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RADIAL SOLUTIONS TO A DIRICHLET PROBLEM INVOLVING CRITICAL EXPONENTS WHEN $N = 6$

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ABSTRACT. In this paper we show that, for each $\lambda > 0$, the set of radially symmetric solutions to the boundary value problem

$$-\Delta u(x) = \lambda u(x) + u(x)|u(x)|, \quad x \in B := \{x \in R^6 : \|x\| < 1\},$$
$$u(x) = 0, \quad x \in \partial B,$$

is bounded. Moreover, we establish geometric properties of the branches of solutions bifurcating from zero and from infinity.

1. INTRODUCTION

We consider the boundary value problem

$$\begin{aligned} -\Delta u(x) &= \lambda u(x) + u(x)|u(x)|, \quad x \in B := \{x \in R^6 : \|x\| < 1\}, \\ u(x) &= 0, \quad x \in \partial B, \end{aligned} \tag{1.1}$$

where Δ denotes the Laplacean operator and λ is a positive real number. Our main result is the following theorem:

Theorem 1.1. a) *For each $\lambda > 0$ there exist positive real numbers $j := j(\lambda)$ and $D := D(\lambda)$ such that if u is a solution to* (1.1) *then $\|u\|_\infty \le D$, and u has at the most j nodal curves.*

b) *Given j there exists $D_1(j)$ such that if (λ_1, u_1) and (λ_2, u_2) are radial solutions to* (1.1) *with the property that u_1 and u_2 have j nodal surfaces, $u_1(0) > D_1(j)$ and $u_2(0) > D_1(j)$ then $\lambda_1 < \lambda_2$ if and only if $u_1(0) > u_2(0)$.*

As shown in [5] Theorem 1.1 also holds for $N = 3, 4$ and 5. The proof of the case $N = 6$ was defered to this paper since it requires arguments of a different kind. The main problem arises from the fact that if $N = 6$ and λ is fixed then the solutions to the initial value problem (1.4) below do not tend to zero in compact subsets of $(0, 1]$ as $v(0) \to \infty$.

Motivated by the results of F. Atkinson, H. Brezis and L. Peletier (see [1]) where they studied the case $N = 3, 4, 5, 6$ in [5] we prove their conjecture that if $N = 3, 4$ then the problem (1.1) has only finitely many radially symmetric solutions. If $N \ge 7$ then for any $\lambda > 0$ it has been proven by S. Solimini in [12] (see also [7]) that (1.1) has infinitely many radially symmetric solutions.

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Problems like (1.1) have been studied very intensively over the last three decades, mainly due to the fact that the well developed variational techniques do not apply because the imbedding of the Sobolev space $H_0^1(B)$ in $L^{2N/(N-2)}(B)$ is not compact. Since 1965 it has been known (see Pohozaev [10]) that for $\lambda \leq 0$ the problem (1.1) has no nontrivial solutions. The existence of a positive solution to (1.1) for certain values of $\lambda > 0$ was shown in 1982 by H. Brezis and L. Nirenberg (see [2]). They introduced a method which have influenced a number of papers and produced important results (see [6], [7] and the references therein). The main ingredient was the mountain pass theorem without the Palais Smale condition.

The proof of Theorem 1.1 depends fundamentally on the following result on the location of the zeros of the solution to

$$\begin{aligned} v_d''(t) + \frac{5}{t}v_d'(t) + v_d(t) + 2v_d(t)|v(t)| = 0, \quad t \in (0,1], \\ v_d(0) = 1, \\ v_d'(0) = 0, \end{aligned} \tag{1.2}$$

where v is a solution to

$$\begin{aligned} v''(t) + \frac{5}{t}v'(t) + v(t) + v(t)|v(t)| = 0, \quad t \in (0,1], \\ v(0) = \frac{1}{2}, \\ v'(0) = 0. \end{aligned} \tag{1.3}$$

In fact we have:

Theorem 1.2. *The zeroes of v_d separate the zeros of v, and conversely.*

The proof of Theorem 1.1 is based on the phase-plane analysis of the solution corresponding to a singular ordinary differential equation. We consider the initial value problem

$$\begin{aligned} v''(t) + \frac{5}{t}v'(t) + \lambda v(t) + v(t)|v(t)| = 0, \quad t \in (0,1], \\ v(0) = d, \\ v'(0) = 0, \end{aligned} \tag{1.4}$$

where $d \in R$. It can be shown using the contraction mapping principle that for every (λ, d) problem (1.2) has a unique solution $v(t) := v(t, \lambda, d)$ on the interval $[0, \infty)$ depending continuously on (λ, d). Of course, radially symmetric solutions to (1.1) are solutions to (1.2) satisfying $v(1, \lambda, d) = 0$. Because v is odd in d, we consider only the case $d > 0$. Our analysis focuses on the study of the level set $M = \{(\lambda, d) \colon v(1, \lambda, d) = 0\}$. Using the fact that the solution to (1.4) does not degenerate and some rescaling properties of $v(t)$, we show that $(v_d(1, \cdot, \cdot), v_\lambda(1, \cdot, \cdot))$ never vanishes on M, where v_d, v_λ denote the partial derivatives of v with respect to d and λ. Hence, M is a differentiable manifold. From the results of [1] we infer that if Γ is a connected component of M then there is a radial eigenvalue μ_j of $-\Delta$ subject to a zero Dirichlet boundary condition, and $\nu_j \in [0, \mu_j)$ such that Γ connects $(\mu_j, 0)$ with (ν_j, ∞). Also $\nu_j = 0$ if and only if $j = 1$. Moreover, using variants of the Sturm comparison theorem we prove that for d large $v_d(1, \lambda, d) \cdot v'(1, \lambda, d) > 0$. Thus, there exists a strictly decreasing function $s \colon (D, \infty) \to R$ such that $v(1, \lambda, d) \in \Gamma$ iff $d = s(\lambda)$. We combine this result with those of [1] and [5] to provide a detailed description of the branches of solutions to (1.1) bifurcating from

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zero and from infinity. Since the only connected component of M that accumulates towards $\{(0,d) : d > 0\}$ is the one corresponding to positive solutions and since for large d such a component is the graph of a strictly decreasing function we have the following corollary.

Corollary 1.3. *There exists* $\hat{\lambda}$ *such that if* $\lambda \in (0, \hat{\lambda})$ *then* (1.1) *has a unique positive solution.*

2. Energy Estimates and Pohozaev's Identity

The following lemma is based on Pohozaev's identity and was extensively used in [5] (see also [3, 4, 11]).

