Journal of Humanistic Mathematics

Volume 6 | Issue 1 January 2016

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Recommended Citation

Dietiker, L. "The Role of Sequence in the Experience of Mathematical Beauty," *Journal of Humanistic Mathematics*, Volume 6 Issue 1 (January 2016), pages 152-173. DOI: 10.5642/jhummath.201601.10 . Available at: https://scholarship.claremont.edu/jhm/vol6/iss1/10

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Cover Page Footnote

This article is based on work presented at the American Educational Research Association annual conference in San Francisco, CA in 2013. The lesson discussed in this proposal was filmed as part of a study funded through the National Science Foundation Career Grant #0546164.

This article is available in Journal of Humanistic Mathematics: https://scholarship.claremont.edu/jhm/vol6/iss1/10

The Role of Sequence in the Experience of Mathematical Beauty

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Abstract

In this article, I analyze the aesthetic dimensions of a *sequence* of mathematical events found in an unusual first grade lesson in order to demonstrate how sequencing may affect an individual's experience of mathematical beauty. By approaching aesthetic as a sense or felt quality of an experience in context [16, 19], this analysis explains how sequence can affect the way mathematical objects or actions are experienced by an individual. Thus, rather than questioning *whether* or *in what ways* a set of mathematical objects are beautiful or not, this paper addresses *under what conditions* is the mathematics in play beautiful. It is argued that with a better understanding of the temporal dimension of mathematical beauty, educational experiences with mathematics can be designed to captivate attention and nurture interest and positive disposition by students toward mathematics.

Keywords: *aesthetic; mathematical story; mathematics education*

Reflect on a moment in which you were moved by mathematical beauty and consider the conditions under which your experience was possible. Can you identify any possible changes to the context of that moment (for example, its timing, location, or what was known at the time) that might have caused a different reaction, such as indifference? Or can you pinpoint reasons why under the otherwise same conditions that someone else might not be similarly moved?

Rather than questioning *whether* or *in what ways* a set of mathematical objects or phenomena are beautiful (or not), this paper addresses *under what conditions* mathematics can be beautiful. It explores the nature

of mathematical beauty by analyzing how the *sequencing* of mathematical events affects the way mathematical objects or actions are experienced by an individual. Specifically, I demonstrate how the sequencing of mathematical experiences can influence aesthetic experience by analyzing an unusual first grade lesson. The particular lesson selected for analysis is special because it contains a unique moment in which the students erupt with excitement upon seeing a new set of geometric figures (referred to as the "aesthetic moment"[1]). By studying this lesson as a sequence of mathematical events, as opposed to analyzing the particular qualities of the mathematical objects involved, I explain how mathematical beauty can be viewed as a quality of a mathematical experience in a particular context that enables an individual or group of individuals to become captivated.

By approaching mathematical aesthetic as a sense or felt quality of an experience in a mathematical context [16, 19], this article explores how the sequence of mathematical events in this lesson may have enabled this stimulating result and explains how different sequences can change the perception of beauty. It focuses attention on the temporal nature of mathematical changes within the lesson sequence, such as how the questions raised and pursued by the class change as the lesson unfolds. To support this focus, this lesson is framed as a mathematical story [6] which focuses attention on how mathematical content unfolds across time. When viewed in narrative terms, the way prior experiences frame those that come later illuminates a temporal dimension of aesthetic moments, such as being moved by mathematical beauty.

To start, I describe with more detail the aesthetic moment of the lesson and explain how it is an example of mathematical beauty. I then introduce the mathematical story framework as a way to interpret both the aesthetic and mathematical dimensions of this mathematics lesson, including the notion of mathematical plot [6]. After this, I describe the mathematics lesson in detail and analyze it for its mathematical plot. That is, the aesthetic moment will be explained by considering how the mathematical characters (the geometric figures) and actions (particularly folding and halving) sequentially changed throughout this lesson, allowing for student anticipation and expectation. After this, I discuss the role of sequence with regard to how

[1] Although this does not mean that all moments are not essentially aesthetic in nature.

the students came to this aesthetic moment and also explore what might have changed if the sequence were altered. Finally, I end with a discussion of mathematical beauty, drawing connections and distinctions with other theories, stories, and analyses (e.g., [4, 11, 12, 14, 18, 19]).

