Humanistic Mathematics Network Journal

Issue 24 | Article 7

5-1-2001

Naïve Thoughts on the Paradox of Gödel

Philip J. Davis
Brown University

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Philip J. Davis
Division of Applied Mathematics
Brown University
Providence, RI 02912

EPIGRAPH

You can't get there from here.

The classic Sam Loyd "Fifteen Puzzle" consists of fifteen movable and numbered square counters placed in a random order in a four by four square frame. One is allowed to slide a counter into an empty space, and the goal is to arrive at the natural ordering of the counters from a given initial arrangement by a sequence of such slides.

Theory shows that starting from half of the possible original positions of the counters, the puzzle is solvable while from the other half it is insolvable. But a simple interchange of any two counters will alter the puzzle from solvable to insolvable or vice versa. Embedding the board in three dimensions makes the puzzle always solvable.

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THE IMPETUS

The impetus for this paper came from a novel by Apostolos Doxiadis, *Uncle Petros and Goldbach's Conjecture*. In the novel, Petros, a mathematician, has set his heart on solving the Goldbach Conjecture. Failing and deeply disappointed, Petros takes psychological refuge in the possibility offered by Gödel's Theorem, that the problem be insoluble. After reviewing this novel for the *SIAM NEWS*, I began thinking about my own reactions over the years to the Gödel Theorem and to one particular aspect of it.

ON THE WORD "NAÏVE" IN THE TITLE

The amount of material related to Gödel's Theorem is enormous: it is far beyond anyone's ability to know it all or understand it. And the Web has multiplied the chatter and blurred it into an incoherent mass of thoughts.

I will approach my topic as I have experienced it in my professional career as an applied mathematician and as a writer. This is the meaning of the word "naïve" in my title.

WHAT IS THE PARADOX?

Simply stated, the Paradox of Gödel is this: although Gödel's Incompleteness Theorem has been touted in some quarters as the most significant mathematical achievement of the 20th century, it seems to be of little significance to the bulk of research mathematicians. Why is this the case?

What is Gödel's Incompleteness Theorem? In nontechnical language one might say: if a mathematical statement has been asserted that seems to make sense, it may not be possible to prove whether the statement is true or false.

Example: A claim has been made that there are an infinite number of 0's in the decimal expansion of π. At the moment, it is not known if this is true or false or if it is undecidable.

I like to put the incompleteness theorem this way: given a mathematical "there" and a "here," you may not be able to get there from here.

Slightly more technically: arithmetic is not completely formalizable. For every consistent formalization of arithmetic, there exist arithmetic truths that are not provable in that system.

"There will always be arithmetic truths that escape our ability to fence them in use the tools of rational analysis" (John Casti).

In what follows, I will use the abbreviation GIT to designate the Gödel Incompleteness Theorem, or more generally, any of its closely related theorems or equivalent formulations.

THE ROMANCE, THE HYPE, AND THE ICONOGRAPHY SURROUNDING GÖDEL'S THEOREM

"An astounding and melancholy revelation" (Ernest Nagel and James R. Newman, *Gödel's Proof*, 1958).

"The most decisive result in mathematical logic" (Boyer, *A History of Mathematics,* 2nd ed.).

"Amazing, shattering" (Morris Kline, *Mathematics: The Loss of Certainty*).

"Mind boggling. One of the pinnacles of human intellectual achievement. Basis for a whole host of related developments in philosophy, computer science, linguistics, psychology. Mankind will never know the final secret of the universe by rational thought (Casti, *Reality Rules*, *II*).

"Only Einstein's theory of General Relativity represents an accomlishment of comparable intellectual grandeur" (Berlinsky, *Black Mischief*).

"GIT a part of a 'golden braid' of math , art and music that penetrates the very nature of human consciousness. Gödel-numbering has opened up vast new worlds" *(*Douglas Hofstadter, *Gödel, Escher and Bach).*

One can find statements in semiotics, theology and eschatology that are based on or allude to the GIT as well as references to Gödel in novels:

> The philosophers at the great universities were, without exception, failed mathematicians. When they were not examining much of the vocabulary of civilized discourse to conclude that it, after all, lacked meaning, they muttered Gödel, Russell, Hilbert, liking to imply that they themselves had chosen philosophy over mathematics to give themselves a wider, though related intellectual field (Renata Adler, *Speedboat*).