Lemma 2.1. *If* $0 \le \tilde{t} < t$ *then*

$$t^5 H(t) - (\tilde{t})^5 H(\tilde{t}) = \int_{\tilde{t}}^{t} r^5 \lambda u^2(r) dr,$$

where

$$\begin{aligned} H(t) &:= t\left(\frac{(v'(t))^2}{2} + \frac{|v(t)|^3}{3} + \lambda\frac{|v(t)|^2}{2}\right) + 2v(t)v'(t) \\ &:= tE(t) + 2v(t)v'(t). \end{aligned} \tag{2.1}$$

Proof. Multiplying the equation in (1.4) simultanously by $r^6 v'(r)$ and $r^5 v(r)$, integrating over $[\tilde{t}, t]$, $0 \le \tilde{t} < t$, and combining the two equations, the proof of the lemma follows (for more details see [5]).

From Lemma 2.1 using the quadratic equation formula we see that

$$\begin{aligned} &tv'(t) + 2v(t) \\ &= \pm\frac{1}{2}\sqrt{16v^2(t) - \frac{8t^2}{3}v^3(t) - 4\lambda t v^2(t) + 8t^{-4}\lambda\int_0^t r^5 v^2(r)dr} \\ &:= \pm\frac{1}{2}\sqrt{R(t)}. \end{aligned} \tag{2.2}$$

Now we define function h by the following equation

$$h(t) = -\frac{tv'(t)}{v(t)}. \tag{2.3}$$

Using (1.2) and Lemma 2.1, as in [5], we obtain

$$h'(t) = \frac{2t^{-5}\int_0^t \lambda r^5 v^2(r)dr - 2t\frac{|v(t)|^3}{3} + t|v(t)|^3}{v^2(t)} > 0. \tag{2.4}$$

Let x_1 denote the first zero of v. Since the left hand side in (2.2) is positive at 0, negative at x_1 and continuous, there exists $\hat{t} \in (0, t)$ such that $R(\hat{t}) = 0$. The uniqueness of $\hat{t}$ follows from (2.4). Since $v'(0, \lambda, d) = 0$ and $v(0, \lambda, d) = d$, we have

$$tv'(t) + 2v(t) = \frac{1}{2}\sqrt{R(t)}, \quad \text{for } t \in [0, \hat{t}], \tag{2.5}$$

and

$$tv'(t) + 2v(t) = -\frac{1}{2}\sqrt{R(t)}, \quad \text{for } t > \hat{t}. \tag{2.6}$$

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Lemma 2.2. *Let x_i and x_{i+1} denote two consecutive zeroes of $v(\cdot,1,\frac{1}{2})$. Then*

$$E(x_{i+1}) \leq \frac{E(x_i)}{6}\left\{5\left(\frac{x_i}{x_{i+1}}\right)^6+1\right\}.$$

□

Proof. From the definition of E (see (2.1)) we have

$$\frac{dE}{dr} = -\frac{5}{r}v'(r)^2. \tag{2.7}$$

Without loss of generality let $v<0$ on $[x_i,x_{i+1}]$. Integrating (2.7) by parts and using (1.3) we obtain

$$\begin{aligned}
E(x_{i+1}) &= E(x_i) - 5\int_{x_i}^{x_{i+1}} \frac{v'(r)}{r}v'(r)dr \\
&= E(x_i) + 5\int_{x_i}^{x_{i+1}} v(r)\frac{v''(r)r - v'(r)}{r^2}dr \\
&= E(x_i) + 5\int_{x_i}^{x_{i+1}} v(r)\frac{-6v'(r) - rv(r) - rv(r)|v(r)|}{r^2}dr \\
&\leq E(x_i) + 15\int_{x_i}^{x_{i+1}} v^2(r)\left(-\frac{2}{r^3}\right)dr - 5\int_{x_i}^{x_{i+1}}\frac{v^2(r)}{r}dr \\
&\leq E(x_i) - 5\int_{x_i}^{x_{i+1}}\frac{v^2(r)}{r}dr.
\end{aligned} \tag{2.8}$$

On the other hand from Pohozaev's identity we have

$$(x_{i+1})^6E(x_{i+1}) = x_i^6E(x_i) + \int_{x_i}^{x_{i+1}} r^5v^2(r)dr. \tag{2.9}$$

Combining (2.8) and (2.9) we infer

$$\begin{aligned}
\left(1-\left(\frac{x_i}{x_{i+1}}\right)^6\right)E(x_i) &\geq \frac{1}{(x_{i+1})^6}\int_{x_i}^{x_{i+1}} r^5v^2(r)dr + 5\int_{x_i}^{x_{i+1}}\frac{v^2(r)}{r}dr \\
&\geq \frac{6}{(x_{i+1})^6}\int_{x_i}^{x_{i+1}} r^5v^2(r)dr.
\end{aligned} \tag{2.10}$$

Using (2.10) in (2.9) we see that

$$E(x_{i+1}) \leq E(x_i)\left\{\left(\frac{x_i}{x_{i+1}}\right)^6 + \frac{1}{6}\left(1-\left(\frac{x_i}{x_{i+1}}\right)^6\right)\right\}, \tag{2.11}$$

which concludes the proof of the lemma. □

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3. The First Three Zeroes of $v(\cdot, 1, \frac{1}{2})$

In this section we estimate the location of the first three zeroes of $v(\cdot, 1, \frac{1}{2})$.

Lemma 3.1. *Let x_1 denote the first zero of $v(\cdot, 1, \frac{1}{2})$. Then $x_1 \leq 4.6$.*

Proof. From

$$-v'(t) = t^{-5} \int_0^t r^5 (1 + |v(r)|) v(r) dr \tag{3.1}$$

on $[0, x_1]$ we see that $v' < 0$ and

$$-v'(t) \geq \frac{t}{6} v(t). \tag{3.2}$$

Hence, integrating (3.2) on $[0, t]$ we obtain

$$v(t) \leq \frac{1}{2} e^{-\frac{t^2}{12}}, \tag{3.3}$$

where we have also used the fact that $v(0) = \frac{1}{2}$.