1. The aesthetic moment in a first grade classroom

Approximately half an hour into the lesson, the teacher turned a page of her flipchart and the students sitting on the floor collectively gasped, leaned forward, and murmured excitedly. The audible enjoyment of the students was so evident that two students across the room, working independently on computers, strained their necks and moved closer to see. This moment marked the start of an extended analysis of geometric figures by the teacher and students lasting approximately ten minutes, throughout which the attention and participation of the first graders was remarkably maintained. This enviable moment, when the teacher had gained the attention and participation of nearly all of her students, stood in stark contrast to the prior portions of this lesson, during which most students appeared cooperative but were only marginally engaged.

What might have enabled this enjoyable mathematical experience? On the surface, there is little explanation for the student reactions to the geometric figures on the flipchart. No special motivational discourse was present that set up or prepared the students to be amazed; the teacher simply asked, "What about these shapes?" as she turned the page. The task itself, finding the line of symmetry, also remained unchanged at the aesthetic moment. In addition, at the point the students audibly gasped, the flipchart showed five different monochromatic shapes with some visible features similar to the figures the class had just finished analyzing (see Figure 1a for the figures analyzed previously and Figure 1b for the figures displayed at the aesthetic moment). Both sets of figures contained a mixture of polygons and closed figures and with curved sides. Geometrically, the distinction was that the first set of shapes contained figures that were symmetric while the second set of shapes contained asymmetric figures.[2]

[2]Unfortunately, the position of the video camera slightly distorts the geometric figures on the flipchart. Note that all of the diagrams in Figure 1a are symmetric and that the triangle in the top left corner of Figure 1b is not isosceles.

(a) The flipchart prior to the aesthetic moment, containing clockwise from the top left corner: a "house" pentagon with two consecutive right base angles, circle, heart, isosceles trapezoid, and square.

(b) The set of asymmetric shapes introduced at the aesthetic moment, containing clockwise from the top left corner: a scalene right triangle, a right trapezoid, another scalene right triangle, a "blob," and a figure with three perpendicular sides and one curved side.

Figure 1: The two consecutive pages of the teacher's flipchart.

2. Framing mathematical beauty

So what is the connection of this aesthetic moment to mathematical beauty? I argue that *at that moment in that lesson in that particular context*, those first grade students experienced mathematical beauty. I claim this for two reasons. First, what excites the children is mathematical in nature. That is, rather than a change in color, focus, media, or discourse, what changed in the lesson was a set of geometric figures on a flipchart. Given this, I contend it is reasonable to assume that the mathematical figures on the chart were responsible for the aesthetic reaction of the students in the context of the lesson.

Second, and perhaps more importantly, I contend that the visible and audible reactions of the students provide ample evidence that they were captivated by the asymmetric figures. The students' attention was not only caught by the entry of these geometric objects, but was also held throughout the remainder of the lesson. I argue that it is this captivation by mathematics that signals a deep attraction toward the objects, and it is this type of reaction that corresponds to the presence of beauty. For example, when I'm moved to remark or gasp in the presence of an object, such as a painting, this signals an attraction, an experience that draws me into studying and analyzing the object. Although the objects that gained the attention of the students were asymmetrical and thus part of a category of geometric objects that are sometimes described as "ugly," I will refer to these geometric figures as "beautiful," not because the students said they were, but because their reactions correspond to the reaction to beauty. Would the students have called the objects beautiful or otherwise? That, unfortunately, is not known as I do not have interviews with the students and the subject of beauty did not come up during the lesson by the students or teacher.

What enabled these "ugly" figures to become objects of mathematical beauty (via their captivation)? Clearly, what captivates individuals can differ for a number of reasons. For example, prior experiences can set up or even eliminate the possibility of surprise and delight. To see this, I contend that the shapes in Figure 1b would likely not get a second look from a mathematician — certainly not a gasp! The depth of knowledge and experiences with symmetry affects an individual's future interpretations. Possibly, if the asymmetric shapes had come earlier in the lesson, they may not have had the same effect. For this reason, I contend that it is not sufficient to only focus on the moment of beauty but is also important to understand what led up to it.

Therefore, the rest of this paper is focused on a way to interpret the aesthetic dimensions of mathematical moments in a sequence. With this understanding, I propose that it is possible for sequences of mathematical events to be deliberately designed so that learners can experience mathematical beauty.