One can find references to Gödel in literary criticism where, according to Simon Blackburn's review of Umberto Eco's *Kant and the Platypus*, "prominent literary intellectuals often like to make familiar reference to the technical terminology of mathematical logic."

One such person is reported as opining that "Gödel showed that every theory is inconsistent unless it is supported from the outside. Derrida showed that there is no outside" *(New Republic Magazine*, Feb 7, 2000).

There are web chats galore on the single topic of "False Applications of the GIT," although what is a "false" application of mathematics and what is a "true" application defy formalization, let alone common agreement.

SOME "APPLICATIONS" OF THE GIT

I use the word "application" here to mean simply that an argument of some sort has been put forward based in some way on the GIT. The GIT serves as a point *d'appui* for both specialists and the laity.

GIT suggests there is no final theory of physics (Stephen Hawking).

GIT suggests that physics—identified with mathematical physics—might be inconsistent.

GIT suggeststhat not everything that is technically desirable is technically possible (Jerome Wiesner).

GIT suggests that "whether we admit it or not all (political, social, military) actions end in the logic of triage (i.e., judgements of priorities of action)" (Hans Magnus Enzensberger, *Civil Wars,* 1994).

GIT suggests humans are not computers. Creativity and intuitive powers are not the product of computer programs (Roger Penrose).

(This last position has been seriously questioned by Murray Gell-Mann.)

"I would be skeptical about the use of the GIT (as in Penrose, 1991) for arguing the limitations of any kind of intelligence" (Steve Smale).

GIT suggests that mathematics will become more and more experimental (Chaitin).

GIT suggests that the foundations of mathematics both philosophically and technically must come to grips with stochasticity (i.e., the probabilistic element) (Gregory Chaitin, David Mumford).

GIT suggests that there may be a "high level way of viewing the mind/brain, involving concepts that do not appear on lower level" (Donald Hofstadter).

GIT suggests that mystical experiences may be the only road to absolute knowledge (Paul Davies).

GIT suggests that since "the consistency of mathematical systems becomes an incalculable question. Thus, even the exercise of mathematics involves an act of faith" (John Polkinghorne, Physicist and Anglican Priest, *One World*).

From these last two quotes, it is an easy step to say the GIT suggests that God exists.

GIT suggests that "a religion based on a plurality of religions may leave us forever struggling with an Axiom of Religious Choice" (Sarah Voss, mathematician and minister, *What Number is God?*).

In a totally different direction, GIT may give aid and comfort to the creators of computer viruses:

"For most plausible definition of 'virus', it is likely that the GIT blocks the possibility of writing a program that accepts all non-viruses and rejects all viruses" (Ernest Davis).

Thus, while each type of virus may be overcome on an individual basis, no panacea can be found. If the use of the word "virus" in both medical and computer contexts is more than mere verbal play, the GIT may suggest that a universal medical cure may be an impossibility.

Finally, there are features or "applications" of GIT that feed into psychology. The nicest, most amusing exposition of this occurs in Apostolos Doxiadis' novel referred to above..

THE PAST AND PRESENT INDIFFERENCE OF THE MATHEMATICIANS TO THE GIT

If the GIT has caused turbulence in philosophy, if it has caused earthquakes in logic, if discussions of the GIT clog the printed page and the websites, how can it be said that GIT is of little significance? And yet, despite the storms that have raged and opinions altered, research mathematicians—with the possible exception of a few number theoreticians—have little regard and less use for the GIT.

One colleague put it this way:

"I've been to many mathematical lectures and scientific meetings all over the world. Not once did the name of Gödel come up."

Another colleague told me:

"I've never lost any sleep over the GIT. But I'm sure that Hilbert did."

Therein lies the paradox.

More generally, of course, research mathematicians have had little use for mathematical logic or for the philosophy or history of mathematics. None of these is required knowledge for a Ph.D. in mathematics (nor hardly even for computer science). The idea, for example, that mathematics proceeds rigorously and rigidly from assumptions to conclusions by a set of allowed logical steps, simply does not correspond to the way that mathematics is either discovered, developed, accepted, justified, applied or presented.