Thus, from (3.3) we see that $v(3) \leq \frac{1}{4}$. On the other hand, for $t \geq 3$ we have

$$R(t) \geq (16 - \frac{8t^2}{12} - 4t + \frac{8t^2}{6}) v^2(r) \geq 0.$$

Thus $\hat{t} \leq 3$. Hence, for $t \geq 3$ from (2.6) we see that

$$-\frac{v'(t)}{v(t)} = \frac{2}{t} + \frac{1}{2tv(t)} \sqrt{R(t)}.$$

In particular

$$\begin{aligned} -\frac{v'(3)}{v(3)} &\geq \frac{2}{3} + \frac{1}{6} \sqrt{16 - \frac{8 \cdot 9}{3 \cdot 4} - 4 \cdot 3 + \frac{8 \cdot 9}{6}} \\ &\geq 1.1937129. \end{aligned} \tag{3.4}$$

Let θ be a differentiable function such that (see [5])

$$\begin{aligned} v(r) &= -\rho(r) \cos \theta(r), \\ v'(r) &= \rho(r) \sin \theta(r), \\ \theta(0) &= 0, \end{aligned} \tag{3.5}$$

with $\rho(r) = \sqrt{v^2(r) + (v'(r))^2}$. An elementary calculation shows that

$$\theta'(r) = \frac{(v'(r))^2 + \frac{5}{r} v(r) v'(r) + (1 + |v(r)|) v^2(r)}{(v'(r))^2 + v^2(r)}. \tag{3.6}$$

Since $\tan \theta(r) = -\frac{v'(r)}{v(r)}$ for $r \in [3, 3.5]$ from (3.6) we have

$$\begin{aligned} \theta'(r) &\geq 1 - \frac{\frac{5}{t} \tan \theta(t)}{\tan^2 \theta(t) + 1} \geq 1 - \frac{\frac{5}{t} \tan \theta(3)}{\tan^2 \theta(3) + 1} \\ &\geq 1 - \frac{2.461314218}{t}. \end{aligned} \tag{3.7}$$

Integrating on $[3, 3.5]$ we infer

$$\theta(3.5) \geq .5 - 2.461314218 \ln \frac{3.5}{3} + \theta(3) \geq .994059.$$

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Reiterating this argument on $[3.5, 4]$ we obtain $\theta(4) \geq 1.1888816$. Similarly on $[4, 4.5]$ we have $\theta(4.5) \geq 1.4852075$, and on $[4.5, 4.6]$ we see that $\theta(4.6) \geq \frac{\pi}{2}$. Thus, $x_1 \leq 4.6$. Hence the lemma is proven. □

Lemma 3.2. *If x_1 denotes the first zero of $v(\cdot, 1, \frac{1}{2})$, then $x_1 \geq 4.4$, and $|v'(x_1)| \leq 0.134$.*

Proof. Since v is a decreasing function on $[0, x_1]$ and $v(0) = \frac{1}{2}$, from (3.1) we have

$$-v'(t) \leq t^{-5} \int_0^t r^5 (1 + \frac{1}{2}) \frac{1}{2} dr = \frac{t}{8}. \tag{3.8}$$

Thus, for $t \in [0, x_1]$ we see that

$$v(t) \geq \frac{1}{2} - \frac{t^2}{16}. \tag{3.9}$$

In particular, from (3.9) we have that $x_1 \geq \sqrt{8}$. Furthermore, by using (3.9) in (3.1) we infer

$$-v'(t) \geq t^{-5} \int_0^t r^5 (1 + \frac{1}{2} - \frac{r^2}{16})(\frac{1}{2} - \frac{r^2}{16}) dr \geq \frac{t}{8} - \frac{t^3}{64} + \frac{t^5}{2560}. \tag{3.10}$$

Hence, on $[0, \sqrt{8}]$ we have

$$v(t) \leq \frac{1}{2} - \frac{t^2}{16} + \frac{t^4}{256} - \frac{t^6}{15360}. \tag{3.11}$$

Reiterating the argument in (3.10) we see that

$$-v'(t) \leq \frac{t}{8} \left(1 - \frac{t^2}{8} + \frac{3t^4}{320} - \frac{11t^6}{16^2 \cdot 36} + \frac{23t^8}{16^4 \cdot 30}\right), \tag{3.12}$$

and

$$v(t) \geq \frac{1}{2} - \frac{t^2}{16} + \frac{t^4}{16^2} - \frac{t^6}{16^2 \cdot 20} + \frac{11t^8}{16^4 \cdot 9} - \frac{23t^{10}}{16^5 \cdot 150}. \tag{3.13}$$

Let

$$\phi(t) = -\frac{v'(t)}{v(t)} - \frac{5}{2t}. \tag{3.14}$$

In particular, from (3.10), (3.11) and the definition of ϕ we see that

$$\phi(2.4) \leq -0.3610558677. \tag{3.15}$$

It can easily be seen that ϕ satisfies the equation

$$\phi'(t) = \phi^2(t) + 1 + v(t) - \frac{15}{4t^2}. \tag{3.16}$$

Using the estimate in (3.3) we obtain

$$1 + v(t) - \frac{15}{4t^2} \leq 1 + \frac{1}{2} e^{-\frac{t^2}{12}} - \frac{15}{4t^2} := g(t). \tag{3.17}$$

Since $g'(t) > 0$ for $t \in [0, 4.5]$ and $g(4.5) \leq 0.9073055 := a^2$, we have

$$\phi'(t) \leq \phi^2(t) + a^2, \tag{3.18}$$

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for $t \in [0, 4.5]$. Thus, integrating (3.18) on $[2.4, x_1]$ we obtain

$$\frac{1}{a}\left(\frac{\pi}{2} - \arctan\frac{\phi(2.4)}{a}\right) \le x_1 - 2.4.$$

Hence

$$x_1 \ge 4.42940506, \tag{3.19}$$

where we have also used (3.13) and the definition of a.

