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This paper is about the last person to be known as a great Rationalist before Kant's Transcendental Philosophy forever blurred the distinction between that tradition and that of the Empiricists. Gottfried Wilhelm von Leibniz is well known both for the Law which bears his name and states that "if two things are exactly the same, they are not two things, but one" and for his co-invention of the Differential Calculus. It is commonly taught that Leibniz and Isaac Newton each independently discovered means to find:

1) the tangent to a curve at a point.
2) the length of a curve, the area of a region, and the volume of a solid.
3) the maximum or minimum value of a quantity.
4) the relation between the velocity and acceleration of a body at an instant and the total distance traveled by that body in a given period of time.[1]

These new mathematical techniques supplanted those of the ancients and provided modern scientists with tools which enabled their science to leapfrog classical science. Since these discoveries were so important to the natural sciences and the ensuing technological development, great prestige came to be attached to them. Competition for this prestige resulted in a bitter dispute in which Leibniz was charged with plagiarism. To chronicle the charges and counter-charges would be an interesting task--but one better left to historians of mathematics.

The 17th century was buzzing with discovery and humming with intellectual activity. Someone would have discovered the calculus before 1700 if neither Leibniz nor Newton had been born.[2] While it is generally agreed that their discoveries were independent, it is known that Newton's discovery preceded that of Leibniz. Though Newton was first, that the centuries have decided in favor of Leibniz's notation because of the relative facility of its use, is testimony to the central position of notation in Leibniz's thinking. For him, thinking is the manipulation of the symbols of notation-facility of manipulation is facility of thought. Anyone who has tried to calculate simple interest using Roman Numerals knows well the importance of an elegant notation.

In the preface to his translations of *The Early Mathematical Manuscripts of Leibniz*, J.M. Child maintains that "the main ideas of [Leibniz's] philosophy are to be attributed to his mathematical work, and not *vice versa*."[3] The esteem in which Leibniz held an elegant notation (the most elegant being the simplest possible way of handling all the possibilities) is all that is offered in support of this. If, by 'main ideas' Child means the form of Leibniz's analysis—that is, that part which has its source in the method which he employs—I would not disagree. But we must ask more than how it is that he performs his analysis, we must also ask what it is that he chooses to analyze. Neither Leibniz's interests nor his optimism knew any bounds. We must remember that in addition to being a great mathematician and logician, he was also the Dr. Pangloss ridiculed by Voltaire in Candide. If recursion of notation is Leibniz's method, *philanthropia* is his motivation. The two are tied together by his views of God and the *philothea* appropriate to the wise man.

The function of God in Leibnizian metaphysics is to be the creator of the universe (the first cause) as well as the source of perfection and order within it (the final cause). Hence, God is, by definition, the perfect creator of the universe. Likewise, by definition, composite substances (bodies) are composed of simple substances[4] and simple substances are unities.[5] Leibniz calls these unified simple substances 'monads' after the Greek *monas*. Since monads are simple, they have no parts. Change occurs when parts are combined together or cleaved apart. "Now where there are no parts, neither extension, nor figure, nor divisibility is possible."[6] Thus monads are changeless, eternal, and by all outer appearances, identical. But the law that bears Leibniz's name tells us that they cannot be identical—for if they were they would be one and the same thing. The dilemma is solved by letting them differ internally. These internal differences lie in the different perceptions and appetitions of the different monads.[7] It is assumed that every monad is subject to change and that change is continuous.[8] But these internal changes "*are inexplicable by mechanical causes*."[9] Mechanical causes apply only to bodies (composite substances) "[T]he perceptions in the monad spring one from the other, by the laws of desires [*appetits*] or of the *final causes of good and evil*, which consist in observable, regulated or unregulated, perceptions; just as the changes of bodies and external phenomena spring one from another, by the laws of *efficient causes*, that is, of motions." He claims that this entails that "there is a perfect harmony between the perceptions of the monad and the motions of the bodies, preestablished at the beginning between the system of efficient causes and that of final causes."[10]