If someone points out that, in principle, all accepted mathematical proofs can be written out in the manner of Russell and Whitehead's proof that $1 + 1 = 2$, I would say the phrase "in principle" is one of the weaseliest expressions in the vocabulary of intellectuals. In principle, a contemporary Robinson Crusoe thrown naked onto an island well supplied with all raw materials could produce an automobile that worked.

Mathematicians work using traditional materials and guidelines and have their own criteria for acceptance. A real conceptual or metaphysical breakthrough occurs perhaps every fifty years. Afterwards, the logicians and philosophers move in and tell the world what, exactly, the mathematicians have been doing.

Mark Steiner comments on the sociological phenomenon of mathematicians who ignore logic completely in their description of the history and philosophy of 20th century mathematics and yet cite the GIT as one of the most important recent results. His recently appeared *The Applicability of Mathematics as a Philosophical Problem* does not mention the GIT.

While the GIT is a piece of mathematics created along traditional lines, but applied to mathematics self-referentially, it must be regarded as an "inside job." On the other hand, since it seems to limit what mathematicians will ever be able to accomplish, limiting the independence of mathematical action, it is also an "outside job."

Numerous people have walked away from certain parts of mathematics, either for conceptual or for utilitarian reasons, e.g., measure theory. They do not accept it. It has no gut meaning for them nor relevance to their scientific work. The GIT paradox is part of this phenomenon.

THE GIT AND THE FAMOUS UNSOLVED (OR ONLY LATELY SOLVED) PROBLEMS

In Doxiadis' novel, his hero uses GIT as an excuse for calling quits to his intense labors on the Goldbach Conjecture.

The mathematical world is full of unsolved problems and conjectures. Most conjectures fail to gain notoriety, primarily because they are not associated with a "great name." Mathematicians lose interest in them, so hence they are not worked over for long periods of time. Many of these, in number theory especially, have been listed in such books as Daniel Shanks' *Solved and Unsolved Problems in Number Theory* and Richard Guy's *Unsolved Problems in Number Theory.*

> *One might very well call such a piece of mathematics a fundamental symbol or myth...*

Neither of these books breathes the name of Gödel. It is probably the case that most of the conjectures listed in these books are decidable one way or another. The difficulty or the depth of a conjecture can only be guessed, but some measure of it may be gleaned from the rewards ($25, $100, etc. 100,000 pre-WWI German marks for Fermat) that are sometimes offered for a solution by some of the proposers.

THE GIT: ONE OF THE FUNDAMENTAL MYTHS OR ARCHETYPES OF MATHEMATICS?

The fact that the GIT has contemporary applications, implications or suggestions relative to a wide variety of fields ranging from cognition, physics, and philosophy, to literature, theology, and politics, gives it a special and remarkable status among mathematical statements. The educated laity seem to be attracted to it as iron to a magnet or as the devout to an icon. The name Gödel can create a best seller or fill a large lecture room. It can also do the reverse. This is part of the paradox and adds to the unique status of the GIT. It would be impossible to make such wide claims for, e.g., Gershgorin's theorem in matrix theory, or indeed for any of the theorems employed routinely in daily research. One would have to go back to the mental world of the Pythagoreans or neo-Platonists (ancient or contemporary) to find statements, contexts and attitudes of equal popularity.

One might very well call such a piece of mathematics a fundamental symbol or myth in the sense of the psychologist Jung. Jungian archetypes carry many interpretations; it is also the case that many explanations have been advanced for the Paradox of Gödel.

A BASKET OF EXPLANATIONS OR DENIALS

My object now is to record a wide variety of reasons that have been given to explain the Paradox of Gödel and then to set forth my own naive reasons.

> The GIT is equivalent to Turing's theorem about the unsolvability of the Halting Problem. I don't think that has any practical consequences for real life computation which deals with finite memory and finite computation times. It just shows the vast gap between what is of metaphysical interest and practical interest. Similar discrepancies abound in economics and in fluid dynamics (Reuben Hersh).

