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A Beautiful Proof by Induction

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Abstract

The purpose of this note is to present an example of a proof by induction that in the opinion of the present author has great aesthetic value. The proof in question is Thomassen's proof that planar graphs are 5-choosable. I give a self-contained presentation of this result and its proof, and a personal account of why I think this proof is beautiful.

A secondary purpose is to more widely publicize this gem, and hopefully make it part of a standard set of examples for examining characteristics of proofs by induction.

Keywords: *proof; mathematical induction; beauty; planar graphs; graph coloring.*

1. Introduction

I have rarely seen proofs by induction being put forward as examples of especially "nice" or "pleasing" proofs. Instead, they are often rather seen as a second rate alternative, to a better "direct" proof. A common example of this phenomenon is the case of the two well-known contrasting proofs of the arithmetic sum formula for the sum of the n first natural numbers, the first one making a geometric interpretation of the terms, and matching terms i and $n-i+1$, to produce an n by $n+1$ rectangle, and the second one proceeding by induction. Another example is the binomial theorem, where the "combinatorial" proof is usually preferred over the inductive one. Though the specific reason for preferring a direct proof over an inductive one may vary from case to case, I am content here to claim that such a preference, for whichever reasons, prevails.

This short note aims to contribute to a vindication of proofs by induction in general, by presenting an extraordinarily pleasing example of a theorem and its proof by induction. The theorem in question is Thomassen's proof that all planar graphs are 5-choosable [8], which is related to the famous four-color theorem. This note also aims to circulate the theorem and proof to a wider audience, perhaps misguided by less pleasing acquaintances with induction. To this end, I have strived to make the account self-contained.

In Section 2 the theorem is presented together with some technical and historical background, and in Section 3, I give a personal account of the aesthetic virtues of the proof, and make comparisons with selected accounts of beauty in mathematics. Section 4 concludes.

2. 5-choosability of planar graphs

2.1. Technical preliminaries

A *graph* is a set V of elements, called *vertices*, and a set E of unordered pairs of vertices, called *edges*. Two vertices that belong to some common edge are said to be neighbors. The standard visual representation of a graph is as a set of dots, one for each vertex, and a line or curve (representing an edge) connecting two dots, if the corresponding pair of vertices belongs to the set of edges. For simplicity, we shall throughout this note assume that the graphs are *connected*, that is, that every vertex can be reached from every other vertex by a sequence of vertices that are connected by edges.

In everyday language, and in many applications, the word *network* is often used, but in the mathematical discipline studying these structures, namely *graph theory*, invariably the term 'graph' is used. This should of course not be confused with the term 'graph' as used in mathematical analysis, as the graph of a function. A *cycle* is a sequence of vertices sequentially connected by edges, where no vertex is repeated, with the exception that the first vertex coincides with the last.

We say that a graph is *planar* if it can be drawn in the plane without crossing edges. It is a *triangulation* if each face is a triangle, and a *near-triangulation* if each *bounded* face is a triangle, that is, there is an outer cycle which encloses the whole graph and every internal face is a triangle. Note that if the outer cycle has only three vertices, then we have a triangulation.

As an example, we may take the *complete graph* on 4 vertices, the graph on 4 vertices with edges between every pair of vertices; see Figure 1. The graph has three bounded faces (contiguous areas, enclosed by edges of the graph), each of which is a triangle, and one unbounded face (the outside of the graph), which is also considered a triangle, since exactly three edges border on it.

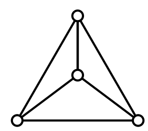

Figure 1: The complete graph on 4 vertices, a triangulation

The graph in Figure 2 is a near-triangulation. The unbounded face is not a triangle, since a total of five edges border on it.

A graph G is *k-colorable* if k colors can be assigned to the vertices of a G in such a way that no two neighboring vertices are assigned the same color. More generally, we say that G is *k-choosable* if for any assignment of lists of k colors to the vertices of G, it is possible to color the vertices with colors from their respective lists in such a way that no two neighboring vertices are given the same color. It should also be pointed out that a k-choosable graph is always k-colorable, since one of the possible list assignments is the one where all lists contain the exact same k colors.

