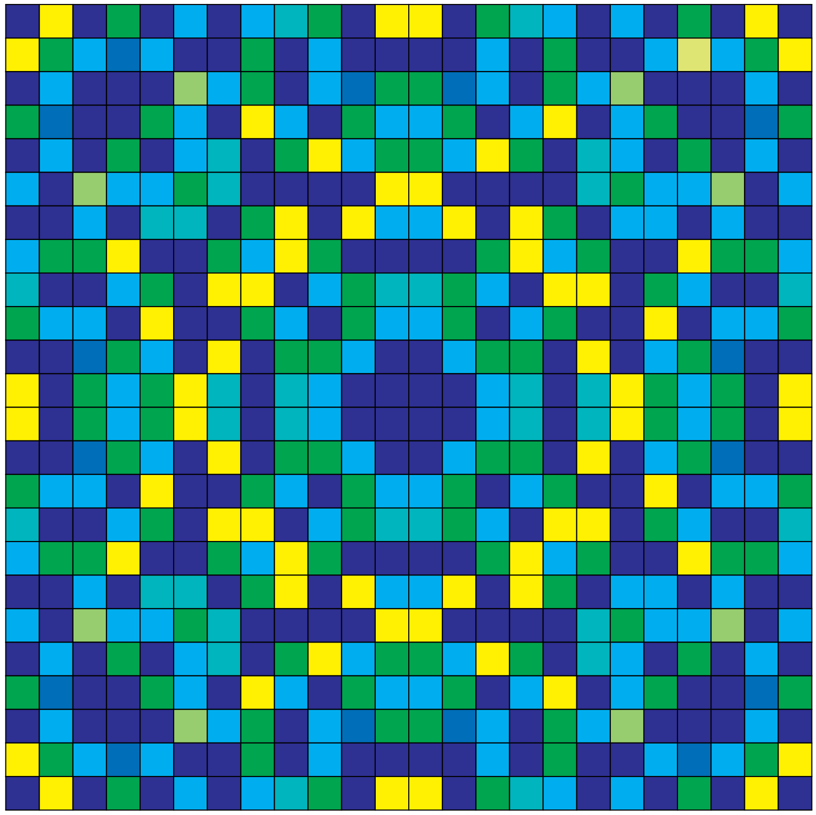


Figure 14: Artwork created by overlapping three 24-queens solutions.

Historically, artistic symmetry has been valued as one of the four objective laws of in the classical conception of beauty. In [1], the ancient Greek philosopher Aristotle wrote, "The chief forms of beauty are order and symmetry and definiteness, which the mathematical sciences demonstrate in a special degree." More recently, Voloshinov [19] makes the argument that symmetry is a "superprinciple" in art, analogously to symmetry as a superprinciple in physics, as the other three objective laws, proportion, the golden section, and rhythm, respectively, are actually applications of symmetry as "the symmetry of similarity", "the symmetry of the parts and the whole," and over time "translational symmetry", respectively. Thus, in this opinion, classical objective beauty is determined by artistic symmetry within a work of art, and so all designs created from overlapped equivalence classes of n-queens solutions are classically beautiful, including the simple kaleidoscopic designs in Figures 1, 12, and 13, as well as the more complex overlapping designs in Figures 7 and 14.

However, some argue that true mathematical symmetry is not pleasing in artwork. Voloshinov writes there is an "aesthetic coldness of ideal symmetry," and McManus [13] describes symmetry as synonymous with "constraint," "boredom," and "simplicity" like in Figure 11, while asymmetry is explained as "motion," "freedom," and "complexity" which can be felt even in the basic designs found in Figure 9. McManus argues that as counterparts both symmetry and asymmetry are essential to the discussion of aesthetics as well as science. For example, the design in Figure 10a retains the symmetry of shape, but not the symmetry of color. Too much asymmetry can also be a problem. Figure 10b may be less predictable than a mathematically symmetric composition, but also illustrates a chaos that can be difficult to comprehend. Dreyfus and Eisenberg note that the human attraction to symmetry is an open question and while it may be a part of our nature, learning to use symmetry is a skill that must be acquired [7, 8]. This may explain why artists and experts have a preference for the novelty of asymmetry as they have had more training in identifying and working with symmetric designs.