Tying the matter a bit more closely to the philosophy of mathematics, Hersh goes on to say:

> It seems to me that most or all issues of mathematical philosophy are important in some sense independent of concrete specific examples or applications.
>
> For instance, is there really an infinite set, or is it just something we imagine? From a philosophical viewpoint, this is a very basic, fundamental question. But for mathematical work, it doesn't make any difference, and many mathematicians couldn't care less about it.

> Gödel's Incompleteness Theorem is a mathematical result with philosophical import. It has limited mathematical import. Which shows that mathematical and philosophical import are not the same thing (Hersh).

> For mathematicians, however, his [Gödel's] theorem was of marginal interest, since Gödel worked with a far more formal definition of proof than that to which they aspired (or still do); so the separation of logic and mathematics continued largely unchanged (Ivor Grattan-Guinness, *The Rainbow of Mathematics*).

> I don't find it [the paradox] paradoxical. You can compare the GIT to Liouville's proof of the existence of transcendental numbers. It is an example of a phenomenon, but it is of no help for interesting number like e or π...One aspect of the matter, not a direct consequence of the GIT, but coming out of that development, is the work on unsolvable decision problems that has had a serious impact in certain fields, e.g., finitely presented groups (Martin Davis).

"The infinite is not the issue. It is the case that the GIT has implications for finite memory and finite computation time" (Ernest Davis).

> As far as applied mathematics goes, there is considerable evidence that all scientifically applicable mathematics depends on weak systems of set theory, even conservative over arithmetic. No new axioms are necessary (Sol Feferman).

Question: does Feferman's observation about weak systems constitute a descriptive hypothesis that limits the structure of physical theories just as the Church-Turing Hypothesis (the Turing machine models all possible computations) limits the nature of computation?

> Most mathematicians don't know or care about logic and they see the GIT as a kind of curiosity. It says nothing about the undecidability of the problems they happen to be working on. It provides no decision procedure for deciding beforehand whether a given statement in mathematics is or isn't decidable.

> If they took the possibility of undecidability seriously, if they agreed, for example, that with probability one, a proposition given at random is undecidable, they would be discouraged away from the field. Mathematicians use their insights, judgements, experience, to enable them to focus on statements which turn out to be decidable(John Casti).

> A parallel from physics The response of mathematicians to GIT has been rather like the response of physicists to general relativity in the period roughly from 1916-1960. Physicists understood that Einstein's results were in some way quite fundamental, but because general relativity seemed so definitely a singular achievement, physicists tended to ignore its implications while ceremoniously paying lip service to its grandeur...

> Mathematicians are instinctively inclined to assume that if the GIT and nearby results are as important as logicians seem to think they are, then it should be possible to use those results to discover something beyond the results themselves. Nothing has yet emerged.

> It is possible for a result to have immense importance for a discipline without leading to anything interesting within the discipline (David Berlinski).

It is not the case that GIT has contributed little to math or computer science. A fair number of interesting problems have been proven unsolvable using a reduction to the GIT. The best known is Hilbert's Tenth Problem. There are numerous other results in number theory, logic, computation theory, discrete math and algebra, e.g., the Paris-Harrigton result which states that the Ramsey theorem is not provable within number theory.

But just wait a bit! Things might change! Mathematicians have tended to ignore the GIT because it seemed to have no connection to other parts of mathematics. However, in the past twenty years, this has changed somewhat. Harvey Friedman's work has shown that incompleteness theorems do have a very real meaning for number theory. But the fact remains that the connection is weak in that it seems to point to nothing more than oddities in the structure of arithmetic. This may or may not change.

ANTI-GÖDELIAN DOUBTS

The GIT seems to have come as a surprise to neither to John von Neumann nor to Norbert Wiener (S.J. Heims: "John von Neumann and Norbert Wiener").

"The mathematical fraternities' actual experiences with its subject give little support to the assumption of the existence of an a priori concept of mathematical rigor" (John von Neumann, *The Mathematician*).

Further down the spectrum there are anti-Gödelian doubts:

"...it is commonplace that Wittgenstein rejected Gödel's proof [i.e., the GIT] because he did not, or even could not, understand it" (Juliet Floyd).