As an example of a coloring from lists, consider the graph in Figure 2, where the vertices are assigned lists of colors of length 3, and the chosen color for each vertex is in bold.

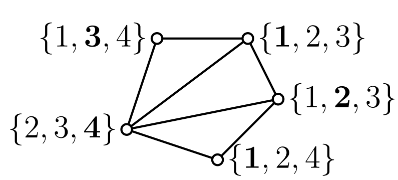

Figure 2: An example of coloring from lists

It is easy to check that no two neighboring vertices are assigned the same color. To establish that the graph in the example is indeed 3-choosable, one would have to prove that for *any* assignment of lists of length 3 to the vertices, there is a way to select one color from the list of each vertex in such a way that no two neighboring vertices are given the same color. The graph in the example is not 2-choosable, since it is not even 2-colorable.

2.2. Short historical note

The most famous problem in the field of graph colorings is indubitably the so-called four-coloring conjecture, stating (in one formulation) that every planar graph is 4-colorable. It was originally formulated in print by de Morgan in a letter to Hamilton in 1852, but the origin of the problem was Francis Guthrie (see Wilson [11]). It was immediately clear that some planar graphs required 4 colors, for instance the complete graph on 4 vertices.

In 1879, Kempe published the first proof of the theorem, but unfortunately the proof was flawed, as observed by Heawood ten years later. Kempe's proof method, however, yielded the result that all planar graphs are 5-colorable.

The conjecture was finally resolved in the positive by Appel and Haken [1], using considerable computer assistance for case analysis. A more detailed account of the history of this problem is given in Wilson [11].

A related, but somewhat less famous problem in the field, posed by Vizing in 1975, and independently by Erdős, Rubin and Taylor [3] in 1979, is whether every planar graph is 5-choosable. At the time the problem was formulated, no examples which required lists of length 5 were known, and it was not until 1993 that Voigt [9] gave the first example of a planar graph which is not 4-choosable. None of the known examples of planar graphs that require lists of length 5 even come close to the simplicity exhibited by the complete graph on 4 vertices, which served as an example for the necessity of 4 colors in the 4-coloring problem. As the counterexamples are not directly relevant to neither the proof nor my analysis, I shall not present them here.

The 5-choosability problem received a fair amount of attention, but it was not until 1994 that the problem was finally solved, when Thomassen [8] proved the theorem detailed in the next section.

2.3. Thomassen's 5-choosability theorem

The following statement is essentially the theorem proved by Thomassen [8], with some slight modifications.

Theorem 2.1. *Let G be a near-triangulation with outer cycle $C : v_1v_2 \cdots v_pv_1$. Assume that v_1 and v_2 are precolored with colors 1 and 2, respectively, and that vertices on the outer cycle are given lists of at least 3 colors, while vertices in the interior of the graph (i.e., not on the outer cycle), are given lists of at least 5 colors. Then the coloring of v_1 and v_2 can be extended to a coloring of the whole graph, such that for each vertex, the color assigned is present in the list on that vertex.*

We note that we can always add extra internal edges to any planar graph G to make it a near triangulation, and that this will only make it *harder* to color the graph, since more pairs of vertices will be neighbors. Also, every list assigned is at most of length 5. In fact, two of the lists have length 1, and the lists on the outer cycle have lists of length at most 3. We see therefore that Theorem 2.1 has as an immediate corollary the following:

Corollary 2.2. *Every planar graph is 5-choosable.*

Clearly, the end goal of proving the theorem was to get the corollary. However much it was felt that the simple formulation "every planar graph is 5-choosable" was true, a proof by induction of this more simple formulation proved very difficult. It should also be noted that the proof presented here, following some contemplation, in fact gives a constructive, effective, way of actually finding the coloring.

The proof of Theorem 2.1 given here is Thomassen's original proof, with some additional comments and expanded arguments. The proof is by induction on the number of vertices, n. Note that the trivial cases $n = 1$ and $n = 2$ are not covered by the theorem, since in these cases, no outer cycle is formed. However, clearly a graph on 1 or 2 vertices is both planar and 5-choosable.

