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POSITIVE SOLUTIONS FOR CLASSES OF MULTIPARAMETER ELLIPTIC SEMIPOSITONE PROBLEMS

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Abstract. We study positive solutions to multiparameter boundary-value problems of the form

$$-\Delta u = \lambda g(u) + \mu f(u) \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $\lambda > 0$, $\mu > 0$, $\Omega \subseteq R^n$; $n \geq 2$ is a smooth bounded domain with $\partial\Omega$ in class C^2 and Δ is the Laplacian operator. In particular, we assume $g(0) > 0$ and superlinear while $f(0) < 0$, sublinear, and eventually strictly positive. For fixed μ, we establish existence and multiplicity for λ small, and nonexistence for λ large. Our proofs are based on variational methods, the Mountain Pass Lemma, and sub-super solutions.

1. Introduction

We study the multiparameter elliptic boundary-value problem

$$\begin{gathered} -\Delta u = \lambda g(u) + \mu f(u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\lambda > 0$, $\mu > 0$, $\Omega \subseteq R^n$; $n \geq 2$ is a smooth bounded domain with $\partial\Omega$ in class C^2 and Δ is the Laplacian operator. We assume $g : [0,\infty) \to \mathbb{R}$ is differentiable, $g(0) > 0$, non decreasing, and there exist $A, B \in (0,\infty)$ and $q \in (1, \frac{n+2}{n-2})$ such that for $x > 0$ and large

$$Ax^q \leq g(x) \leq Bx^q. \tag{1.2}$$

Also, we assume there exists $\theta > 2$ such that for $x > 0$ and large

$$xg(x) \geq \theta G(x) \tag{1.3}$$

where $G(x) = \int_0^x g(t)dt$.

Further, we assume $f : [0,\infty) \to \mathbb{R}$ is differentiable, $f(0) < 0$, non decreasing, eventually strictly positive, and there exists $\alpha \in (0,1)$ such that

$$\lim_{u\to\infty} \frac{f(u)}{u^\alpha} = 0. \tag{1.4}$$

We establish the following results:

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Theorem 1.1. *Let $\mu > 0$ be fixed. There exists $\lambda^* > 0$ such that if $\lambda \in (0, \lambda^*)$,* (1.1) *has a positive solution u_λ satisfying $\|u_\lambda\|_\infty \geq c^* \lambda^{-\frac{1}{q-1}}$, where $c^* > 0$ is independent of λ.*

Theorem 1.2. *There exists $\mu_0 > 0$ such that for $\mu \geq \mu_0$,* (1.1) *has at least two positive solutions for λ small.*

Theorem 1.3. *Let $\mu > 0$ be fixed. Then* (1.1) *has no positive solution for λ large.*

We note that for fixed $\mu > 0$, when λ is small $\lambda g(0) + \mu f(0) < 0$, and hence (1.1) is a semipositone problem. It has been well documented in recent years (see [8, 12, 13]), that the study of positive solutions for semipositone problems is mathematically very challenging. We establish Theorem 1.1 using the Mountain Pass Lemma. In Theorem 1.2, the second positive solution is established via sub-super solutions. The nonexistence result in Theorem 1.3 is proved by using the fact that $\lambda g(u) + \mu f(u)$ is bounded below by a piecewise linear function. We will prove Theorem 1.1 in Section 2, Theorem 1.2 in Section 3, and Theorem 1.3 in Section 3. Our results apply, for example, to the case when $f(u) = (u+1)^{\frac{1}{3}} - 2$ and $g(u) = u^3 + 1$.

We refer the reader to [10] where the case $n = 1$ was studied in detail. In particular, using a modified quadrature method, analysis of positive solution curves and their evolution as λ, μ vary was established. See [25] for related results for single parameter semipositone problems.

2. Proof of Theorem 1.1

We extend g and f as $g(x) = g(0)$ and $f(x) = f(0)$ for all $x < 0$. Throughout this paper we will denote by W the Sobolev space $W_0^{1,2}(\Omega)$ and by L^r the space $L^r(\Omega)$, for $r \in [1, \infty)$. Let $J : W \to \mathbb{R}$ be defined by

$$J(u) := \int_\Omega \frac{|\nabla u|^2}{2} dx - \int_\Omega H_\lambda(u) dx, \tag{2.1}$$

where $H_\lambda(u) = \lambda G(u) + \mu F(u)$ with $G(t) = \int_0^t g(s) ds$ and $F(t) = \int_0^t f(s) ds$. For future reference we note that there exist real numbers $\tilde{A}, \tilde{B}, \tilde{C}$ such that

$$\begin{gathered} G(x) \leq B \frac{|x|^{q+1}}{q+1} + \tilde{B} \quad \text{for all } x \in \mathbb{R}, \\ G(x) \geq A \frac{x^{q+1}}{q+1} + \tilde{A} \quad \text{for all } x \in [0, \infty), \\ F(x) \leq |x|^{\alpha+1} + \tilde{C} \quad \text{for all } x \in \mathbb{R}. \end{gathered} \tag{2.2}$$

In addition, defining $h_\lambda(x) = \lambda g(x) + \mu f(x)$ it follows from (1.2) that for any $\theta_1 \in (2, \theta)$, there exists θ_2 such that

$$x h_\lambda(x) \geq \theta_1 (\lambda G(x) + \mu F(x) - \theta_2) \quad \text{for all } x \in \mathbb{R}. \tag{2.3}$$

Also from (1.2) and (1.4) we see that there exists θ_3 such that

$$\begin{gathered} |g(x)| \leq \theta_3 (|x|^q + 1) \quad \text{for all } x \in \mathbb{R}. \\ |f(x)| \leq \theta_3 (|x| + 1) \quad \text{for all } x \in \mathbb{R}. \end{gathered} \tag{2.4}$$

It is well known that J is class C^1 and that u is a critical point of J if and only if u is a solution of (1.1). We prove J has a critical point using the Mountain Pass

Lemma (see Ambrosetti and Rabinowitz in [5]). We now recall the Mountain Pass Lemma.