> The GIT is based on a chimera. The formalization of mathematics assumes its representation in a set of recognizable signs that are beyond questioning. The metamathematics however is stymied by the ambiguity (incoherence) in how those signs are actually viewed. The metamathematical argument of the GIT collapses into confusion. Since the whole enterprise of formalization is not feasible, GIT is redundant. No wonder mathematicians are not bothered by it in their work (Miriam Yevick).

THE WAY I SAW THE PARADOX

I first heard of the GIT around 1941, when I was a college undergraduate. The GIT was then ten years old. It caused no alarm in me. The bottom line seemed quite reasonable. There were mathematical problems I could not solve. I had heard that there were problems that no one had yet been able to solve. I knew that there were problems, which, as stated, provably had no solution.

An example: working in the plane, connect three houses by curves, to the "electric, gas, and water works" so that the curves do not intersect. (But in real life we connect them in 3-d).

Another example: the squaring of the circle by ruler and compass. Or, perhaps more significantly, the "demonstration that Euclid's Parallel Postulate" cannot be derived from the other postulates. Thus, there appear to be many problems that were impossible to solve in the way they have been formulated.

This being the case, and arguing by analogy, GIT seemed to me to be reasonable. As in the Fifteen Puzzle cited in the Epigraph, you might not be able to get "there" from "here." Of course, these examples are specific problems within mathematics and the GIT is a theorem about theorems. Up a metalevel, or is it?. But the proof of the independence of the parallel axiom is a proof that there can be no proofs of dependence. So the disparity of levels did not bother me. However regarded, these analogies were strong enough for me. (But not strong enough, apparently, for Frege, Russell, Hilbert, *et al.* Is this yet another paradox?)

The idea of mapping formulas onto integers (Gödel Numbering) seemed ingenious, but a bit dubious. The Gödel numbers are so large! What kind of existence can be attributed to them ? Do they really function in the way that 1, 2, 3 do? Are these numbers being used in different ways that really do not mesh with one another or with the numbers of everyday arithmetic? (I was, and still, am a "weak finitist.")

And then came the *coup de grace*. Nothing but a complicated form of the Liar Paradox. Hence a self-referential swindle, a trick of language.

So while I was quite willing to accept the bottom line of the GIT, I did not care much for the proof. I did not need the whole Gödelian apparatus to convince myself that I couldn't lift myself up by my own bootstraps either physically, mentally, or mathematically.

To add to my undergraduate skepticism, why was the world famous logician W.V.O. Quine, with whom I was even then studying mathematical logic, in the Department of Philosophy at Harvard and not in the Department of Mathematics? Obviously, the Harvard Mathematics Department considered mathematical logic to be irrelevant to their interests. In point of fact, this was my first perception of the Paradox of Gödel.

MY CURRENT VIEWS

To discuss GIT and the Paradox, as Reuben Hersh pointed out above, one might very well go into the logic, the philosophy and metaphysics of mathematics, metamathematics and cognition. What is a legitimate mathematical object, existentially? What are legitimate constructions or operations? What is truth? What is proof? How can we recognize what makes sense and what doesn't? What does it mean to "know"? What sense does it make to say that "it will never be known whether the statement X is true or false"? What does it mean to explain anything?

I shall bypass all these. I will not look for an explanation in terms of logical structures or the relative strengths and weaknesses of axiom systems. I will go for what might be called a historical view of the matter.

Toward this end, one should realize that at various times actual mathematical practice has been other than what it is claimed to be in an ideal and hence limited sense. Over the years, I came to believe that the "standard" view of mathematics as consisting of hypothetical-deductive structures is a totally inadequate description of how I (personally) have understood and internalized mathematics; how I applied mathematics to itself or to the outside world, or how I created new mathematics.

Historically, there are many times and places in mathematics where mathematics has said "impossible," "no way." Some of these impossibilities are hinted at in the persistence of old mathematical terminology, e.g., negative, irrational, imaginary numbers. Another impossibility: no general formula involving a finite number of simple operators and root extractions can be found for the solution of the quintic equation. Yet, the history of mathematics displays all these and many, many more impossibilities and contradictions (e.g., Heaviside's operational calculus; Dirac's delta function) being bypassed, legitimized, co-opted, often by the method of context extension.