Proof of Theorem 2.1. We proceed by induction on the number n of vertices of G. If $n = 3$, then $G = C_3$, a cycle on three vertices, and the result clearly holds, since there are at least three available colors for the only vertex v_3 that is not pre-colored. From now on, we assume that $n \geq 4$.

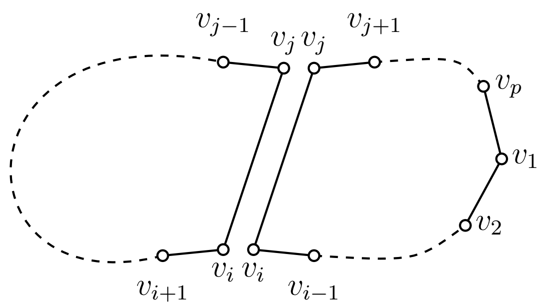

Figure 3: Splitting the graph over a chord from v_i to v_j

Suppose the outer cycle C has an edge v_iv_j (a *chord*) between two vertices v_i and v_j that are not adjacent along the outer cycle. We may then, by relabeling if necessary, assume that $2 \leq i \leq j-2 \leq p-1$ (note that $v_{p+1} = v_1$). We now apply the induction hypothesis to color the graph consisting of $v_1v_2 \cdots v_iv_jv_{j+1} \cdots v_pv_1$ and its interior (the right part of the diagram in Figure 3). This fixes the colors on vertices v_i and v_j. We now apply the induction hypothesis to color the cycle $v_jv_iv_{i+1} \cdots v_{j-1}v_j$ and its interior, treating the two neighboring vertices v_j and v_i as being pre-colored (that is, taking the roles of vertices v_1 and v_2 in the statement of the theorem). Since this completes the coloring of G, we may from now on assume that G has no chord.

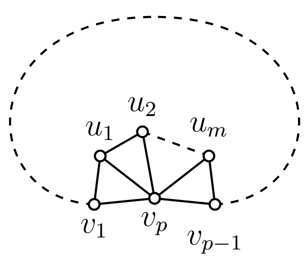

Figure 4: Neighbors of the vertex v_p

Consider the vertex v_p. Let $v_1, u_1, u_2, \ldots, u_m, v_{p-1}$ be the neighbors of v_p, in that order (see Figure 4). Note that since G has no chord, all the u_i are internal vertices, and since G is triangulated, $C_p : v_1v_2 \cdots v_{p-1}u_m \cdots u_2u_1v_1$ is a cycle in G. If v_p is removed, C_p is the outer cycle of the remaining graph, G_p, and so the induction hypothesis applies to G_p.

More specifically, suppose c_1 and c_2 are two colors from the list of assigned colors to v_p, different from the fixed color 1 on vertex v_1. If we remove the colors c_1 and c_2 from all the lists on the vertices u_i, they still have 3 colors left, and the induction hypothesis still applies. With this particular assignment of lists, we color G_p, and observe that we can then also color v_p with either c_1 or c_2, depending on which color happens to be assigned to the vertex v_{p-1}. This concludes the proof. $\square$

3. Aesthetic appraisal

3.1. Aesthetic generalities

First, the testimony of the present author is that this proof evokes a very strong positive emotion. It is of course hard to self-report wherein exactly this positive emotion consists, but I feel that it is at least *immediate* and *disinterested*, in the sense that the emotion is not preceded by rational evaluation, and that I have no personal stake in the particular outcome of the matter. Regarding the latter point, in other words, I am not happy that it turned out that in fact lists of length 5 suffice, nor would I have been happy if it had turned out that sometimes lists of length 6 are needed.

These two components are the two principal components of a standard philosophical account of *taste*, which underlies the concept of the aesthetic (see Shelley [7]), so I take this as an indication that the emotion results from making an aesthetic judgement.

3.2. Personal aesthetic appraisal of aspects of the proof

Going on from the immediate emotion evoked by this proof, I shall now expand on a more reasoned view on why I think this is a beautiful proof.

3.2.1. Surprise and simplicity

There is an element of pleasant surprise to this proof. Given the historical setting, with the four-color conjecture and its eventual computer-assisted proof, and the complicated nature of the counterexamples informing Vizing's and Erdős, Rubin and Taylor's problem, one might have expected the solution to the problem to be very complicated. Instead, Thomassen's proof is relievingly simple.

