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Review: Truncated Toeplitz operators of finite rank

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Bessonov, R. V. [Bessonov, Roman V.] (RS-AOS2)
Truncated Toeplitz operators of finite rank. (English summary)
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Let θ be an inner function and let $K_\theta = H^2 \ominus \theta H^2$ denote the corresponding *model space*. A *truncated Toeplitz operator* (TTO) is an operator $A_\varphi\colon K_\theta \to K_\theta$ of the form $A_\varphi f = P_\theta(\varphi f)$, where P_θ denotes the orthogonal projection from L^2 onto K_θ. The general study of truncated Toeplitz operators was initiated by D. E. Sarason in a seminal article [Oper. Matrices **1** (2007), no. 4, 491–526; MR2363975] from 2007; a more current survey of the field is [S. R. Garcia and W. T. Ross, in *Blaschke products and their applications*, 275–319, Fields Inst. Commun., 65, Springer, New York, 2013; MR3052299].

While TTOs share some of the fundamental properties of Toeplitz operators (e.g., $A_\varphi^* = A_{\overline{\varphi}}$), they differ in many crucial ways. For instance, although the only compact Toeplitz operator on H^2 is the zero operator, it turns out that compact *truncated* Toeplitz operators exist in abundance. Sarason identified all rank-one truncated Toeplitz operators [D. E. Sarason, op. cit. (Theorem 5.1)]; they are all expressible as simple tensors involving the reproducing kernel $k_\lambda = \frac{1-\overline{\theta(\lambda)}\theta}{1-\overline{\lambda}z}$ and its "conjugate" $\widetilde{k}_\lambda = \frac{\theta-\theta(\lambda)}{z-\lambda}$ in the sense of C-symmetry [S. R. Garcia and M. Putinar, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315; MR2187654; Trans. Amer. Math. Soc. **359** (2007), no. 8, 3913–3931; MR2302518]. Here $\lambda \in \mathbb{D}$ or λ is a point of $\mathbb{T}$ at which θ has an angular derivative in the sense of Carathéodory (ADC).

Sarason also constructed a class of finite-rank truncated Toeplitz operators, involving simple tensors of derivatives of the functions k_λ and $\widetilde{k}_\lambda$, and asked whether every finite-rank truncated Toeplitz operator is a linear combination of the types already described. The paper under review provides an affirmative answer to Sarason's problem.

To be more specific, it is proved that the general finite-rank truncated Toeplitz operator on K_θ is a finite linear combination of the operators $\overline{D}^n[k_\lambda \otimes \widetilde{k}_\lambda]$ and $D^n[\widetilde{k}_\lambda \otimes k_\lambda]$ for $n \geq 0$ and $\lambda \in \mathbb{D} \cup \Omega_n$, where Ω_n denotes the set of all points $\lambda \in \mathbb{T}$ such that every function in K_θ and its derivatives up to order n have non-tangential limits at λ. A description of the set Ω_n is given in the well-known article of P. R. Ahern and D. N. Clark [Amer. J. Math. **92** (1970), 332–342; MR0262511].

The proof of the main result involves a number of technical ingredients, such as the description of the predual X_a of the space of all bounded TTOs on K_θ by A. D. Baranov, R. V. Bessonov and V. Kapustin [J. Funct. Anal. **261** (2011), no. 12, 3437–3456; MR2838030], Clark's unitary perturbations of the compressed shift [J. Analyse Math. **25** (1972), 169–191; MR0301534], A. G. Poltoratskiĭ's result on the existence of non-tangential boundary values σ_α-a.e. for all functions from K_θ [Algebra i Analiz **5** (1993), no. 2, 189–210; MR1223178], and Sarason's characterization (in the spirit of Brown and Halmos) of TTOs [D. E. Sarason, op. cit.]. *Stephan R. Garcia*

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Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.