I began to wonder about the notion of proof, a process absolutely fundamental within a certain view of

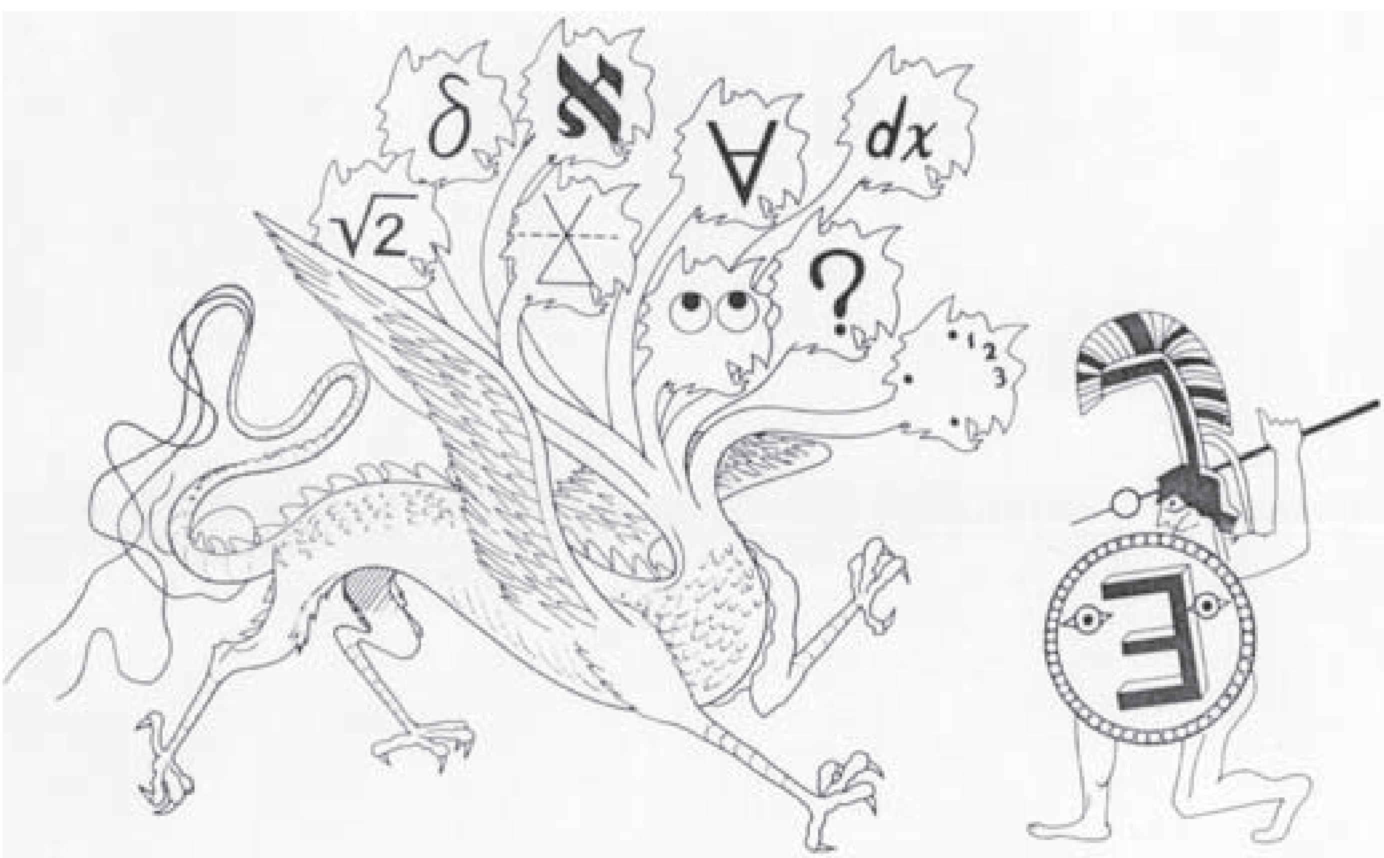

Figure 1
The Hydra of Mathematical Impossibility is slain by the Hercules of context extension. (From Davis and Park, 1987.)

mathematics, but proof was not identical understanding. Moreover, proof was subsequent to deciding that initially there was something there to prove. And the axioms were statements designed post hoc long after a substantial corpus of mathematics was in place. (In the case of arithmetic with Frege in 1884! Was there no valid arithmetic till then?).I began to feel that what was important was "mathematical evidence,"of which proof is only one component.

