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NON-NEGATIVE SOLUTIONS FOR A CLASS OF RADIALLY SYMMETRIC NON-POSITONE PROBLEMS

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ABSTRACT. We consider the existence of radially symmetric non-negative solutions for the boundary value problem

$$-\Delta u(x) = \lambda f(u(x)) \qquad \|x\| \leq 1,\ x \in R^N (N \geq 2)$$
$$u(x) = 0 \qquad \|x\| = 1$$

where $\lambda > 0$, $f(0) < 0$ (non-positone), $f' \geq 0$ and f is superlinear. We establish existence of non-negative solutions for λ small which extends some work of our previous paper on non-positone problems, where we considered the case $N = 1$. Our work also proves a recent conjecture by Joel Smoller and Arthur Wasserman.

1. INTRODUCTION

Here we consider the existence of radially symmetric non-negative solutions for the boundary value problem

$$-\Delta u(x) = \lambda f(u(x)) \qquad \|x\| < 1, \quad x \in \mathbf{R}^N, \quad N \geq 2 \tag{1.1}$$

$$u(x) = 0 \qquad \|x\| = 1 \tag{1.2}$$

where $\lambda > 0$ and $f: [0, \infty) \to \mathbf{R}$ is such that $f' \geq 0$ As is well documented, the study of (1.1)–(1.2) is equivalent to the problem

$$-u'' - (n/r)u' = \lambda f(u); \qquad r \in (0, 1) \tag{1.3}$$

$$u'(0) = 0 \tag{1.4}$$

$$u(1) = 0, \tag{1.5}$$

where $n = N - 1$. We will assume that

$$\lim_{u \to +\infty} (f(u))/u = +\infty, \text{ i.e., } f \text{ is superlinear}, \tag{1.6}$$

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$$f(0) < 0 \text{ (non-positone)}, \tag{1.7}$$

and for some $k \in (0,1)$,

$$\Lambda = \lim_{d \to +\infty} (d/f(d))^{N/2} \{F(kd) - [(N-2)/(2N)]df(d)\} = +\infty \tag{1.8}$$

where $F(x) = \int_0^x f(r)\,dr$.

If $f(0) > 0$ (positone) and $\lambda > 0$ small, it is known that (1.1)–(1.2) has two solutions: one near zero, the other bifurcating from infinity. However, the popular method of sub-super solutions used in positone problems seems rather difficult to apply when $f(0) < 0$, since $v \equiv 0$ is no longer a sub-solution. In fact, it is a super solution. This is why we have been motivated to undertake this study. Our main result is given in Theorem 1.1.

Theorem 1.1. *Under the above assumptions, there exists* $\lambda_0 > 0$ *such that if* $0 < \lambda < \lambda_0$, *then* (1.1)–(1.2) *has a non-negative solution* u_λ *such that* $u_\lambda > 0$ *and decreasing on* $[0,1]$ *and* $u'_\lambda(1) < 0$.

Castro and Shivaji [3] have made an extensive study of the one-dimensional problem $(N = 1)$. Our proof of Theorem 1.1 is based on the shooting method. That is, to prove that (1.3)–(1.5) has a solution, we consider the problem (1.3)–(1.4) subject to $u(0) = d$. By analyzing this problem depending on the parameter d, we show that for an adequate value of d, u satisfies also (1.5). To prove Lemma 3.2 we use an identity of Pohozaev type (see §2) used by Castro and Kurepa [1, 2] to study oscillatory solutions of other radially symmetric problems. For other applications and extensions of this type of identity, see Ni and Serrin [4].

Our work also proves a recent conjecture by Smoller and Wasserman [6]. In their paper they proved an existence result applicable to functions of the type $f(u) = u^q - \varepsilon$ where $\varepsilon > 0$, $1 < q < N/(N-2)$ and conjectured that an optimal result would be to extend it to $1 < q < (N+2)/(N-2)$. In fact, our work includes this optimal result since if $f(u) = u^q - \varepsilon$ where $\varepsilon > 0$, $1 < q < (N+2)/(N-2)$ then (1.8) is satisfied with k chosen larger than $[(q+1)(N-2)/(2N)]^{(1/(q+1))}$. Note here that if $q < (N+2)/(N-2)$ then $(q+1)(N-2)/(2N) < \{(N+2)/(N-2)+1\}(N-2)/(2N) = 1$.

We will restrict our proofs in this paper to the case $N > 2$. When $N = 2$ the proof is easier along the same lines as in the case $N > 2$.