Surely a perfect creator would create the best possible product if not a perfect product. The best possible universe is that in which "there is the greatest variety together with the greatest order;...the most results...; the most of power, knowledge, happiness and goodness in the creatures that the universe could permit."[11] This universe is a plenum—that is "*nature never makes leaps*"[12] —and because of this "everything is connected and each body acts upon every other body..."[13] Bodies are connected together by efficient causes. God, as the creator of the universe, is the only being with a complete knowledge of it. That God does have perfect knowledge of the universe is required by the principle of sufficient reason: "nothing happens without its being possible for him who should understand things, to give a reason sufficient to determine why it so and not otherwise."[14] Other monads perceive the whole universe with some degree of confusion. Some of the least confused are aware that they are representing the perceived objects of the universe to themselves. Leibniz anticipates Kant's Transcendental Unity of Apperception[15] and grants *consciousness* only to beings with reflective awareness (that is awareness of being aware of bodies external to it). This reflection gives reason access to necessary truths. Leibniz is here constrained by his criterion of necessity. Leibnizian necessity is analytic—necessary propositions are those whose denials are self contradictory. Section 30 of *The Monadology* begins: "It is also by the knowledge of necessary truths and by their abstractions, that we rise to acts of reflection, which makes us think of that which calls itself 'I', and to observe that this or that is within us: and it thus implies that in thinking of ourselves, we think of being, of substance, simple or composite, of the immaterial and of God himself..." Perhaps he is suggesting that the mind concentrates on the subject because the truth value of a necessary proposition is determined without reference to an object.

Kant employs a second test of necessity to tell a more plausible story. If something is universal, that is if it applies to everything, there is some sense in which it is necessary.[16] Kant notes that " [i]t must be possible for the 'I think' to accompany all my representations."[17] Though the proposition 'I think that X is my representation' is analytic, "it reveals the necessity of a synthesis of the manifold given in intuition."[18] Descartes never could have forseen the uses to which his *cogito ergo sum* would be put.

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...his life's goal [was] to create an 'alphabet of human thought,' the symbols of which if "correctly and ingeniously established...will be capable of being read without any dictionary" and will provide "a fundamental knowledge of all things."

Leibniz calls those monads that are the souls of the reflective animals—those which are said to be rational—spirits and says that they have access to "immaterial things and truths."[19] "...As regards the rational soul, or *spirit*, there is something in it more than in the other monads, or even in simple souls. It is not only a mirror of the universe of creatures, but also an image of the Divinity. The *spirit*...imitates, in its department and in its little world, where it is permitted to exercise itself, what God does in the large world."[20] Reason is the pathway to the city of God.

In streamlining notation, Leibniz is making that pathway more accessible to his fellows. Thus, his life's goal to create an 'alphabet of human thought,' the symbols of which if "correctly and ingeniously established...will be capable of being read without any dictionary" and will provide "a fundamental knowledge of all things."[21] Leibniz did not finish this project before his death, but his work was carried on by people like Frege and Russell and Whitehead who formalized first order logic. That this is not possible was not proven until 1931 when Kurt Goedel showed that all first order systems contain statements that can neither be proven true nor shown to be false(e.g. “this sentence is false”). The ‘universal calculus’ is the first of the two *calculi* with which Leibniz would preoccupy himself. Another is the differential calculus for which he and Newton are remembered. That calculus is far more specific—not covering “the conceptual atoms from which absolutely all molecular concepts can be formed,”[22] but only the rates of change at an instant—that is, variation of variation of bodies in space. We set out to show that Leibniz’s mathematics and philosophy are not clearly differentiated bodies of work, but are, respectively, his methods and his aims.

One approach to this task is to explore the philosophical pre-suppositions which Leibniz makes when he chooses to use the mathematical techniques which he does. Answering this question requires first a detailed examination of the problems and of Leibniz’s solutions. Most of the English secondary literature on this topic seems to depend on J.M. Child’s 1920[23] translation of the manuscripts “...unearthed by Dr. C.I. Gerhardt in a mass of papers belonging to Leibniz that had been preserved in the Royal Library of Hanover...”[24] Child’s work contains both the daily notes which Leibniz made to himself in late 1675 while he was in the process of making his discovery and two later reflections upon that period: the first a postscript, later cancelled, to a letter written from Berlin in April of 1703 to James Bernoulli; the second, Leibniz’s own version of the story to be told to posterity: *Historia et Origo Calculi Differentialis* which was “probably finished just before the death of Leibniz...in November 1716.”[25] This last is of the same period as *The Monadology* and *The Principles of Nature and of Grace* (both 1714) as well as the *Metaphysical Foundations of Mathematics* (1715) which also warrants our attention.