3. Interpreting mathematical sequences as mathematical stories

In order to learn how this lesson's sequence of mathematical activity may have enabled the first graders to experience mathematical beauty through the introduction of the geometric figures in Figure 1b, this lesson was reinterpreted as a mathematical story. The reliance of a literary metaphor to make sense of mathematical experiences may not be surprising; literary stories are known for both their aesthetic qualities and their ability to deliver a message. Stories that are too boring are soon abandoned while stories that captivate can compel us to keep reading/watching. Although other forms of art also offer metaphorical possibilities to recognize mathematical qualities that attract or repel, such as sculpture or line art, literature additionally relies on logic to hold it together. That is, when a story is read, a new imaginary world is conjured by a reader within which the story's truth is defined. Therefore, analyzing the mathematical story told during this lesson can offer insight into how its mathematical content unfolded (its logic dimensions) and compelled these students to engage and react (its aesthetic dimensions).

Specifically, *a mathematical story* is defined as the temporal unfolding of mathematical content [5, 6]. Similar to a literary story, a mathematical story is the ordered sequence of connected mathematical events experienced by the participants in the audience connecting the beginning with its end. Developed from Bal's [1] narratological framework in [5, 6], this reframing of mathematical sequences reveals how mathematical characters are acted upon through mathematical action in mathematical settings. For example, *mathematical characters* are the mathematical objects brought into existence (objectified) through reference in the story, such as a square or right triangle. *Mathematical action* describes the work of an actor (such as a student or teacher) in changing the mathematical objects of study, such as halving a square by folding along a line of symmetry. Mathematical characters and actions are brought into being in a constructed "space" such as a white board or a coordinate plane, referred to as the *mathematical setting*.

Note that the interpretation of this first-grade lesson as a story reframes its sequence of mathematical content as a form of art [7] that can be deliberately sequenced for effect. Studying the aesthetic dimensions of mathematical experiences is not new (see, for example, [16, 19]), and several other researchers have used the metaphor of story (or narrative) to describe the aesthetic qualities of mathematics classrooms and texts (a mixture of in-

teresting examples include [3, 9, 10, 11, 17]). Specifically, Netz [11] offers a compelling demonstration of how a narrative framework can enable the aesthetic dimensions of different sequences to be compared and contrasted. Netz also argues that mathematical sequence is not a given but is, in part, a decision of an author. This suggests that authors of mathematical texts (including textbooks) can sequence mathematical content intentionally for dramatic effect.

The mathematical story framework is particularly relevant in the consideration of mathematical beauty because it offers a specific way to describe how mathematical sequences can move individuals — namely, its mathematical plot. The mathematical plot describes the way the mathematical sequence captivates and holds the interest of its audience. For example, when a mathematical story hints of a future revelation, it can spur the formulation and pursuit of questions ("How many lines of symmetry will this shape have?") similar to how a reader of a literary story might wonder how the story will end (i.e., "Will Romeo and Juliet live happily ever after?") [13]. Formally, the *mathematical plot* describes the dynamically changing tension between what is already known and what is desired to be known by the participants as the story progresses [6]. It enables the description of how a mathematical sequence can generate suspense (by setting up anticipation for a result) and surprise (by revealing a different result than the one anticipated). Something that is desired to be known by an individual is represented by a mathematical question. Some questions of mathematical stories may span the entire story while others may exist only for a brief time as small puzzles or mysteries.

4. The lesson as a mathematical story

I do not claim that this particular sequence of mathematical activities would result in the same aesthetic response by all individuals. Instead, the question under consideration is why this sequence was aesthetically compelling *to these students in this context*. Certainly, aesthetic experiences vary by individual [18] and the meaning generated during a story is a mixture of a reader's goals, experience, and purpose, and thus is individual in nature [15]. Since the audience of the enacted mathematical story described earlier was arguably the students, I maintain that the students' evaluation of the experience is the appropriate measure of its aesthetic value. However, since the experience of each child in the room is unavailable beyond the excited

reaction apparent on the videotape, this analysis (and thus the mathematical story presented here) represents my interpretation of the mathematical changes throughout the lesson.

4.1. Setting the scene

The selected first-grade math lesson was part of a sequence of lessons observed in a southeastern part of the United States in Spring 2011. This lesson was based on Lesson 4 of Unit 7 of the Grade 1 textbook *Math Expressions* [8]. Ms. Argonne (a pseudonym) was in her third year of teaching and this was the second year she had used this textbook. Prior to this lesson, the students used unit tiles to solve word problems involving the doubling of integers (between 1 and 10) and the halving of even numbers. The class contained 11 female and 9 male students. Although racial data was not available, approximately two-thirds of the class appeared to be White. At the start of the lesson, the students were seated at tables in groups of four. Off to the side was a colorful carpet area with an easel and flipchart. During the portion of the lesson leading up to the aesthetic moment, a female and male student were working independently off to the side of the room.