There is a certain bittersweet quality to this simplicity, in that it indicates that there are still well-studied, seemingly difficult results out there, whose proofs do not require a complex apparatus of new theory and endless toil (like the only presently known proof of Fermat's last theorem by Wiles). This is more like a genius business idea, or a brilliant pop song or a great rock riff. Somehow, people keep coming up with new great and simple business ideas or music compositions, and in retrospect it appears as though it would have been possible to come up with them oneself. This proof certainly falls into that category, and I believe that a large part of my appreciation of the simplicity of the proof comes from this fact, rather than from the element of surprise.

3.2.2. Artisanry

Simple though the proof is, it is clear that the specific statement of the theorem is skillfully tailored to allow for a successful proof by induction. As I read the proof and first realize how the induction step in the case with the chord relies on having two pre-colored vertices, and then realize how the case without the chord relies on the triangulated nature of the graph, a sense of artistic mastery is evoked. Expanding on this, I consider the theorem to be proved to be Corollary 2.2, and formulating the statement of Theorem 2.1 to be part of the proving process. Here, there is to me evidence of a dialectic process, of engaging in a back-and-forth between proof attempts and reformulations of the theorem, in some ways similar to the process that Lakatos [5] describes.

In other words, I don't believe that the statement of the theorem that Thomassen proved was something that was expected in advance to be true in this exact form. It follows that it was not simply a matter of employing induction to prove it. Instead, a more reasonable account of this process is that proof by induction was a proof method that was felt to be promising, and that the particular provable statement had to be discovered by varying the conditions so that the induction step could in fact be carried out. Though I have no specific insight into how exactly this process unfolded (which would be interesting in itself to study), I imagine that I can make a reasonable guess, and based on this guess I experience something both impressive and inspiring.

3.3. Empirical grounding

Is my account here simply the personal view of a single mathematician? I do not hope to have given the reasoned account any mathematician would present regarding this particular proof. What I do believe, however, is that there is wide consensus among mathematicians that this indeed is a singularly pleasing proof. The empirical data I have to back up this claim is rather modest, and not collected in a methodical fashion, but whenever I have discussed this proof with other mathematicians, they have agreed (in general terms) about the proof being aesthetically pleasing.

3.4. Relation to previous accounts of beauty in mathematics

In this section, I shall briefly describe some previous accounts of beauty in mathematics, and relate these accounts to my personal account above. I have not been explicitly guided by these accounts in formulating my own view on the example presented in the present paper, but they will serve here to situate my experience in the literature.

Hardy and Rota are very often quoted on this subject, and I would be remiss not to mention them. I will then briefly treat general lists of adjectives, and finally relate my experience in this case to a more structured account of aesthetics in mathematics, based on Kant.

3.4.1. Hardy's view

In his book *A mathematician's apology* [4], Hardy proposes that surprise is at the heart of what constitutes the experience of beauty in mathematics. For Thomassen's result, I think there is little surprise involved in the result itself, and as I hope to have made clear above, I feel that the surprise that I experienced originated from the simplicity of the proof. This surprise was certainly informed by my previous knowledge of the difficulties involved in proving the four-color theorem, and so supports the claim that surprise is relative to background knowledge. Again, I don't believe my aesthetic experience comes from the feeling of surprise, so I feel that this part of Hardy's view is not convincing.

However, I'd also like to mention what is perhaps the most famous quote from Hardy's book [4]:

> "A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. [...] The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way." (pages 84–85)

Being partial to combinatorics, I see in this quote a relation to constructive mathematics, and recognize the constructive nature of Thomassen's proof as contributing to the sense of beauty. Of course, the induction used in the proof and the constructiveness of it are intimately related: The induction in a sense describes how to construct the coloring needed in a stepwise fashion.