Important to the discovery of the infinitesimal calculus was the friendship which Leibniz made with Christiaan Huygens shortly upon his arrival in Paris in 1672. Leibniz looked closely at Euclid’s axiom that the whole is greater than the part[26] and determined that it was not valid when applied to the angle of contact between a circle arc and its tangent.[27] That the whole seems not to be greater than the part is the result of a long lived debate over the definition of ‘angle.’ The Ancients did not require that the line segments which bound an angle be straight. Euclid defines rectlineal angles as a species of angles in general.[28] There was also great uncertainty about into which of Aristotle’s categories angles should be put. Euclid’s view was that angles, as magnitudes, belong in the category of quantity. If angles are magnitudes it must be possible to relate the quantity of any two angles as a ratio (e.g. 45 degrees : 90 degrees = 1:2). Assume a circle centered at (0, -R) tangent to the X axis and normal to the Y axis at the origin. Construct a secant line from the origin to (R, -R). Together, with the X axis, it forms a 45 degree rectilineal angle L_R, the magnitude of which can be compared with that of the curvalineal, horn-like keratoeides angle L_C formed between the X axis and the circle arc from the origin to (R, -R). If we successively reduce the length of the secant we reduce L_R by the same ratio. We can form a series of congruent ratios: $L_R:L_C = L_{R1}:L_{C1} = L_{Rn}:L_{Cn}$ where the length of the secant is 1/n of the original. At the point of tangency, both angles are of zero magnitude and yet expected to stand in the same constant ratio. Euclid maintains that all rectilineal angles are greater than any curvilineal angle (III, 16).[29] Thus, the rectilineal angle is in the contradictory position of having to be both greater than and equal to the ‘horn-shaped’ angle. The failure of the axiom to hold for the angle of contact led Leibniz to believe, with Hobbes,[30] that it is not an axiom at all, but a provable theorem.[31] From early childhood Leibniz wanted to allow only definitions and statements of identity to be axiomatic.[32] He reduced Euclid’s axiom to a syllogism containing only definition and identity:

“Whatever is equal to a part of another, is less than that other: (by the definition)
But the part is equal to a part of the whole: (i.e., to itself, by identity)
Hence the part is less than the whole. Q.E.D.”[33]

Since any part ‘A’ is equal to itself, we can conclude that A - A = 0. And build from that:

“A - A + B - B + C - C + D - D + E - E = 0
\/ \/ \/ \/
+ L + M + N + P

If now A, B, C, D, E are supposed to be quantities that continually increase in magnitude, and the differences between successive terms are denoted by L, M, N, P, it will then follow that

A + L + M + N + P - E = 0
i. e. L + M + N + P = E - A

that is, the sums of the differences between successive terms, no matter how great their number, will be equal to the difference between the terms at the beginning and the end of the series." Leibniz went on to take the second differences—that is the differences of the differences—and the third. Once he got going this way, it was not possible to avoid discovering "that the differences of the powers of the natural numbers, when taken continuously, finally vanished."[34]

Leibniz went to Huygens with the news of his discovery of a general method for summing infinite series. Huygens was interested and suggested both a problem which he had considered earlier as a test exercise as well as some readings in the relevant literature.[35] Thus began the relationship to which Leibniz says he owes his "introduction to higher mathematics."[36] Huygens lead Leibniz to Gregoire's *Opus geometricum* which considers geometrical representations of sums of series.[37] Leibniz met other mathematicians in his travels. It was in London where he met John Pell who told him that Nicolaus Mercator had already documented the discovery of the vanishing differences of powers.[38] Leibniz's work in the summing of difference series led to the conception of the 'harmonic triangle' "in which the oblique rows are successive difference sequences, so that their sums can be easily read off from the scheme."[39] This representation made obvious the mutually inverse relation between difference sequences and sum sequences. This became quite significant when Leibniz applied a similar scheme to geometry and discovered that "the determinations of quadratures and tangents are also mutually inverse operations."[40] The area under a curve which crosses above the X axis at the origin can be approximated by inscribing rectangles and then taking the summation of their areas. If we assume the rectangles to be of width 1, the area under the curve is approximately equal to the sum of the Y values. Leibniz realized that if the rectangles became infinitely narrow, the approximations would become exact.[41] While the rectangles still have width, atop of each is a right triangle—the hypotenuse of which is a section of the circle arc. Blaise Pascal had noticed that these triangles are nearly "similar to the triangles formed by ordinate, tangent and subtangent, or ordinate, normal and sub-normal"[42] when proving "the theorem of Archimedes for measuring the surface of a sphere."[43] From this Leibniz was able to develop a general theorem for determining "the whole moment of the curve."[44] Leibniz spent the better part of two years "finding, analytically (that is, by manipulation of formulas) all sorts of relations between quadratures."[45] One such relation was that "the moments of the differences about a straight line perpendicular to the axis are equal to the complement of the sum of the terms, and the moments of the terms are equal to the complement of the sum of the sums."[46] Leibniz first expressed this relation in a notation borrowed from Bonaventura Cavalieri.[47] For example: "omn.xl = x omn.l - omn.omn.l, where l is taken to be a term of a progression, and x is the number which expresses the position or order of the l corresponding to it; or x is the ordinal number and l is the ordered thing."[48] ("'Omn.' is the abbreviation of *'omnes linae'*, 'all lines.'")[49] Utility rather than necessity was the mother of the invention of the integral notation of the calculus: "It will be useful to write ∫ for omn., so that ∫l = omn.l, or the sum of the l's. Thus.... ∫(xl) = x∫(l) - ∫∫l is equivalent to the 'omn.xl = x omn.l - omn.omn.l' of the above example.[50]