4.2. The mathematical story in six acts

What follows is a summary of the sequential events of the mathematical story as they unfolded throughout the lesson. Six acts were identified by changes in the mathematical characters (e.g., the geometric objects under consideration), actions (e.g., the changing of the objects, such as folding a paper square), or setting (e.g., paper cutouts or shapes drawn on a board or flipchart). For each act, I both narrate the mathematical story and discuss its significance. I also share relevant evidence of aesthetic reactions by students. Quotations are direct quotes from the teacher or students unless otherwise stated.

ACT A: At the start of the mathematical story, students were given a worksheet with colorful symmetric shapes (non-square rectangle, equilateral triangle, square, and circle) and were prompted to cut them out and "fold them in half." The students worked individually but interacted in groups of four, folding shapes repeatedly and comparing their results with each other. This folding activity was not obvious for most students and presented some challenge. Students asked the teacher and each other questions that were in essence, "How do I fold this in

half?" and "Is this shape folded in half?" During this time, the teacher circulated the classroom, praising students and asking questions such as, "How do you know [the fold] halves the shape?", "Are both sides equal?", and "Is there another way?" Whenever the teacher asked about the validity of an answer, the students refolded the paper shape to show that the two parts coincided. At times, to find an additional way to fold, several of the first graders erroneously folded the halved shapes in half again, to which the teacher would ask, "If you open it up, how many parts did you divide [the shape] into?" [student: "4"] followed by "If you folded in half, how many parts should you have?" [student: "2"]. Sometimes, the folding resulted into two parts that were not congruent, but this result was usually unchallenged by the teacher or other students. Overall, nearly all students displayed pride in how many possible folds they could find and the creative ways they found folds, as evidenced in their smiles, declarations ("I found seven!"), and displays to their fellow group members ("Look what I did").

ACT B: Ms. Argonne then pulled the class together to debrief the folding activity, asking volunteers to demonstrate on the board how they halved each figure. When soliciting volunteers, most hands went in the air, indicating that sharing out was desirable, confirming the sense of pride students displayed in Act A. One by one, selected volunteers came up to the front board and drew a line inside a geometric diagram to represent a fold. At one point, a student drew a diagonal to halve a non-square rectangle. The teacher accepted this answer along with the student's claim that the two parts were "equal" even though the parts were not reflections of one another.[3] Overall, three different folds were presented and accepted by the teacher for the square (horizontal, vertical, and diagonal), three for the non-square rectangle (horizontal, vertical, and diagonal), only one for the equilateral triangle (vertical),

[3]Certainly, the diagonal does bisect the rectangle. However, as it will be explained throughout this section, it is my interpretation that the teacher desired students to find reflected halves and not merely congruent parts. Unfortunately, it is not uncommon for new teachers to identify diagonals of non-square rectangles as lines of symmetry. She may have intentionally accepted this answer to de-emphasize the need for symmetry, but it is also possible that she would have rejected this answer had she been able to determine that the two parts were not reflections of each other.

and three for the circle (horizontal, vertical, and diagonal). During these presentations, the teacher repeatedly asked the presenters, "How many parts did you make?", "Are the parts equal?", and "Is there another way?"

By the end of Act B, the word "half" has multiple meanings for these students. The teacher and students have repeatedly linked halving to "resulting in two equal parts," but the teacher's gestures and word choices as well as the student explanations also have demonstrated a need for two halves to coincide when folded, and therefore, to be reflections of each other and, thus, congruent.[4] For example, while justifying a fold of a square, one student explained, "When I [folded it in a way that would not halve the shape] it would only be a little part there and some of the paper would be sticking out so I folded it in the middle and there would be two parts." The need to coincide also came up in the discussion of why the circle could have so many folds, for which a student reasoned that the lack of "straight sides" meant that parts would not jut out when folded. Also, when the teacher challenged the class to consider a case where a fold along a mid-segment of an equilateral triangle (although they did not use the term mid-segment), a student explained that the entire triangle was not folded in half because the top part (triangle) was "pointy" while the bottom part (trapezoid) was not. The need for symmetry was also indicated with gestures. For example, when the teacher asked, "would they be equal? Or the same?", she placed her hands next to each other palms forward, followed by palms together, communicating both congruence and reflective symmetry across a fold. Yet it was not always clear whether symmetry was always necessary, as demonstrated when the diagonal was accepted as a fold for the non-square rectangle.