Lastly, there is evidence that humans are naturally drawn to symmetry because of the way the brain uses symmetry to understand relationships between objects. Symmetry has been explored in psychological studies as an important part of our perception of human beauty (see Grammer [11]); research also links symmetry with health and attractiveness (see Fink [10]). In [2], the controversial philosopher Avital describes symmetry as the relationship between an object depicted in a work of art and all such objects that exist outside the artwork; thus even though nature does not have perfect mathematical symmetry, a painting can. He argues that mathematicians and artists are both "concerned with the connecting of universals" and that the difference is in the "level of connectivity they achieve." Findings from Enquist and Arak [9] support these conclusions. They programmed artificial neural networks to analyze signals and saw "the preference for symmetry is a consequence of the need to recognize signals irrespective of their position or orientation." They note that humans find "symmetric patterns attractive in contexts unrelated to signalling" and hypothesize this "may result from the universal need among organisms to recognize objects," an idea introduced by Zee in [22]. Avital further explains, the external reference in figurative art will elevate an artwork to something bigger while abstract art is only self-referential and descends from an abstract concept to a single example. In this framework, Avital would categorize Figures 3 and 15 as artwork, while leaving the designs in Figure 2 as merely "aesthetic objects."

Despite these deliberations, the central unanswered question is posed by Weyl [20]:

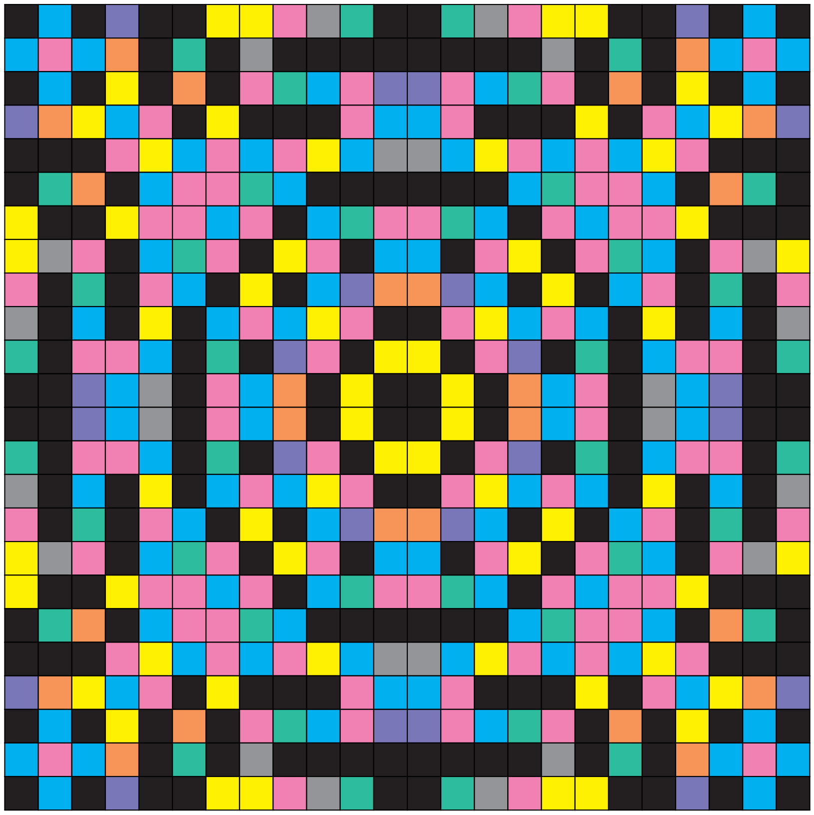


Figure 15: Stained Glass Window.

> One may ask whether the aesthetic value of symmetry depends on its vital value: Did the artist discover the symmetry with which nature according to some inherent law has endowed its creatures, and then copied and perfected what nature presented but in imperfect realizations; or has the aesthetic value of symmetry an independent source?

Many mathematicians who have been surprised, even awed, by the symmetry found in their research may agree with Weyl, as do I. And so we conclude with his response to the question above:

> I am inclined to agree with Plato that the mathematical idea is the common origin of both: the mathematical laws governing nature are the origin of symmetry in nature, the intuitive realization of the idea in the creative artist's mind its origin in art; . . . .

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