2. Preliminaries and notations

First of all we extend f to $(-\infty,\infty)$ by defining $f(x) = f(0)$ for $x < 0$. By (1.6) we see that $\lim_{d\to\infty} F(d) = \infty$. Hence (see (1.7)) there exist positive real numbers $\beta < \theta$ such that

$$0 = f(\beta) = F(\theta). \tag{2.1}$$

Since (see (1.8)) $\Lambda = \infty$, we see that there exists $\gamma > (\theta/k)$ such that

$$2NF(kd) - (N-2)df(d) \geq 0 \qquad \text{for } d \geq \gamma. \tag{2.2}$$

Now for each real number d, the initial value (1.3), (1.4), $u(0) = d$ has a unique solution $u(t, d, \lambda)$. This solution depends continuously on (d, λ) in the sense that if $\{(d_n, \lambda_n)\} \to (d, \lambda)$, then $\{u(\ , d_n, \lambda_n)\}$ converges uniformly to $u(\ , d, \lambda)$ on $[0, 1]$. To see this, we observe that for each (d, λ) the map

$$u(s) \to d + \lambda \int_0^s t^{-n} \int_0^t r^n(-f(u(r)))\, dr \tag{2.3}$$

defines a contraction on $C([0, \varepsilon], R)$ for ε small enough.

Next given $d \in R$, $\lambda \in R$, we define

$$E(t, d, \lambda) = \frac{(u'(t, d, \lambda))^2}{2} + \lambda F(u(t, d, \lambda)), \tag{2.4}$$

$$H(t, d, \lambda) = tE(t, d, \lambda) + \frac{N-2}{2} u(t, d, \lambda)u'(t, d, \lambda). \tag{2.5}$$

Multiplying (1.3) by $r^N u'$ and integrating over $[\hat{t}, t]$, and then multiplying (1.3) by $r^n u$ and integrating over $[\hat{t}, t]$, we obtain

$$\begin{aligned} t^{N-1} H(t, d, \lambda) = \hat{t}^{N-1} H(\hat{t}, d, \lambda) + \int_{\hat{t}}^t r^n \lambda [NF(u(r, d, \lambda)) \\ - \frac{N-2}{2} f(u(r, d, \lambda))u(r, d, \lambda)]\, dr. \end{aligned} \tag{2.6}$$

This identity is a form of "Pohozaev identity." For more details see Castro and Kurepa [1] and Pucci and Serrin [5].

Further, for $d \geq \gamma$ let $t_0 := t_0(d, \lambda)$ be such that $d \geq u(t_0, d, \lambda) \geq kd$ for all $t \in [0, t_0)$ and $u(t_0, d, \lambda) = kd$. Multiplying by r^n (1.3)–(1.4) and $u(0) = d$ gives $u'(t, d, \lambda) = -\lambda t^{-n} \int_0^t r^n f(u(r, d, \lambda))\, dr$. Hence $-\lambda t f(kd) \geq Nu'(t, d, \lambda) \geq -\lambda t f(d)$, and integrating on $[0, t_0]$ we have

$$C_1\{d/(\lambda f(kd)\}^{1/2} \geq t_0 \geq C_1\{d/(\lambda f(d))\}^{1/2} \tag{2.7}$$

where $C_1 = \{(1-k)2N\}^{1/2} > 0$. Also choosing $\hat{t} = 0$, $t = t_0$, (2.6) gives

(2.8)

$$\begin{aligned} t_0^n H(t_0, d, \lambda) &= \lambda \int_0^{t_0} r^n \{NF(u(r, d, \lambda)) \\ &\qquad - [(N-2)/2] f(u(r, d, \lambda))u(r, d, \lambda)\}\, dr \\ &\geq \lambda \int_0^{t_0} r^n \{NF(kd) - [(N-2)/2] f(d)d\}\, dr \\ &\geq \lambda \{NF(kd) - [(N-2)/2] f(d)d\} t_0^N / N \\ &\geq \lambda \{NF(kd) - [(N-2)/2] f(d)d\} \cdot \{C_1^N / N\} \cdot \{d/(\lambda f(d))\}^{N/2} \\ &= C_2 \lambda^{(1-N/2)} \{F(kd) - [(N-2)/(2N)]\, df(d)\} \cdot \{d/f(d)\}^{N/2} \end{aligned}$$

where $C_2 = (C_1)^N > 0$.

3. Main lemmas and proof of Theorem 1.1

Lemma 3.1. *If* $\lambda \in (0, \lambda_1 := N(\gamma - \beta)/f(\gamma))$, *then* $u(t, \gamma, \lambda) \geq \beta$ *for all* $t \in [0, 1]$.