This elegant, economical notation freed Leibniz to easily develop an elaborate scheme of analytic truths. Elaborate as this scheme is and though it is "correctly and ingeniously established" it is not an 'alphabet of human thought' and it certainly does not provide "a fundamental knowledge of all things."[51] That Leibniz was able to dig as far as he did into the foundation of analytic thought is amazing. I can't help but believe in his belief that:

> [E]verything in the whole wide world proceeds mathematically, that is infallibly, so that if one had enough insight into the inner parts of things and also enough memory and understanding to take in all the circumstances and calculate them, he would be a prophet; he would see the future in the present as in a mirror.[52]

This gave him the zeal to proceed with his analysis. Though this belief relates mathematics with the world, it is a *philosophical* belief—not a mathematical one. In this sense it is not unlike the question of the ontological status of the *infinitesimal* — a philosophical question about a mathematical method. It is for this reason that I disagree with Child when he maintains that "the main ideas of [Leibniz's] philosophy are to be attributed to his mathematical work, and not *vice versa*."[53] For Leibniz, mathematics and philosophy are inexorably intertwined.

NOTES

1 Anton, Howard; *Calculus With Analytic Geometry*; Wiley; NY; 1980.
2 See for example J.M. Child; *The early mathematical manuscripts of Leibniz*, Open Court,1920, in which the author credits Barrow with the discovery (pp 7-8).
3 Child, p. iii.
4 *Monadology* [1714], *SS* 2. in *Leibniz Selections*, Weiner, P., ed. Scribners 1951.
5 *The Principles of Nature and Grace, based on Reason* (*Principles*) [1714], *SS* 1, in Wiener.
6 *Monadology, SS* 3.
7 *Principles, SS* 2.
8 *Monadology, SS* 10.
9 *Monadology, SS* 17.
10 *Principles, SS* 3.
11 *Principles*, *SS* 10.
12 *New Essays* [1704], p. 378 in Wiener.
13 *Principles*, *SS* 3.
14 *Principles, SS* 7.
15 *Critique of Pure Reason*, Kemp Smith, translator, St. Martins, 1965 (B ed.), Transcendental Deduction *SS* 16.
16 Robinson, Richard, *Necessary Propositions*, Mind (1958) pp. 291.
17 *Critique of Pure Reason SS* 16, (B 131).
18 *Critique*, B 135.
19 *Principles, SS* 5.
20 *Principles, SS* 14.
21 *De Arte Combinatoria*, [1666] (p. 90)
22 *Leibniz, Logical Papers*, G.H.R. Parkinson, Trans. & Ed., Clarendon, Oxford, 1966 p. XV.
23 Child, J.M., *The early Mathematical Manuscripts of Leibniz, Translated from the Latin texts published by Carl Immanuel Gerhardt with critical and historical notes*. Open Court, 1920.
24 Child, Introduction, p. 3.
25 Child, p. 4.
26 *Elements I*, Common Notion 5. (New Numbering) Axiom 8 (Old System), Heath, Sir Thomas L., Trans., Dover 1956.
27 Hofmann, Joseph Ehrenfreid, *Leibniz in Paris 1672-1676, His Growth to Mathematical Maturity*, Cambridge, 1974, p.12.
28 Heath, Definitions 8 & 9.
29 Heath, p. 177.
30 Thomas Hobbes, *De corpore* I Ch. 8 *SS* 25.
31 Hofmann, p. 13.
32 *Historia Et Origo*, in Child, 1920, p. 29.
33 *Hist. Et Origo*, p. 30.
34 *Hist. Et Origo*, p. 36.
35 Hofmann, p. 15.
36 *Hist. Et Origo*, p. 36.
37 Hofmann, p. 166.
38 *Hist. Et Origo,* p. 36.
39 Bos, H.J.M., *Newton, Leibniz and the Leibnizian Tradition*, in I. Grattan-Guinness, ed. *From the Calculus to Set Theory, 1630-1910*, Duckworth, London, 1980, p. 61.
40 *op. cit*. pp. 61-2.
41 Ibid
42 Bos, p. 63.
43 *Hist. Et Origo* p. 38.
44 *op. cit*. p. 40.
45 Bos, p. 66.
46 MSS. October 26,1675, in Child, 1920, p. 70.
47 Bos, p. 66.
48 MSS. October 29, 1675, p. 80.
49 Bos, p. 66.
50 MSS. October 29,1675, p. 80.
51 *De Arte Combinatoria*, (90).
52 *On Destiny or Mutual Dependence*, in Wiener, p. 571.
53 Child, Introduction, p. 4.

"It is the theory that decides what we can observe."
--Albert Einstein