I began to wonder about the concept known as consistency. To be inconsistent, mathematically speaking, is to commit the primal sin. If one allows one contradiction, one can demonstrate anything at all. But can one really say with absolute objectivity, finality, and without relativistic allusions, what consistency consists of when there is a historical record of a constant patching up of mathematical inconsistencies in a way that makes them disappear? (Imre Lakatos)

An up-to-date example: consider the arithmetic system that is embodied in the popular and useful scientific computer package known as MATLAB. MATLAB yields the following two contradictory statements:

(The symbol == means "is the equality true or false?")

	Input	Output
(1)	l e-50 == 0,	false
(2)	2 + l e-50 == 2,	true.

Yet, MATLAB arithmetic is a (finite, but large) mathematical structure. Operations can be carried out. Certain inputs lead to certain outputs. These might be called MATLAB truths or theorems. The computation itself is the proof or the validation of these truths. They are deemed useful by the scientific community. The structure has its own integrity in that it consists of just what it consists of and it does just what it does. Yet, when judged by certain other ideal structures MATLAB embodies contradictions. While the God of Consistency does not thunder nor shake the earth in the presence of these logical irrationalities, one might well ask whether these contradictions can lead to error or disaster when MATLAB is employed in physical applications. They can, but "knowledgeable" programming makes the likelihood small. In any case, "ideal" mathematical computations (if indeed they can be carried out) might also lead to disaster.

Incidentally, I believe that Wittgenstein deplored "the superstitious fear of mathematicians for contradictions" (quoted in Karl Menger's *Reminiscences of the Vienna Circle*).

Rejecting mathematical platonism, formalism, logicism, and constructivism, I adopted a position that has been called variously "social constructivism," "quasi-empiricism," or "humanism."

THE PARADOX OF GÖDEL: MY EXPLANATION

Is this an explanation? Not really; just some thoughts conjured up by thinking about the paradox.

Mathematics is a living organism. The modes of its discoveries, developments, justifications, and interpretations cannot be formalized in a few paragraphs– if at all. They are time dependent and hence cannot be set down once and for all.

The development of mathematics either as a manufactured or a discovered corpus, goes forward to a great extent without set global goals. As it goes forward, year by year, what it turns up can be quite fortuitous, serendipitous, perhaps even interesting; in such a case, the arrival at a theorem is automatically accompanied by evidence of its validity or relevance, sometimes even by its proof.

Mathematics moves forward from statements already in place that suggest other statements. One of the goals then becomes to arrive at a proof of the suggestions. But the researcher is borne forward by a trust in a kind of "principle of continuity" (which admittedly can be dead wrong, see the "Fifteen Puzzle") implying that statements "close" to proved statements are themselves provable or disprovable.

In the older Eastern tradition, explicit proof is often missing. In the Western tradition, the notions of what is proof and what is provability evolved slowly and simultaneously with the discovery or creation of much material that was in fact provable. Alongside this, there grew the dominating or establishment view of mathematics as a logically deductive enterprise. The steady supply of proofs and the demand for more, interacting upon one another, grew together. The characterizing notion of mathematics as proved theorems grew with equal steps with the success in proving those theorems even as the notion of what constituted a valid proof altered and changed with time. The concept of what constitutes a proof has no finality; it develops alongside the material on which it operates.

Contemporary histories of mathematics present what is often called "Whiggian history." That is, they promote the current established view of mathematics as deductive structures, and they interpret the mathematical past as leading inevitably to the established present.