Proof. Let $t_1 := \sup\{t \leq 1; u(r, \gamma, \lambda) \geq \beta$, for all $r \in (0, t)\}$. Since $u'(t, \gamma, \lambda) = -\lambda t^{-n} \int_0^t s^n f(u(s, \gamma, \lambda))\, ds$, u is decreasing on $[0, t_1]$. Also if $\lambda \in (0, \lambda_1)$, $t \in [0, t_1]$, we have

$$|u'(t, \gamma, \lambda)| \leq \lambda t f(\gamma)/N < \gamma - \beta.$$

Hence $u(t_1, \gamma, \lambda) > \gamma - (\gamma - \beta)t_1$. In particular, if $t_1 < 1$ this gives $u(t_1, \gamma, \lambda) > \beta$ contradicting the definition of t_1. Thus $t_1 = 1$ and the lemma is proven.

Lemma 3.2. *There exists* $\lambda_2 > 0$ *such that if* $\lambda \in (0, \lambda_2)$, *then* $\{u(t, d, \lambda)\}^2 + \{u'(t, d, \lambda)\}^2 > 0$ *for* $t \in [0, 1]$, $d \in [\gamma, +\infty)$.

Proof. Now for $t \geq t_0$, (2.6) and (2.8) gives

(3.1)
$$t^n H(t) \geq C_2 \lambda^{(1-N/2)} \{Fkd) - [(N-2)/(2N)]\, df(d)\}\{d/f(d)\}^{n/2} + \lambda \int_{t_0}^{t} r^n \{NF(u(r, d, \lambda)) - [(N-2)/2] f(u(r, d, \lambda)) u(r, d, \lambda)\}\, dr.$$

Now by (1.8), our definition of $f(x)$ for $x < 0$ and the fact that $f(0) < 0$, there exists a constant $B < 0$ such that $G(s) = NF(s) - [(N-2)/2] f(s) s \geq B$ for all s. Further using (1.8), we may assume without loss of generality that γ is large enough so that $\{F(kd) - [(N-2)/(2N)] df(d)\}\{d/f(d)\}^{N/2} \geq 1$ for $d \geq \gamma$. Hence by (3.1) we have, for $t \in [t_0, 1]$,

$$t^n H(t) \geq C_2 \lambda^{(1-N/2)} \{F(kd) - [(N-2)/(2N)] df(d)\} \cdot \{d/f(d)\}^{N/2} + \lambda B\{t^N - t_0^N\}/N \tag{3.2}$$

$$\geq C_2 \lambda^{(1-N/2)} + \lambda B/N$$

$$= \lambda\{C_2 \lambda^{-N/2} + B/N\}. \tag{3.3}$$

That is, there exists λ_2 such that for $\lambda \in (0, \lambda_2)$, $H(t)$ (and hence $[u(t, \lambda, d)]^2 + [u'(t, \lambda, d)]^2$) is positive for every $t \in [0, 1]$ and every $d \in [\gamma, +\infty)$ and the lemma is proven.

Lemma 3.3. *Given any* $\lambda > 0$, *there exists* $d > \gamma$ *such that* $u(t, d, \lambda) < 0$ *for some* $t \in [0, 1]$.

Proof. Let $\rho > 0$ and ω be such that $\omega'' + (n/t)\omega' + \rho\omega = 0$, $\omega(0) = 1$, $\omega'(0) = 0$ and the first zero of ω is $\frac{1}{4}$. By (1.6), there exists $d_0(\lambda) \geq \theta/k$ such that if $x \geq d_0$ then

$$(f(x)/x) \geq (\rho/\lambda). \tag{3.4}$$

Suppose now that for every $d > \gamma$, $u(t, d, \lambda) \geq 0$ for all $t \in [0, 1]$. First we show that there exists $d_1(\lambda) \geq d_0(\lambda)$ such that for $d > d_1(\lambda)$ and $t_1 \in (0, 1]$

$$\text{if } u'(t_1, d, \lambda) = 0 \text{ then } u''(t_1, d, \lambda) < 0. \tag{3.5}$$

In fact, let $d_1(\lambda) > d_0(\lambda)$ be such that if $d > d_1(\lambda)$ then $t^n H(t) > 0$ for all $t \in [t_0, 1]$ (see (3.2)). Suppose there exists $t_1 \in [0, 1]$ with $u'(t_1, d, \lambda) = 0$, $u''(t_1, d, \lambda) \geq 0$. By (1.3) we have $u(t_1, d, \lambda) \leq \beta$. Since $kd > \beta$ we have $t_1 > t_0$. Thus $t_1^n H(t_1) = t_1^N \lambda F(u(t_1, d, \lambda)) < 0$, which contradicts the definition of $d_1(\lambda)$. Thus (3.5) holds. By (3.5) we have that if $d > d_1(\lambda)$ and $t \in [0, 1]$ then $u(t, d, \lambda) \cdot u'(t, d, \lambda) \leq 0$. Hence, by (3.2) and the fact that $E(t, d, \lambda) \geq \lambda F(kd)$ for $t \in [0, t_0]$, there exists $d_2(\lambda) > d_1(\lambda)$ such that for $d > d_2$