Since each of the given geometric figures had a different number of possible folds that were accepted in this mathematical story, this hunt for folds extended the focus of "how can this shape be folded" to "how many ways can I fold this shape in half?" Despite the ambiguity of

[4]Note that in this lesson, there was no indication that a geometric shape could be halved so that the two parts are not congruent. Although this definition of halving is nonstandard and problematic, the mathematical story framework focuses on the mathematics that emerges and not what *should* emerge.

the term "halving" in this lesson, by the end of Act B, it would be reasonable for students to assume that all geometric figures can be folded into two parts that are reflections of each other.

ACT C: Next, the teacher and students moved to a new area of the classroom with a colorful carpet and an easel with chart paper. The teacher started by saying, "When you guys were folding your shapes, you were finding a line of symmetry [which] is a line that divides a shape into two equal parts, ... or two halves."[5]

With this statement, it would be reasonable for students to assume that all geometric figures can be halved (a conclusion of Acts A and B) by drawing or folding a line of symmetry. At this point, it was evident that this mathematical story was about the symmetry of geometric figures.

ACT D: The teacher next directed attention to five new figures previously drawn on a flipchart (see Figure 1a), and asked, "How do we find a line of symmetry [of these new shapes]?" All of these shapes, which included a heart, circle, square, isosceles trapezoid, and a pentagon that resembled a house with a pitched roof, appeared symmetric. Students showed enthusiasm to volunteer to present, but once a student was selected, the attention of the other students waned. Students were yawning, leaning against a wall, and looking around the classroom. Selected students came up to the flipchart, drew lines of symmetry on the chart, and explained their answers. In the case of the isosceles trapezoid, a student drew the diagonal and explained that the two parts are "the same sized triangles" and "equal." This answer was accepted by the teacher, indicating that she also may not have recognized in Act B that the two parts of the non-square rectangle formed by the diagonal were not reflections of each other (since there was also evidence that the teacher desired symmetry). The accepted answers confirmed the potential student assumption that all geometric figures have a line of symmetry. The assumption that halving is the same as finding a line

[5]By stating this, the teacher lumped together cases in which reflection did not exist (i.e., the diagonal of the non-square rectangle) with those that did. This provides evidence that the teacher may not have noticed the lack of symmetry in the case of the diagonal of the non-square rectangle and may have been unaware of the contradiction.

of symmetry was also confirmed when the teacher skipped the circle because "you guys just did the circle," referring to the presentations of halving a circle in Act B.

The mathematical action of "halving" (which the teacher now called "drawing a line of symmetry") remained unchanged throughout this act, which supported the focus of the students on the question, "Can I draw a line of symmetry?" With this focus, the expectation for more symmetric figures was set. The students were positioned as sleuths searching for these lines of symmetry; the geometric shapes represented symmetric mysteries needing to be solved.

ACT E: Next, the teacher flipped the chart paper and revealed five asymmetric figures (see Figure 1b) and asked, "What about these?" As described earlier, nearly all of the students gasped and showed visible excitement. The students' recognition that the pursuit of a line of symmetry may not fruitful is what Barthes (1974) refers to as jamming. In narrative, jamming occurs when there is an indication that a question of the story by a reader is suddenly and unexpectedly threatened. This dawning realization spurs a new mathematical question, although never explicitly stated: "Are all shapes foldable [i.e., symmetric]?" To answer this, volunteers came up to the flipchart one at a time and attempted to draw a line of symmetry for each of these figures.

In contrast to earlier acts, the students' attention to these asymmetric objects was sustained even after volunteers were selected to go up to the flipchart. Also, during the presentations, several students quietly initiated discussions with neighbors on the carpet, remaining on task and pointing repeatedly at the flipchart as they discussed the shapes.

Also during this act, the relationship between symmetry and halving was further strengthened. For example, when a student erroneously drew a diagonal for the right trapezoid as a line of symmetry, the teacher asked, "Are these parts equal?" and explained that the diagonal was not a line of symmetry because the parts were not halves, thus, making it clear that all lines of symmetry create two halves.

ACT F: The teacher ends the lesson by stating the moral of the mathematical story: "They have to have equal parts, so not all shapes have a line of symmetry, ok? ... A lot of them do have lines of symmetry, but not all of them do."