3.4.2. Rota's view

In opposition to Hardy, Rota [6] contends that surprise is in fact not the source of experiences of beauty in mathematics, his counterexample being Morley's theorem, which he claims to be surprising but not beautiful. Instead, Rota claims that beauty is merely a proxy for enlightenment, and that when mathematicians talk of beauty, they do this to try and avoid talking of enlightenment. Is the theorem in this paper and/or its proof enlightening? Rota far from defines enlightenment in a matter so precise as to make it possible to unambiguously decide whether a particular proof is enlightening, but the core idea seems to be that enlightenment is what answers questions of the type "what is this good for?". Over and above the mere truth of a statement, there is some sort of meaning to it, knowledge of which would be enlightenment.

I (unfortunately) really can't say that I share Rota's contention that talk of beauty is mistaken, and is really talk of enlightenment. Despite this, we may of course try to evaluate the dimension of enlightenment in the example in the present paper. I think that just reading the statement of the theorem starts you wondering why on earth one would want to take a starting configuration as described, and why one would want to restrict the statement only to near-triangulations. It seems adequate to ask of this "what is it good for", and I feel that as the proof unfolds, an answer to this question is certainly provided: Those assumptions are good for precise technical things that make the proof work.

3.4.3. Lists of characteristics of beauty

In general, many lists of adjectives (more or less well grounded) have been proposed to capture the nature of beauty in mathematics. I shall be content here to mention one of these, selected because of its being grounded in the confessions of working mathematicians. Wells [10] collected the beauty judgements of 86 readers of the *Mathematical Intelligencer* (and I assume that at least a sizable portion of this sample were indeed mathematicians, in some sense). The mechanisms of beauty most frequently quoted by these mathematicians were *simplicity*, *brevity*, *surprise* and *depth*. The element of surprise has been treated above, so I shall now say a few words about the remaining three qualities in relation to the example in the present paper.

As I have described above, I consider in this case the simplicity and brevity (which are not the same, but close enough for a joint treatment here) to be part of the feeling of surprise. Indeed, brevity/simplicity also contributes independently to my personal aesthetic judgement of the proof as described above. I do, however, feel that neither the theorem nor its proof are "deep", given how I take the term to be understood. Neither of them reveals a connection between different areas of mathematics or casts new light on an entire area of study. This does not detract from my aesthetic experience in the present example.

3.4.4. Kant's aesthetics of mathematics

After listing terms intended to pin down the experience of beauty in mathematics, I will now turn to a more structured account, inspired by Kant, of the mechanisms of the aesthetic in mathematics. An independent exegesis of Kant's aesthetics of mathematics is beyond the scope of this paper, but it seems to be a common contention that Kant's aesthetics does not account for, nor even allow, beauty in mathematics. In opposition to this, Breitenbach [2], gives an account of Kant's view on the experience of mathematical beauty that can be described in the following way:

> [...] it is generated by the awareness that our capacities for imaginative synthesis fit together with our conceptual capacities in a way that makes it possible for us to learn something genuinely new about a priori concepts by pure acts of imagination. It is the awareness of this harmony, elicited by the process of mathematical demonstration rather than the finished product that is the

> basis of aesthetic experience in mathematics on Kant's account. (page 969)

Though perhaps a controversial reading of Kant, I find that this account closely resembles the feeling I have for the proof in question. As I have described in Section 3.2, in particular regarding what I call artisanry, this proof in a sense seems to carry its creation story on its surface, and my positive attitude towards it is, in my opinion, informed by an experience of this process of creation.

I think that in some proofs, for example proofs by direct calculation, it is clear that there is little 'imaginative synthesis' involved, and indeed in many less appealing proofs by induction, this is also the case. The proof of the arithmetic sum formula, for instance, is a perfectly standard argument, fit for being an exercise in a textbook, given that the formula itself is known. This proof has little imaginative content and, as predicted by Breitenbach's account, does not elicit an experience of beauty. Thomassen's proof, on the other hand, is an absolutely non-standard proof by induction, and the adaptation of the assumptions to allow for a successful induction is to my mind clearly an imaginative effort.

4. Concluding remarks

As I hope to have shown, there are pleasing proofs by induction. I also think that the theorem and proof presented here illustrate many important points regarding mathematical induction, that deserve to be studied in greater detail. I hope to revisit this issue in a subsequent paper.

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