Now, we estimate $E(x_1)$. From Lemma 2.1 we have

$$\begin{aligned} {x_1}^6 E(x_1) &= \int_0^{1.5} r^5 v^2(r)dr + \int_{1.5}^{\sqrt{8}} r^5 v^2(r)dr + \int_{\sqrt{8}}^{x_1} r^5 v^2(r)dr \\ &\le \frac{1.5^6}{6 \cdot 2^2} + (0.3785)^2\left(\frac{8^3 - 1.5^6}{6}\right) \\ &\quad + (0.21667)^2\left(\frac{x_1^6 - 8^3}{6}\right), \end{aligned} \tag{3.20}$$

where we have estimated $v(1.5)$ and $v(\sqrt{8})$ by using (3.3). Since $E(x_1) = (v'(x_1))^2/2$, from (3.19) and (3.20) it follows that

$$|v'(x_1)| \le 0.133715, \tag{3.21}$$

which together with (3.19) proves the lemma. □

Lemma 3.3. *Let $\overline{x_1} > x_1$ be the first zero of $v'(\cdot, 1, \frac{1}{2})$. Then*

$$5.97 \le \overline{x_1} \le 6.171$$

and

$$|v(\overline{x_1})| \le 0.0712.$$

Proof. Since $\theta'(t) \ge 1$, (see (3.6)) on $[x_1, \overline{x_1}]$ we see that $\overline{x_1} - x_1 \le \frac{\pi}{2}$. Hence

$$\overline{x_1} \le x_1 + \frac{\pi}{2} \le 6.1707963. \tag{3.22}$$

Furthermore, from (3.21) and (3.22) we infer

$$1 + |v(t)| - \frac{15}{4t^2} \le 1 + |v'(x_1)| - \frac{15}{\overline{x_1}^2} \le 1.0352348. \tag{3.23}$$

Thus, arguing as in (3.16) we obtain

$$\overline{x_1} \ge x_1 + \frac{\pi}{2\sqrt{1.0352348}} \ge 5.973284. \tag{3.24}$$

In order to estimate $v(x_1)$ we use Lemma 2.1. Since $v'(\overline{x_1}) = 0$, we have

$$\overline{x_1}^6\left(\frac{v^2(\overline{x_1})}{2} + \frac{v^3(\overline{x_1})}{3}\right) \le \frac{(v'(x_1))^2}{2}{x_1}^6 + \int_{x_1}^{\overline{x_1}} r^5 v^2(r)dr.$$

Hence

$$v^2(\overline{x_1}) \le \frac{3(v'(x_1))^2}{2(\frac{\overline{x_1}}{x_1})^6 + 1} \le (0.0711735)^2, \tag{3.25}$$

where we have used (3.19) and (3.22). This together with (3.22) and (3.24) proves the lemma. □

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Lemma 3.4. *If x_2 is the second zero of $v(\cdot, 1, \frac{1}{2})$, then*

$$7.5 \leq x_2 \leq 8.1,$$

and

$$v'(x_2) \leq 0.0612.$$

Proof. Let $w(t) := t^{\frac{5}{2}} v(t)$. It can easily be shown that w satisfies

$$w''(t) + \left(1 - \frac{15}{4t^2} + |v(t)|\right) w(t) = 0. \tag{3.26}$$

Since

$$1 - \frac{15}{4t^2} + |v(t)| \geq 1 - \frac{15}{4x_1^2} \geq .8088,$$

(see (3.19)), from the Sturm comparison theorem and Lemma 3.1 we infer

$$x_2 \leq x_1 + \frac{\pi}{\sqrt{.8088}} \leq 8.0931. \tag{3.27}$$

On the other hand, for $t \in [x_1, x_2]$ using (3.25) and (3.27) we have

$$1 - \frac{15}{4t^2} + |v(t)| \leq 1 - \frac{15}{4x_2^2} + |v(\overline{x_1})| \leq 1.0139201.$$

Hence,

$$x_2 \geq x_1 + \frac{\pi}{\sqrt{1.0139201}} \geq 7.5494033, \tag{3.28}$$

where we have also used (3.19).

Next, from (3.28), Lemma 3.1 and Lemma 3.2 we see that

$$(v'(x_2))^2 \leq \frac{(v'(x_1))^2}{6} \left\{ 5 \left(\frac{4.6}{7.549}\right)^6 + 1 \right\} \leq (.061175)^2,$$

which together with (3.27) and (3.28) concludes the proof of the lemma. □

Lemma 3.5. *Let x_3 denote the third zero of $v(\cdot, 1, \frac{1}{2})$. Then*

$$10.66 \leq x_3 \leq 11.35,$$

and

$$|v'(x_3)| \leq 0.035.$$

Proof. Since on $[x_2, x_3]$ we have

$$1 - \frac{15}{4t^2} + |v(t)| \geq 1 - \frac{15}{4x_2^2} \geq .9325697,$$

by the Sturm comparison theorem and (3.28) we see that

$$x_3 \leq x_2 + \frac{\pi}{\sqrt{.9325697}} \leq 11.346371. \tag{3.29}$$

On the other hand, imitating the arguments of Lemma 3.3 it can easily be shown that if $\overline{x_2} > x_2$ is such that $v'(\overline{x_2}) = 0$, then $v(\overline{x_2}) \leq 0.0476045$. Thus

$$1 - \frac{15}{4t^2} + |v(t)| \leq 1.04761 - \frac{15}{4x_3^2} \leq 1.0184761.$$

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Hence,

$$x_3 \geq x_2 + \frac{\pi}{\sqrt{1.0184761}} \geq 10.66237, \tag{3.30}$$

where we have also used (3.28). Furthermore,

$$(v'(x_3))^2 \leq \frac{(v'(x_2))^2}{6} \left\{ 5 \left(\frac{8.0932}{10.66} \right)^6 + 1 \right\} \leq (.035)^2. \tag{3.31}$$

In particular, from (3.30) and (3.31) we see that

$$|v'(x_3)| \leq \frac{4.5}{x_3^2}. \tag{3.32}$$

Thus, the lemma is proven. □

4. The First Four Zeroes of $v_d(\cdot, 1, \frac{1}{2})$

In this section we estimate the first four zeros of v_d, where v_d denotes the partial derivative of v with respect to d.

