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In 1935, L. Zippin showed that every separable rimcompact completely metrizable space has a metrizable compactification with a countable (not necessarily infinite) remainder [Z]. A Tychonoff space X with a compactification γX such that $|\gamma X - X| \leq \omega$ is called a *Zippin space* and γX is called a *Zippin* compactification. If, in addition, $\gamma X - X$ is metrizable, X is called a *strongly Zippin space* and γX a *strongly Zippin compactification.* In this paper, an attempt is made to characterize spaces that are Zippin or strongly Zippin.

We succeed in this goal only in small part, but we do obtain a number of conditions on a space that are either necessary or sufficient for such compactifications to exist. For the most part, proofs are omitted. A more complete version of this paper will appear elsewhere.

At the Fourth Prague Topological Symposium, T. Hoshina also presented a paper on this topic. His results and mine overlap, but are not identical.

All topological spaces considered are assumed to be Tychonoff spaces. Any such space has a maximal compactification βX, called the *Stone-Čech* compactification of X that maps continuously onto any compactification γX of X with a mapping that extends the identity map [GJ, Chapter 6]. If the topology of X has a base of open sets with compact boundary, then X is called *rimcompact* (the term *semicompact* is used in [Z] and *semibicompact* is used in [M]). Every rimcompact space has a compactification ΦX maximal among the compactifications with a zero-dimensional remainder. ΦX is called *the Freudenthal compactification* of X [I, pp. 109-122] [M].

If P is a property of topological spaces, then X has P at ∞ if $\beta X - X$ has P. It is noted in [HI, Sec. 3] that if P is compactness, local compactness, σ-compactness, or the Lindelöf property, then X has P at ∞ if and only if $\gamma X - X$ has P for any compactification γX of X. A space that is σ-compact at ∞ is said to be *Čech-complete* or an *absolute* G_δ. It is well known that a metrizable space is Čech-complete if and only if it admits a complete metric [E, p. 190]. X is Lindelöf at ∞ if and only if every compact subset K_1 of X is contained in a compact set K_2 for which there is a countable family $\{U_i\}$ of open sets containing K_2 such that any open set containing K_2 contains some U_i. In particular, every metrizable space is Lindelöf at ∞ [HI, Sec. 3]. Also, if X is Lindelöf at ∞ and has a compactification with 0-dimensional remainder, then X is rimcompact by [I, p. 114].

It follows that every Zippin space is rimcompact and Čech-complete. (See also [RI] [R2]). As is noted in [I, p. 109]:

> $C\ell_{\gamma X}(\gamma X-x) = (\gamma X-X) \cup R(X)$ *for any compactification* γX *of* X, where R(X) is the set of points of X that fail to have a compact neighborhood.

Thus, by [CN, Sec. 6], we have:

1. *Proposition If* X *is a Zippin space then*
   (a) X *is rimcompact.*
   (b) X *is Čech-complete.*
   (c) $|R(X)| \leq$ *exp exp* ω.

   If X *is strongly Zippin, then, in addition:*
   (d) R(X) *is a Lindelöf space.*

The upper bound in (c) cannot be lowered. For if Q is the space of rational numbers, then βQ is a strongly Zippin compactification of $\beta Q-Q = R(\beta Q-Q)$, and $|\beta Q| = |\beta Q-Q| = \exp \exp \omega$ [GJ, Chap. 9].

Whether the conditions of Proposition 1 are sufficient to insure that a space X is a Zippin space remains an open question. Below, two kinds of sufficient conditions are obtained; those that make R(X) a "large" part of X, and those that make it in a sense "small". I begin with the former.

A space X such that every family of pairwise disjoint of open sets is countable is said to satisfy the *countable chain condition* (CCC). A space X is called *metacompact* or *weakly paracompact* if every open cover has a point-finite open refinement. As is well known, every paracompact, and hence every metrizable space is metacompact [E, pp. 225-228].

As in [LM], a space X is called *dense separable* if every dense subspace of X is separable.

2. *Theorem. Suppose* X *is a Zippin space such that* X-R(X) *is separable. Then:*
   (a) X *satisfies the* CCC.
   (b) *If* X *is metacompact or strongly Zippin, then* X *is a Lindelöf space.*
   (c) *If* X *is strongly Zippin, then* X *is separable.*
   (d) *If* X-R(X) *is dense separable, so is* X.

3. *Corollary. Suppose* X *is a metrizable space such that* (X-R(X)) *is separable. Then the following are equivalent.*

(a) X *is a strongly Zippin space.*

(b) X *is a Zippin space.*

(c) X *is separable, rimcompact, and Čech-complete.*

Next, a characterization of a special class of strongly Zippin spaces is given. It is established by decomposing the remainder of X in its Freudenthal compactification ΦX.

4. *Theorem. If* R(X) *is locally compact, then* X *is a strongly Zippin space if and only if* X *is rimcompact, Čech-complete, and* R(X) *is a Lindelöf space. Indeed, such a space has a strongly Zippin compactification with remainder homeomorphic to either a countable discrete space or its one-point compactification.*

I conclude with some remarks, examples, and questions.

A. By modifying [LM, Example 5.3], an example can be given of a Zippin space that is not strongly Zippin. It can be shown, however, that if R(X) is Lindelöf and X is a Zippin space, then X is strongly Zippin.

B. Clearly every closed subspace of a (strongly) Zippin space is (strongly) Zippin, and every open subspace of a Zippin space is rimcompact and Čech-complete by Proposition 1. The existence of open subspaces of βQ-Q that are not Lindelöf shows that an open subspace of a strongly Zippin space need not be strongly Zippin. I do not know, however, if an open subspace of a (strongly) Zippin space has to be a Zippin space.

C. Recall that a continuous closed surjection $f\colon X \to Y$ such that $f^{-1}(y)$ is compact for every $y \in Y$ is called a *perfect* map. If Y = [0,1]-Q, then the projection map of Y×[0,1] onto Y is perfect, Y is a strongly Zippin space, but Y×[0,1] is not rimcompact and hence is not a Zippin space (although it is the product of a compact space and a strongly Zippin space). I do not know, however, if a perfect image of a (strongly) Zippin space must be (strongly) Zippin.

D. It follows easily from [GM, Example 5.3, ff.] that no connected Zippin space has a countable partition into compact sets.

E. It is easily verified that if R(X) = X is connected, then the remainder of X in any compactification is connected, whence X cannot be a Zippin space. (See [R 1, Corollary 3]). Indeed, if X is also Lindelöf at ∞, it cannot even be rimcompact. In particular, a countably infinite product of copies of R is not rimcompact.

F. It was shown by McCartney in [Mc, 3.6] that X has a maximal Zippin compactification if and only if X has a compactification with zero-dimensional remainder and $\beta X-X$ has only countably many components. Indeed, if this latter holds, then ΦX is the maximal Zippin compactification. For a simpler proof see [D].

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