5. The mathematical plot: The logical and aesthetic threads of the mathematical story

The mathematical plot of this lesson is the traceable thread of the logical and aesthetic dimensions of the content that explains how the later events of the mathematical story were anticipated and interpreted. The students' expectation can be understood by examining how the mathematical focus of the mathematical story shifted throughout its sequence. The shifts in mathematical characters, actions, and settings, and focal mathematical questions under investigation during each act are presented in Table 1 on the following page. Note that this table reflects my interpretation of the mathematical characters, actions, settings, and questions under pursuit throughout the lesson as evidenced by the student and teacher discourse. For example, the mathematical action in Act D is described as remaining unchanged as the students shifted from halving to finding lines of symmetry. This is because, in this lesson, the act of drawing a line of symmetry in Act D was the same in both goal and process as the act of halving previously. Since the prior experiences with halving in this lesson focused on finding parts that were symmetric and because the teacher explicitly linked the act of finding a line of symmetry to what the students were doing previously, I conclude that the mathematical action in Acts D and E remained unchanged in this mathematical story.

As shown in the table, each act leading up to the aesthetic moment at the start of Act E had a mathematical story element that remained unchanged, allowing a gradual transition of mathematical content. At the beginning of the mathematical story, the focus was on individual geometric paper shapes that could be folded in half in multiple ways. The transition to drawing the "folds" on the board connected the folding activity with drawing line segments within geometric figures. The naming of these line segments as lines of symmetry in Act C shifted the focus of the mathematical story from being about the geometric shapes to being about the lines of symmetry that could be found within geometric figures. The exclusive focus on symmetric mathematical characters in Acts A through D made it reasonable for students to assume that doing so is always possible. By the end of Act D, it was reasonable for students to expect more symmetric mathematical characters at the start of Act E.

Table 1: Shifts in the mathematical story throughout the acts leading up to the aesthetic moment.

Story Element	Act A	Act B	Act C	Act D	Act E
Mathematical Characters	Square, non-square rectangle, equilateral triangle, circle	Remained unchanged	Lines of symmetry	Heart, circle, square, isosceles trapezoid, "house" pentagon and their lines of symmetry	Five asymmetric figures
Mathematical Actions	Folding in half (i.e., creating two coinciding and congruent parts)	Drawing lines to represent folds	No action is evident	Remained unchanged, but now the result is called a "line of symmetry"	Remained unchanged
Mathematical Settings	Paper shapes	Diagrams on a white board	Diagrams on chart paper	Remained unchanged	Remained unchanged
Focal mathematical questions	How can I fold [a shape] in half?	How can I draw a fold on a diagram?	What is the fold called?	How can I find a line of symmetry?	Are all shapes foldable [symmetric]?

Along with the logical (mathematical) thread, the mathematical plot traces the changing tension between what was known and what was desired to be known as a consequence of the story sequence. The evident tension at the beginning of the mathematical story, with the challenge of bisecting shapes by folding, notably lessened by the beginning of Act D. This lack of tension was evident in the reactions by the students throughout Act D; the only apparent challenge was trying to be selected as a volunteer. The mathematical conflict seemed to have been resolved. The underlying inquiry in place ("Can I fold/find a line of symmetry?") appeared to have been answered with resounding success. Instead, when the new asymmetric figures were revealed, this expectation was violated, generating surprise and confusion. The students were captivated by the new possibility before them — geometric objects can be asymmetric! The question "Can I fold/find a line of symmetry?" that had earlier appeared answered now suddenly had a new answer: "Not always." The plot had turned, leading to a new focus of inquiry: "Are all shapes foldable [symmetric]?"

An inspiring result of this plot turn was that these students became committed to resolving the new conflict, namely, how to draw a line of symmetry for these new and different figures. This turn in the mathematical plot generated audible and visible excitement, which was maintained for approximately 10 minutes until the moral of this mathematical story was stated by the teacher (i.e., not all geometric shapes have lines of symmetry). This moral signaled the end of the mathematical story and offered students resolution to the emotional tension raised in Act E. Although the answer to the question "Are all shapes foldable [symmetric]?" might seem obvious near the start of Act E to a math expert after some attempts at bisecting asymmetric shapes with a line, the students maintained their desire to find a way to bisect these shapes throughout the act. Since the students did not revert to their passive participation (as seen in Acts B and D), I posit that these students held onto their working conjecture that all geometric shapes should be symmetric. Thus, the teacher's announcement at the end of the mathematical story that not all shapes are symmetric both answered the mathematical question (the logical) and released the remaining tension of wanting to know (the aesthetic).