Lemma 4.1. *Let τ_1 denote the first zero of $v_d(\cdot, 1, \frac{1}{2})$. Then $\tau_1 \geq 4.04$.*

Proof. Since $v_d(0) = 1$ and $v(0) = \frac{1}{2}$ we have

$$-v_d'(t) = t^{-5} \int_0^t r^5 (1 + 2|v(r)|) v_d(r) dr \leq \frac{t}{3}, \tag{4.1}$$

for $t \in [0, \tau_1]$. Thus

$$v_d'(t) \geq 1 - \frac{t^2}{6}. \tag{4.2}$$

On the other hand, using (3.9) and (4.2) in (4.1) we infer

$$-v_d'(t) \geq t^{-5} \int_0^t r^5 (1 + 1 - \frac{r^2}{8})(1 - \frac{r^2}{6}) dr = \frac{t}{3} \left(1 - \frac{11t^2}{64} + \frac{t^4}{160} \right). \tag{4.3}$$

Integrating (4.3) we obtain

$$v_d(t) \leq 1 - \frac{t^2}{2} + \frac{11t^4}{768} - \frac{t^6}{2880}, \tag{4.4}$$

for $t \in [0, \sqrt{6}]$. Reiterating the argument and using (3.11) and (4.4) in (4.1) we see that

$$\begin{aligned} -v_d'(t) &\leq t^{-5} \int_0^t r^5 (2 - \frac{r^2}{8} + \frac{r^4}{128})(1 - \frac{r^2}{6} + \frac{11r^4}{768}) dr \\ &\leq \frac{t}{3}(1 - \frac{11t^2}{64} + \frac{11t^4}{640} - \frac{19t^6}{16^2 \cdot 6} + \frac{11t^8}{16^2 \cdot 112}), \end{aligned} \tag{4.5}$$

and

$$v_d(t) \geq 1 - \frac{t^2}{2} + \frac{11t^4}{12 \cdot 64} - \frac{11t^6}{18 \cdot 640} + \frac{19t^8}{16^2 \cdot 144} - \frac{11t^{10}}{64^3 \cdot 560}. \tag{4.6}$$

Let

$$\psi(t) = -\frac{v_d'(t)}{v_d(t)} - \frac{5}{2t}. \tag{4.7}$$

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In particular, from (4.5), (4.6) and the definition of ψ we see that

$$\psi(1.8) \leq -0.2597377. \tag{4.8}$$

It can easily be seen that ψ satisfies the equation

$$\psi'(t) = \psi^2(t) + 1 + 2v(t) - \frac{15}{4t^2}. \tag{4.9}$$

From the estimate for v on $[0, \sqrt{8}]$ (see (3.11)) we obtain

$$1 + 2v(t) - \frac{15}{4t^2} \leq 1 - \frac{t^2}{8} + \frac{t^4}{128} - \frac{t^6}{7680} - \frac{15}{4t^2} := G(t). \tag{4.10}$$

Now, we calculate the maximum of $G(t)$ on $[0, \sqrt{8}]$. Since

$$G'(t) = -\frac{1}{2t^3}\left(\frac{t^4}{2}(1 - \frac{t^2}{8} + \frac{t^4}{320}) - 15\right).$$

we see that $1 - \frac{t^2}{8} + \frac{t^4}{320}$ has its minimum at $t = \sqrt{20}$. For $t \in [2, 2.4]$ we have that $\frac{t^4}{2} - \frac{t^6}{16} + \frac{t^8}{640} - 15 \leq 0$, hence $G'(t) \geq 0$. Also, for $t \in [0, 2]$ we see that $\frac{t^4}{2} + \frac{t^8}{640} - 15 \leq 0$. Therefore $G'(t) \geq 0$ for $t \in [0, 2.4]$. Thus, $G(t) \leq G(2.4) \leq 0.8632751 := b^2$. Hence, for $t \in [1.8, 2.4]$ we have

$$\psi'(t) \leq \psi^2(t) + b^2. \tag{4.11}$$

Furthermore, by integrating (4.11) on $[1.8, 2.4]$ and using (4.9) we obtain

$$\frac{1}{b}(\arctan \frac{\psi(2.4)}{b} - \arctan \frac{\psi(1.8)}{b}) \leq 0.6. \tag{4.12}$$

Thus,

$$\psi(2.4) \leq -0.1254803. \tag{4.13}$$

For $t > 2.4$ with $v(t) > 0$, using (3.3) we infer

$$1 + 2v(t) - \frac{15}{4t^2} \leq 1 + e^{-\frac{t^2}{12}} - \frac{15}{t^2} := G_1(t). \tag{4.14}$$

Since

$$G_1'(t) = -\frac{t}{6}e^{-\frac{t^2}{12}} - \frac{15}{2t^3}$$

we see that $\max G_1(t) = 1.0597904 = G_1(3.2)$, hence by integrating

$$\psi'(t) \leq \psi^2(t) + G_1(3.2)$$

on $[2.4, t]$ we see that if $v_d(\tau_1) = 0$ then

$$\frac{1}{\sqrt{G_1(3.2)}}\left(\frac{\pi}{2} - \arctan \frac{\psi(2.4)}{\sqrt{G_1(3.2)}}\right) \leq \tau_1 - 2.4.$$

Thus

$$\tau_1 \geq 4.043663, \tag{4.15}$$

where we have also used (4.13). Hence, the lemma is proven. □

Lemma 4.2. *If τ_2 is the second zero of $v_d(\cdot, 1, \frac{1}{2})$, then*

$$\tau_2 \geq 7.059.$$

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Proof. In order to estimate τ_2 we note that $z(t) := t^{\frac{5}{2}} v_d(t)$ satisfies

$$z''(t) + \left(1 - \frac{15}{4t^2} + 2|v(t)|\right) z(t) = 0. \tag{4.16}$$

First we estimate v on $[\tau_1, x_1]$. Since

$$v''(t) = -\frac{5}{t}v'(t) - |v(t)|v(t) \leq -\frac{5}{t}v'(t), \tag{4.17}$$

after integration on $[\tau_1, x_1]$ we infer

$$v'(t) \geq v'(x_1)\frac{x_1^5}{t^5}. \tag{4.18}$$

Hence

$$v(x_1) - v(\tau_1) \geq \frac{v'(x_1)x_1^5}{4}(\tau_1^{-4} - x_1^{-4}).$$

Therefore,

$$v(\tau_1) \leq 0.1046821. \tag{4.19}$$

Hence on $[\tau_1, x_1]$ we have

$$1 - \frac{15}{4t^2} + 2|v(t)| \leq 1 - \frac{15}{4x_1^2} + 2v(\tau_1) \leq 1.032143. \tag{4.20}$$

For $t > x_1$ by using (3.19) we obtain

$$1 - \frac{15}{4t^2} + 2|v(t)| \leq 1 - \frac{15}{4x_1^2} + 2v(\overline{x_1}) \leq 1.0850936. \tag{4.21}$$