$$E(t, d, \lambda) \geq \lambda F(d_0) + 2d_0^2 \tag{3.6}$$

for all $t \in [0, 1]$. Now let $d > d_2$. Since $(d\omega)'' + (n/r)(d\omega)' + \rho(dw) = 0$ and $u'' + (n/r)u' + \lambda(f(u)/u)u = 0$, we get

$$u(t)\{t^n v'(t)\} - v(t)\{t^n u'(t)\} = \int_0^t s^n \left\{ \frac{\lambda f(u/s)}{u(s)} - \rho \right\} ds \tag{3.7}$$

where $v = d\omega$. Hence if $u(t, d, \lambda) \geq d_0$ for all $t \in [0, \frac{1}{4}]$ then by (3.4) and facts that $v(\frac{1}{4}) = 0$, $v'(\frac{1}{4}) < 0$ we obtain a contradiction to (3.6). Thus there exists $t^* \in (0, \frac{1}{4})$ such that

$$u(t^*, d, \lambda) = d_0. \tag{3.8}$$

Also u is decreasing on $(0, t^*)$ and (3.6) implies $u'(t^*, d, \lambda) \leq -2d_0$. Since we are assuming $u(t, d, \lambda) \geq 0$ for all $t \in (t^*, 1]$, we have $u' \leq 0$ for $t \in [t^*, 1]$. Therefore $0 \leq u(t, d, \lambda) \leq d_0$ for $t \in (t^*, 1]$ and by (3.6) we have $u'(t, d, \lambda) \leq -2d_0$ for all $t \in (t^*, 1]$. Hence integrating we have

$$u(t^* + \tfrac{1}{2}, d, \lambda) - d_0 \leq -2d_0 \cdot (\tfrac{1}{2}),$$

that is, $u(t^* + \frac{1}{2}, d, \lambda) \leq 0$ where $t^* + \frac{1}{2} < 1$, with $u'(t^* + \frac{1}{2}, d, \lambda) \leq -2d_0$. Thus there exists $T \in (0, 1)$ such that $u(T, d, \lambda) < 0$ which is a contradiction, hence the lemma is proven.

Proof of Theorem 1.1. Let $\lambda_0 = \min\{\lambda_1, \lambda_2\}$ and $\lambda \in (0, \lambda_0)$. Let $\hat{d}(\lambda) := \hat{d} = \sup\{d \in [\gamma, +\infty);\ u(t, d, \lambda) \geq 0$ for all $t \in [0, 1]\}$. By Lemma 3.3 we have that $\hat{d} < +\infty$. Now we claim that:

(A) $u(1, \hat{d}, \lambda) = 0$,
(B) $u(t, \hat{d}, \lambda) > 0\ \forall t \in [0, 1)$,
(C) $u'(1, \hat{d}, \lambda) < 0$, and
(D) u is decreasing on $[0, 1]$.

Suppose there exists $T_1 < 1$ such that $u(T_1, \hat{d}, \lambda) = 0$. Then Lemma 3.2 gives $u'(T_1 \hat{d}, \lambda) \neq 0$ and without loss of generality we can assume $u'(T_1, \hat{d}, \lambda) < 0$. Thus there exists $T_2 \in (T_1, 1)$ such that $u(T_2, \hat{d}, \lambda) < 0$, a contradiction to the definition of $\hat{d}$. This proves (B). That is, $u(1, \hat{d}, \lambda) \geq 0$. Suppose $u(1, \hat{d}, \lambda) > 0$. Then there exists $\eta > 0$ such that $u(t, \hat{d}, \lambda) \geq \eta$ for all $t \in [0, 1]$. Thus there exists $\delta > 0$ such that $u(t, \hat{d} + \delta, \lambda) \geq \eta/2$ for all $t \in [0, 1]$, which contradicts the definition of $\hat{d}$. Hence $u(1, \hat{d}, \lambda) = 0$ and

(A) is proven. Finally (C) follows from Lemma 3.2 and (D) follows by Gidas, Ni and Nirenberg [7].

Remark. Unlike the case $N = 1$ (see Castro and Shivaji [1987, Theorem 1.2]), for $N \geq 2$ the problem (1.1)–(1.2) does not have non-negative solutions with interior zeros. This follows because if there exists $t_0 \in (0, 1)$ for which $u'(t_0) = u(t_0) = 0$ then $E(t_0) = 0$. By (1.3) we obtain $dE/dt = -n(u')^2/t \leq 0$. But $E(1) \geq 0$. Thus $E = 0$ for all $t \in [t_0, 1]$ which is possible only if $u' = 0$ and hence $u = 0$ for all $t \in [t_0, 1]$. But from (1.3) we see that this is impossible with $f(0) < 0$.

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