6. The role of sequence in the perception of mathematical beauty

In the previous section, the aesthetic moment of the lesson was explained with its mathematical plot, which supported a growing expectation for symmetric shapes. The student reactions (e.g., gasps and wonder) were a result of a sequence of events that enabled these geometric objects to be perceived as exotic and special. How might a slight alteration of that sequence of events shift the perception of the asymmetric objects? For example, consider what would change if the order of Acts D and E were reversed (so the mathematical story would be A-B-C-E-D-F). By the end of Act C, the students would still have experience with folding paper shapes and representing the folds with lines on a white board. However, in this altered sequence, students would not have yet looked at a geometric diagram to find its line of symmetry. Upon entering Act E, the student question "can I draw a line of symmetry?" would not yet be answered.

This change in sequence is significant for how students enter Act E and meet the asymmetric figures for the first time for several reasons. First, since the question "can I draw a line of symmetry?" has not yet been considered, then when students first encounter the asymmetric figures, the students are not already assuming that they can (or should be able to). The fact that their process of halving would help to find a line of symmetry would not yet be recognized. So when viewing the asymmetric shapes, there would be little surprise when they are not able to visually locate the line of symmetry.

In addition, since only one set of shapes would have been encountered prior to Act E, this altered sequence would not establish a pattern of symmetric geometric figures. Although the expectation for symmetry might be started with the introduction of the word in Act C, this expectation would not be as strong since there would not be repeated evidence that the shapes encountered in the mathematical story would be symmetric or otherwise.

Finally, and perhaps most importantly, the tension felt in this altered sequence changes. In the original lesson, the shift in tension was the result of following Act D, during which the students could see that drawing lines of symmetry for different geometric figures was not difficult, with Act E, during which this process was challenged. Yet, if Act E is moved earlier, the absence of a violation of expectation could instead lead to student frustration on what is meant by "symmetry." While tension likely would still exist in Act E, it

might be instead based on an assumption that the reason the students could not find a line of symmetry was because they didn't "get it." The focus of their attention would be shifted from the asymmetric geometric figures and their attributes to the students' failing process of finding a line of symmetry.

I acknowledge that this adjusted sequence was not observed in the class and therefore my claims about the potential aesthetic reactions based on the adjusted sequence are only conjectured and based on my own interpretation. Yet, the changes to the expectations and assumptions of the students would likely diminish the opportunity for aesthetic surprise and increased interest when encountering the asymmetric figures.

7. On being captivated by mathematics

While limited, this study offers evidence that mathematical plots can be designed to build mathematical anticipation and enable students to enjoy mathematical beauty, which can stimulate sustained mathematical inquiry in mathematics classrooms. This paper relies on the assumption that beauty is not an inherent quality of an object, but instead describes an individual's experience [4]. By using this classroom example as a case of mathematical beauty, I suggest that at least some cases of mathematical beauty occur when the mathematical experience opens up the horizon of what is possible mathematically and, at the same time, captivates an individual to move into that new and unfamiliar "space." Using an analysis of the mathematical plot of a first grade lesson, I have argued that at least one of the contextual conditions of the perception of mathematical beauty is the sequence of prior experiences.

Yet it is important to consider other potential reasons for the students' reactions: Could the students' gasps of wonder simply be the result of the juxtaposition of the asymmetric shapes with the symmetric shapes in this lesson? Removing the juxtaposition by introducing the asymmetric shapes at the beginning of the lesson, without the build-up with the paper folding and lines of symmetry, likely would mean that these same geometric objects would have been given no more than a passing interest. Perhaps the blob might have been interesting to students, but there is a large gap between what is 'interesting' and what makes a geometric object so compelling that it spurs the sustained attention of 5 and 6 year-olds for over 10 minutes! Therefore, in this mathematical story, the juxtaposition appears at least partly responsible for the surprise and sustained interest.

However, the juxtaposition of symmetry and asymmetry does not appear to be a sufficient condition. In the alternative sequence discussed earlier, which still contains a juxtaposition of symmetry (in Acts A, B, and C) with asymmetry (in Act E), the aesthetic response witnessed in the classroom that day would not have likely occurred. Instead, this minor sequential change of the lesson would alter the perception of the asymmetric objects from exotic and alluring to annoying and problematic! Finding the line of symmetry of the objects in Figure 1b would be impossible regardless of the sequence; however, in the original sequence students were captivated by the asymmetric objects and moved to persevere, convinced perhaps that there must be a way to tame these objects into submission.