Thus by the Sturm comparison theorem from (4.20)–(4.21) using also (4.15) we see that

$$\tau_2 \geq \tau_1 + \frac{\pi}{\sqrt{1.0850936}} \geq 7.0595584, \tag{4.22}$$

which concludes the proof of the lemma. □

Lemma 4.3. *Let $\tau_3 < \tau_4$ denote the third zero and the fourth zero of $v_d(\cdot, 1, \frac{1}{2})$. Then*

$$\tau_3 \geq 10.07,$$

and

$$\tau_4 \geq 13.09.$$

Proof. Since on $[\tau_2, x_2]$ we know that $|v(\tau_2)| \leq |v(\overline{x_1})|$ it follows that

$$1 - \frac{15}{4t^2} + 2|v(t)| \leq 1 - \frac{15}{4x_2^2} + 2|v(\overline{x_1})| \leq 1.0850936.$$

On the other hand, for $t > x_2$ we have

$$1 - \frac{15}{4t^2} + 2|v(t)| \leq 1 - \frac{15}{4x_3^2} + 2|v(\overline{x_1})| \leq 1.0660916.$$

Hence,

$$\tau_3 \geq \tau_2 + \frac{\pi}{\sqrt{1.0850936}} \geq 10.075454. \tag{4.23}$$

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Similarly

$$\tau_4 \geq \tau_3 + \frac{\pi}{\sqrt{1.0850936}} \geq 13.091349. \tag{4.24}$$

Thus, the lemma is proven. □

5. Proof of Theorem 1.2

In order to prove the separation of zeroes for v and v_d we first show

Lemma 5.1. *If x_i and x_{i+1} are two consecutive zeroes of $v(\cdot, 1, \frac{1}{2})$ on $[x_3, \infty)$, then $x_{i+1} - x_i \leq 3.46$. Moreover,*

$$\frac{x_i}{x_{i+1}} \geq .75.$$

Proof. Without loss of generality we can assume that $v'(x_i) > 0$. Let $\overline{x}_i \in (x_i, x_{i+1})$, $i \geq 3$, be such that $v'(\overline{x}_i) = 0$. Then, from (3.6) we see that $\theta' \geq 1$ on $[x_i, \overline{x}_i]$. Thus

$$\overline{x}_i - x_i \leq \frac{\pi}{2}. \tag{5.1}$$

On the other hand on $[\overline{x}_i, x_{i+1}]$, using the fact that $x_i \geq 10$ we infer

$$\theta'(t) \geq 1 - \frac{5}{2t} \sin 2\theta \geq 1 - .25 \sin 2\theta. \tag{5.2}$$

Hence, by integrating (5.2) we obtain

(5.3)

$$\begin{aligned} x_{i+1} - \overline{x}_i &\leq \int_{\frac{\pi}{2}}^{\pi} \frac{d\theta}{1 - .25 \sin 2\theta} = \int_0^{\frac{\pi}{2}} \frac{dy}{1 - .25 \sin y} = \frac{2}{\sqrt{1 - \frac{1}{16}}} \arctan \frac{1.25}{\sqrt{1 - \frac{1}{16}}} \\ &\leq 1.8833. \end{aligned}$$

Therefore, combining (5.1) with (5.3) we have

$$x_{i+1} - x_i \leq 3.455.$$

Furthermore,

$$\frac{x_i}{x_{i+1}} \geq \frac{x_i}{x_i + 3.455} \geq .7524,$$

which proves the lemma. □

Lemma 5.2. *Let $x_3 < x_4 < \ldots$ denote the zeroes of $v(\cdot, 1, \frac{1}{2})$ on $[10, \infty)$. If $|v'(x_3)| \leq \frac{M}{x_3^2}$ then*

$$|v'(x_i)| \leq \frac{M}{x_i^2}.$$

Proof. We prove the latter inequality by induction. As shown in (3.32) it holds for $i = 3$. Suppose $|v'(x_i)| \leq \frac{M}{x_i^2}$. By Lemma 5.1 we have

$$\frac{5}{6}\left(\frac{x_i}{x_{i+1}}\right)^6 + \frac{1}{6} \leq \frac{x_i^4}{(x_{i+1})^4}, \tag{5.4}$$

because $\left(\frac{x_i}{x_{i+1}}\right)^2 \geq .558$. Hence, by (2.1) we see that $|v'(x_i)| \leq \frac{M}{x_{i+1}^2}$, which concludes the proof of the lemma. □

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Lemma 5.3. *For any $j \geq 3$ the following holds*

$$x_j \leq x_3 + (j-3)\pi + .68, \tag{5.5}$$

and

$$\tau_{j+1} \geq \tau_4 + (j-3)\pi - .23. \tag{5.6}$$

Proof. Since for $t \in [x_i, x_{i+1}]$ we have

$$1 - \frac{15}{4t^2} + |v(t)| \geq 1 - \frac{15}{4x_i^2},$$

by the Sturm comparison theorem it follows that

$$\begin{aligned} x_{i+1} &\leq x_i + \frac{\pi}{\sqrt{1-\frac{15}{4x_i^2}}} \leq x_i + \frac{\pi}{1-\frac{15}{4x_i^2}} \\ &= x_i + \pi\left(1 + \frac{15}{4x_i^2} + \left(\frac{15}{4x_i^2}\right)^2 + \cdots\right) \\ &= x_i + \pi + \frac{15\pi}{4x_i^2}\cdot\frac{1}{1-\frac{15}{4x_i^2}}. \end{aligned} \tag{5.7}$$

Since $x_i \geq 10.66$ we obtain

$$x_{i+1} \leq x_i + \pi + \frac{15}{4x_i^2}\cdot\frac{400}{385} \leq x_i + 3.25. \tag{5.8}$$

On the other hand using Lemma 4.2 we see that

$$1 - \frac{15}{4t^2} + |v(t)| \leq 1 - \frac{15}{4x_{i+1}^2} + \frac{M}{x_i^2} \leq 1 + \frac{M}{x_i^2}. \tag{5.9}$$

Hence, from the Sturm comparison theorem it follows that

$$\begin{aligned} x_{i+1} &\geq x_i + \frac{\pi}{\sqrt{1+\frac{M}{x_i^2}}} \geq x_i + \frac{\pi}{1+\frac{M}{2x_i^2}} \\ &= x_i + \pi\left(1 - \frac{M}{2x_i^2} + \left(\frac{M}{2x_i^2}\right)^2 - \cdots\right) \\ &= x_i + \pi - \frac{\pi M}{2x_i^2}\cdot\frac{1}{1+\frac{M}{2x_i^2}} \\ &\geq x_i + \pi - .12 \geq x_i + 3, \end{aligned} \tag{5.10}$$

where we have used the fact that $x_i \geq 10$ and $M = 4.5$ (see (3.32)). Thus,

$$x_{i+1} \geq x_3 + 3(i-3).$$

Hence

$$\begin{aligned} \frac{400}{385}\sum_{j=3}^{n}\frac{15\pi}{4(x_3+3(j-3))^2} &\leq \frac{400}{385}\cdot\frac{15\pi}{4}\cdot\frac{1}{3}\int_{-3}^{\infty}\frac{ds}{(x_3+3s)^2} \\ &\leq \frac{400}{385}\cdot\frac{15\pi}{4\cdot 18} \leq .68. \end{aligned} \tag{5.11}$$