If it should have turned out that a good many of the statements deemed interesting by mathematicians or scientists were unprovable, or undeterminable whether they were unprovable, then the view of the mathematical enterprise as it developed in the 19th and 20th centuries, an enterprise that set increasing store by deductive proof, would have become untenable.

> *"The concept of what constitutes a proof has no finality; it develops alongside the material on which it operates.*

In such a case, mathematics would not have disappeared. It would be an art with a special vocabulary and modus operandi, a form of rhetorical discussion, a set of procedures, suggestions or rules as to how the world might be organized, and the GIT would be both true and irrelevant.

AS REGARDS THE FUTURE.

In my opinion the most significant mathematical development of the 20th century has been the computer in all its ramifications, mathematical, scientific and social. Having done scientific computation in the pre-electronic days as well as with contemporary very-high-level "tool kits," I still tend to think of the computer as a "mathematical instrument." But this view and a related view that the computer is a "logical engine," an "algorithm cruncher," though historically accurate, may now be as obsolete as the horse and buggy. What is replacing it?

Programmed computation, i.e., algorithms, and deductive proof have common features. But the future dominance of the algorithm–and the GIT is algorithmic in structure–has been questioned. Since it is fundamental to all digital communication in the same way the elementary particles of physics are fundamental to a Hawaiian wedding luau, there are some signs that the algorithm may have to share the center stage of technological and instructional emphasis or even to retire to the wings.

Here are a few straws in the wind:

Computer scientist Peter Wegner thinks that in the future the emphasis will shift from algorithmic models of computation to interactive models. We have now reached the point where a single computer has become a basic "elementary particle" of information interaction, to be combined with myriads of other individual computers and acted on non-algorithmically by the whole exterior environment, human and non-human.

"Interactive systems are grounded in an external reality both more demanding and richer than the rule based world of noninteractive algorithms" (Wegner).

"The conventional metaphor [for computation] will be replaced by the notion of a community of interacting entities" (Lynn A. Stein).

By way of a parallel within mathematics:

> Mathematics has been regarded traditionally as 'theorems.' It is now becoming the study of structures. Until the 20th century, there have been only two structures: geometry and arithmetic. Now there are many (David Mumford).

(MATLAB , mentioned earlier, is just one of the more fairly recent ones.)

While by no means neglecting the algorithm, we must surely add to the idea of structures the notion of stochasticity as a prime element of the future composition of mathematics.

"The intellectual world as a whole will come to view logic as a beautiful elegant idealization but to view statistics as the standard way in which we reason and think" (David Mumford).

Working with the material of a structure and employing its rules, the mathematical culture goes forward from "here" by successive steps and arrives at a "there." Often probabalistically and non-algorithmically and even multivalently. The delivery of the "there" from the "here" may be regarded as proof in an extended sense. Whether the "there" is in any way interesting or appealing or suggestive or useful or whether it corresponds to a "there" desired in advance is altogether another issue.

> By now the ideas elaborated by Gödel, Church, Turing, and Post have passed entirely into the body of mathematics where themes and dreams and definitions are all immured, but the essential idea of an algorithm blazes forth from any digital computer, the unfolding of genius having passed inexorably from Gödel's Incompleteness Theorem to Space Invaders VTT rattling on an arcade Atari (David Berlinski: *The Advent of the Algorithm: The Idea that Rules the World*).

The GIT is moving off center stage, a place it never really occupied. Like the ideas of Freud which now appear more in literature than in therapy, it will survive as an archetypal statement from which all kinds of inferences–mainly non-mathematical–will continue to be drawn.

AUTHORS CONSULTED ORALLY OR E-WISE

The author wishes to thank these individuals who graciously answered the questions I put to them:

David Berlinski, John Casti, Ernest Davis, Martin Davis, Apostolos Doxiades, Sol Feferman, Juliet Floyd, Harvey Friedman, Ivor Grattan-Guinness, Reuben Hersh, Douglas Hofstadter, David Mumford, Franco Preparata, Pet Wegner, Miriam Yevick.

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"The development of mathematics towards greater precision has led, as is well known, to the formalization of large tracts of it, so that one can prove any theorem using nothing but a few mechanical rules."

--Kurt Gödel