Thus, context clearly matters, a point also made by Wells [20] when noting how the evaluation of a mathematical theorem can change with time. But what else can support an individual's experience of mathematical beauty? Even though I have proposed that mathematical beauty is not an attribute of a mathematical object, I do not mean to suggest that the objects in question do not matter. Literary theory again supports this recognition, as different characters (for example, Dumbledore and Snape in the *Harry Potter* series) within the same literary plot appeal to different people for different reasons. In a mathematical story about symmetry, such as the one discussed in this paper, it is possible that the asymmetric figures needed to have the same number of shapes and approximately the same collection of straight and curved sides as the symmetric figures to allow the lack of symmetry to captivate the attention of the children. I strongly suspect that the setting (i.e., representation) of the mathematical story matters as well, just as a literary story set in a remote desert will offer an individual different things to wonder about than one set in a major urban metropolis.

Another factor to the aesthetic moment described in this paper, I argue, is the fact that the children were first graders. Their young age limited their likely experiences with analyzing groups of shapes, and their lack of knowledge about symmetry certainly helped allow the geometric objects to seem mysterious. In addition, their attitudes about school were likely less cynical than you would find in a middle or high school student and natural curiosity was an acceptable trait.

Yet I do not want to suggest that similar aesthetic reactions to mathematics are not possible with older students. In my current research, I have been observing and collecting data in high school algebra classrooms around the

United States to learn more about how mathematical stories in algebra classrooms can offer aesthetic moments. Although not the norm of every lesson, my research team has been delighted to gather multiple examples of algebraic stories that captivated ninth-grade students similarly to the first grade lesson described in this paper. Algebraic lessons on topics that often are taught in an uninspiring manner, such as solving systems of linear equations, have led to displays of wonderment, excitement, and captivation similar to the one described in this paper, with public student exhortations of "Oh my god!" "Whoa!" and "You about to have me praise the Lord in here!" As with the example in this paper, these reactions marked aesthetic moments during which the students were captivated with the mathematics and not by aspects outside of mathematics (such as a funny contextual word problem or appealing music). The sequential analysis of written mathematics curriculum for their potential aesthetic dimensions for middle and high school has been published elsewhere, including mathematical stories that involve repeating decimals [5], probability [7], and the roots of quadratic equations [6].

This paper contributes to a growing number of articles regarding what it means to experience mathematical beauty. The role of mathematical aesthetic in terms of a sense of fit, balance, completion or order written about extensively elsewhere (see, for example, [14, 16, 19]). That "ugly" objects, such as asymmetric figures, can also attract is described by Le Lionnais [18]. Thus, in addition to these qualities of mathematical aesthetic, this paper suggests that an individual's sense of mathematical beauty involves the framing and assumptions of the individual, which are created through prior experiences. That is, earlier experiences frame an experience by building anticipation, which directs an individual's focus of attention and influences the information being sought (e.g., looking for an answer to the question, "How can I fold it into two congruent parts?").

Is mathematical beauty always context dependent, however? Interestingly, surveys of mathematical beauty (which are generally limited to mathematicians) offer some evidence that there are mathematical phenomena that are nearly universally appraised as beautiful. Wells [20] for example found that the relationships $e^{i\pi} - 1 = 0$ and Euler's formula for a polyhedron $V + F = E + 2$ are typically identified as very beautiful by mathematicians. This suggests that some mathematical truths might transcend the limitations of context and the different experiences of an individual. Perhaps in cases such as these, where there appears to be widespread agreement on the

beautiful nature of mathematical objects, the experience of beauty results from the fact that the objects have rare qualities not found in most mathematical objects. Due to this, it is possible that for some mathematical relationships or objects, almost anytime in any sequence of experience, they will be surprising.

This discussion leads me to raise the question of whether any mathematical object or phenomenon could possibly be construed by an individual as mathematically beautiful given the right context. In other words, does there exist a mathematical story that would similarly captivate students with a non-special number, such as 68? Or with a function, such as $y = -\frac{1}{2}x + 6$? In mathematics education, when a mathematical lesson is neither stimulating nor interesting, is it a result of a poorly designed mathematical story or is it the result of poor delivery? Learning more about the ways mathematics can captivate the interest of students (and when they do not) can potentially enable the development and implementation of new, inviting, and compelling mathematical stories. With a better understanding of the temporal dimension of mathematical beauty, educational experiences with mathematics can be designed to captivate attention and nurture interest and positive disposition by students toward mathematics.

Acknowledgments

This article is based on work presented at the American Educational Research Association annual conference in San Francisco, CA in 2013. The lesson discussed in this proposal was filmed as part of a study funded through the National Science Foundation Career Grant #0546164.

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