Similarly

$$\tau_{i+1} \leq \tau_i + 3.2, \tag{5.12}$$

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and

$$\tau_{i+1} \geq \tau_i + \pi - \frac{M\pi}{\tau_i^2}(.96) \geq x_i + 2.9. \tag{5.13}$$

Hence

$$\begin{aligned}\sum_{j=4}^{n} \frac{M\pi(.96)}{(\tau_4 + (j-4)2.9)^2} &\leq \frac{M\pi(.96)}{2.9} \int_{-4}^{\infty} \frac{ds}{(\tau_4 + 2.9s)^2} \\ &\leq \frac{M\pi(.96)}{2.9^2\tau_4} \leq \frac{M\pi(.96)}{2.9^2} \cdot \frac{1}{13} \leq .23.\end{aligned} \tag{5.14}$$

Thus the lemma is proven. □

By the Sturm comparison theorem we see that $\tau_j < x_j$ for $j = 1, 2, \ldots$. On the other hand, by Lemmas 3.5, 4.3 and 5.3 we have

$$\begin{aligned} x_j &\leq x_3 + (j-3)\pi + .68 \\ &\leq \tau_{j+1} - \tau_4 + x_3 + .68 + .23 \\ &\leq \tau_{j+1} - 13.09 + 11.35 + .68 + .23 \\ &< \tau_{j+1}. \end{aligned} \tag{5.15}$$

Hence, from (5.15) and the results of sections 3 and 4 on the first three zeroes of v and the first four zeroes of v_d we see that for $j = 1, 2, \ldots$ we have

$$\tau_j < x_j < \tau_{j+1} < x_{j+1}.$$

Thus, the zeroes of $v(\cdot, 1, \frac{1}{2})$ separate the zeroes of $v_d(\cdot, 1, \frac{1}{2})$ and conversely, which concludes the proof of Theorem 1.2.

6. Proof of Theorem 1.1

The crucial ingredient in the proof of Theorem 1.1 is Lemma 6.3 below showing that the zeroes of $v(t, \lambda, d)$ separate the zeros of $v_d(t, \lambda, d)$.

It can easily be shown that the following homogeneity relation holds

$$v(t, 1, \frac{d}{\lambda}) = \frac{1}{\lambda} v(\frac{t}{\sqrt{\lambda}}, \lambda, d). \tag{6.1}$$

Let $t_j(\lambda, d)$ denote the j-th zero of $v(\cdot, \lambda, d)$ and $\tau_j(\lambda, d)$ be the j-th zero of $v_d(\cdot, \lambda, d)$. From (6.1) we see that

$$t_j(\lambda, d) = \sqrt{\lambda} t_j(1, \frac{d}{\lambda}), \tag{6.2}$$

and

$$\tau_j(\lambda, d) = \sqrt{\lambda} \tau_j(1, \frac{d}{\lambda}). \tag{6.3}$$

Using (6.1) and the results of [1] we have

$$\lim_{d \to \infty} t_j(\lambda, d) = \sqrt{\lambda} t_j(1, \frac{d}{\lambda}), \tag{6.4}$$

and

$$\lim_{d \to \infty} t_1(\lambda, d) = 0, \tag{6.5}$$

for λ in compact subsets of $(0, \infty)$.

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Lemma 6.1. *Let $s_1 := s_1(d)$ denote the first zero of $v'(\cdot, 1, d)$. Then $s_1 \to 0$, as $d \to \infty$.*

Proof. For $t \le s_1$ we have

$$\begin{aligned} -t^5 v'(t) &= \int_0^t r^5(1+|v(r)|)v(r)dr \\ &= \int_0^{t_1} r^5(1+|v(r)|)v(r)dr + \int_{t_1}^t r^5(1+|v(r)|)(-v(r))dr. \end{aligned} \tag{6.6}$$

Also, from Lemma 2.1 we have

$$t^6 \frac{(v'(t))^2}{2} + 2t^5 v(t)v'(t) + t^6(\frac{|v(t)|^3}{3} + \frac{\lambda v^2(t)}{2}) \ge 0.$$

Hence, using the quadratic formula we see that

$$2t^{10}v^2(t) \ge t^{12}(\frac{|v(t)|^3}{3} + \frac{\lambda v^2(t)}{2}) \ge t^{12}\frac{|v(t)|^3}{3}.$$

Thus,

$$|v(t)| \le 6t^{-2} \tag{6.7}$$

for $t \in [0, t_1]$.

Since, by the results from [1] we know that $t_1(1, d) \to 0$ as $d \to \infty$ and since (6.7) holds, we have

$$\int_0^t r^5(1+|v(r)|)v(r)dr \le 6\left(\frac{t_1^4}{4} + 6\frac{t_1^2}{2}\right) \to 0, \tag{6.8}$$

as $d \to \infty$.

Suppose s_1 is bounded away from zero, hence there exists $m > 0$ such that

$$s_1 > m. \tag{6.9}$$

Let $0 < y < \min\{t_1(1, -\frac{1}{2}), m\}$. On $[t_1, s]$ we see that $v'' = -\frac{5}{r}v' - v - |v|v \ge 0$. Hence, since v is convex on $[t_1, s]$ and since for d large $v(\cdot, 1, d) \to v(\cdot, 1, -\frac{1}{2})$ we have

$$\begin{aligned} \int_{t_1}^t r^5(1+|v(r)|)v(r)dr &\ge \int_{t_1}^t r^5 \frac{v(y)}{y-t_1}(r-t_1)dr \\ &\ge K|v(y)|\left(\frac{t^7}{7} + \frac{t_1^7}{6}\right), \end{aligned} \tag{6.10}$$

where K is a constant. Since $t_1 \to 0$, as $d \to \infty$, from (6.6), (6.8) and (6.10) with $t = s_1$ we get a contradiction, which proves the lemma. □

Lemma 6.2. *If $\tau_j(\lambda, d)$ denotes the j-th zero of $v_d(\cdot, \lambda, d)$, then*

$$\lim_{d\to\infty} \tau_j(\lambda, d) = \sqrt{\lambda}\tau_{j-1}(1, -\frac{1}{2}), \tag{6.11}$$

and

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$$\lim_{d\to\infty} \tau_1(\lambda, d) = 0, \tag{6.12}$$

for λ *in compact subsets of* $(0,\infty)$.

Proof. Let $\sigma_1 := \sigma_1(d)$ denote the first zero of $v_d'(\cdot,1,d)$. By the Sturm comparison theorem and Lemma 6.1 we know that $\sigma_1 < s_1$. Let w be the multiple of $v_d(\cdot,1,d)$ that satisfies

(6.13)
$$w''(t) + \frac{5}{t}w'(t) + w(t) + 2w(t)|v(t,1,d)| = 0, \quad t \in [\sigma_1,\infty),$$
$$w(\sigma_1) = 0,$$
$$w'(\sigma_1) = -1.$$

Since $v_d(\cdot,1,-\frac{1}{2}) = -z$ satisfies

$$z''(t) + \frac{5}{t}z'(t) + z(t) + 2z(t)|z(t,1,-\frac{1}{2})| = 0, \tag{6.14}$$

and

$$|z'(t)| \leq \frac{t}{3} \tag{6.15}$$

as $d \to \infty$, we have

$$|z(\sigma_1) + 1| \leq \frac{\sigma_1^2}{6} \to 0. \tag{6.16}$$

Thus by the continuous dependence on initial conditions and coefficients we see that $w(\cdot,1,d)$ converges to $z(\cdot,1,-\frac{1}{2})$ uniformly on compact subsets of $[0,\infty)$ as $d \to \infty$. Since the zeros of $w(\cdot,1,d)$ are the zeros of $v_d(\cdot,1,d)$ we see that $\tau_j(1,d) \to \tau_{j-1}(1,-\frac{1}{2})$ as $d \to \infty$. Hence the lemma follows from (6.3) and the Sturm comparison theorem.

Lemma 6.3. *For* d *sufficiently large and for* λ *in compact subsets of* $(0,\infty)$*, the zeroes of* $v(\cdot,\lambda,d)$ *separate the zeroes of* $v_d(\cdot,\lambda,d)$*. Moreover,*

$$v'(1,\lambda,d)\cdot v_d(1,\lambda,d) > 0.$$

□

Proof. The separation of zeroes follows from the Sturm comparison theorem, Lemma 6.2 and Theorem 1.2. Hence

$$\tau_1(\lambda,d) < t_1(\lambda,d) < \ldots < \tau_k < t_k = 1 < \tau_{k+1}. \tag{6.17}$$

Also, because $v_d'(\tau_1) < 0$ we have $\operatorname{sgn} v_d'(\tau_i) = (-1)^{i+1}$. Hence $v(\tau_i)v_d'(\tau_i) < 0$. Finally, if $v'(1) > 0$, then $v < 0$ on $(t_{k-1}, t_k = 1)$. Thus, $v_d'(\tau_{j-1}) > 0$. Hence $v_d(1) > 0$. Similarly, if $v'(1) < 0$, then $v_d(1) < 0$, which concludes the proof of the lemma. □

Next, we recall various properties of the bifurcation diagram of the set of radial solutions to (1.1). We summarize these properties in the following lemma.

Lemma 6.4. a) Γ *is a connected component of* $\{(\lambda,d); v(1,\lambda,d) = 0\}$ *if and only if there exists a positive integer* k *and a continuous function* $\beta_k : (0,\infty) \to (0,\infty)$ *such that* $\Gamma = \{(\beta_k(d),\beta_k(d),d); d \in (0,\infty)\} := \Gamma_k$.

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b) *If* $\{(\lambda_n, d_n)\}$ *is a sequence in* Γ_k *with* $d_n \to \infty$ *then* $\{\lambda_n\}$ *converges to* $t_{k-1}(1, -\frac{1}{2})$.

c) *If* $(\lambda, d) \in \Gamma_k$ *then* $\lambda < \mu_k$, *where* μ_k *is a radial eigenvalue of* $-\Delta$ *subject to a zero Dirichlet boundary condition.*

d) *For each* $k \geq 1$ *there exists* $q(k) > 0$ *such that if* $(\lambda, d) \in \Gamma_k$ *then* $\lambda \geq q(k)$ *and* $q(k) \to \infty$ *as* $k \to \infty$.

e) $\{(\lambda, d); v(1, \lambda, d) = 0, v(\cdot, \lambda, d)$ *has exactly* k *zeroes in* $(0, 1]\}$ *is connected.*

f) *For each positive integer* k *there exist a positive number* r *and a differentiable decreasing function* $s : (r, \infty) \to (0, \infty)$ *such that* $(\lambda, d) \in \Gamma_k$ *if and only if* $\lambda = s(d)$.

Proof. The proof of parts a)–e) is identical to those in [5, Lemma 5.2]. Part f) follows from the implicit function theorem, Lemma 6.3 and part b).

Now, we conclude the proof of Theorem 1.1.

From part d) of Lemma 6.4 we see that for fixed $\lambda > 0$ there can not exist radial solutions with arbitrarily large number of zeros. This proves the existence of $j(\lambda)$. Also if $\lambda \neq t_i(1, -\frac{1}{2}), i = 1, 2, ...,$ then the existence of $D(\lambda)$ follows from parts d) and b) of Lemma 6.4.

Suppose, now, that $\lambda = t_i$ for positive integer i and that $v(1, \lambda, d_n) = 0$ for some sequence $\{d_n\} \to \infty$. By part d) of Lemma 6.4, without loss of generality we can assume that $v(\cdot, \lambda, d_n)$ has j zeroes in $(0, 1]$. Thus, by parts a) and e) of Lemma 6.4 we see that $\{(\lambda, d_n)\} \subset \Gamma_j$. Since, by Lemma 6.4 part f), $\lambda = s(d_n)$ for n large we have a contradiction to the fact that s is decreasing. Thus, the existence of $D(\lambda)$ for all $\lambda > 0$ is established.

Part b) of Theorem 1.1 follows from part f) of Lemma 6.4. Hence, Theorem 1.1 is proven. □

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