Lemma 2.1 (Mountain Pass Lemma). *Let E be a real Banach space and $J \in C^1(E,\mathbb{R})$ satisfy the Palais-Smale condition. Suppose $J(0)=0$ and*

(I) *there are constants $\rho, \alpha > 0$ such that $J/_{\partial B_\rho} \geq \alpha$ and*
(II) *there is an $e \in E\backslash\overline{B_\rho}$ such that $J(e) \leq 0$.*

Then J possesses a critical value $c_0 \geq \alpha$. Moreover, c_0 can be characterized as

$$c_0 = \inf_{\sigma\in\Gamma} \max_{t\in\sigma[(0,1)]} J(t),$$

where $\Gamma = \{\sigma \in C([0,1],E) : \sigma(0)=0, \sigma(1)=e\}$ and B_ρ is a ball in E with center 0 and radius ρ.

We recall that $J : W \to \mathbb{R}$ is said to satisfy the Palais-Smale condition if every sequence (v_n), such that $(J(v_n))$ is bounded and $\nabla J(v_n) \to 0$, has a convergent subsequence.

Due to (2.3) a standard argument (see [5]) shows that for each $\lambda > 0$, the functional J satisfies the Palais-Smale condition.

In Lemma 2.2 we show that J satisfies the first and second conditions of the Mountain Pass Lemma and obtain a critical estimate on J. In Lemma 2.3 we obtain a crucial regularity estimate which we will use to prove that the solution obtained from the Mountain Pass Lemma is positive.

In the next lemma we prove that J satisfies the remaining conditions of the Mountain Pass Lemma and obtain an estimate on the critical level.

Lemma 2.2. *There exists $\overline{\lambda} > 0$ and $C > 0$ such that if $\lambda \in (0,\overline{\lambda})$ then J has a critical point u_λ of mountain pass type satisfying*

$$J(u_\lambda) \geq \frac{C^2}{8}\lambda^{-\frac{2}{q-1}}.$$

Proof. By the Sobolev imbedding theorem there exist positive constants K_1, K_2 such that

$$\|u\|_{L^{q+1}(\Omega)} \leq K_1\|u\|_{W_0^{1,2}(\Omega)}, \quad \text{and} \quad \|u\|_{L^{\alpha+1}(\Omega)} \leq K_2\|u\|_{W_0^{1,2}(\Omega)}, \tag{2.5}$$

for all $u \in W_0^{1,2}(\Omega)$. Let $C = ((q+1)/(4BK_1^{q+1}))^{1/(q+1)}$ and $r = C\lambda^{-\frac{1}{q-1}}$. Let $\|u\|_{W_0^{1,2}} = r$. This and (2.2) yield

$$\begin{aligned}
J(u) &= \frac{1}{2}\,r^2 - \int_\Omega H_\lambda(u)dx \\
&\geq \frac{1}{2}\,r^2 - \frac{\lambda B}{q+1}\int_\Omega |u|^{q+1}dx - \lambda\tilde{B}|\Omega| - \mu\int_\Omega |u|^{\alpha+1}dx - \mu\tilde{C}|\Omega| \\
&\geq \frac{1}{2}\,r^2 - \frac{\lambda BK_1^{q+1}}{q+1}r^{q+1} - \lambda\tilde{B}|\Omega| - \mu K_2^{\alpha+1}r^{\alpha+1} - \mu\tilde{C}|\Omega| \\
&= \lambda^{-2/(q-1)}\Big(\frac{C^2}{4} - \lambda^{(q+1)/(q-1)}\tilde{B}|\Omega| - \mu K_2^{\alpha+1}C^{\alpha+1}\lambda^{(1-\alpha)/(q-1)} \\
&\quad - \mu\tilde{C}|\Omega|\lambda^{2/(q-1)}\Big) \\
&\geq \lambda^{-2/(q-1)}\frac{C^2}{8}
\end{aligned} \tag{2.6}$$

for λ sufficiently small.

Let v_1 denote an eigenfunction corresponding to the principal eigenvalue λ_1 of $-\Delta$ with Dirichlet boundary conditions with $v_1 > 0$ and $\|v_1\|_{W_0^{1,2}} = 1$. Let

$$F(\beta) = \min\{F(s); s \in [0,\infty)\}. \tag{2.7}$$

For $s \geq 0$

$$\begin{aligned} J(sv_1) &= \frac{s^2}{2}\|v_1\|^2_{W_0^{1,2}(\Omega)} - \lambda\int_\Omega G(sv_1)dx - \mu\int_\Omega F(sv_1)dx \\ &\leq \frac{s^2}{2} - \lambda\Big(As^{q+1}\int_\Omega \frac{v_1^{q+1}}{q+1}dx + \tilde{A}|\Omega|\Big) - \mu F(\beta)|\Omega| \\ &\to -\infty \text{ as } s \to \infty, \end{aligned} \tag{2.8}$$

since $q > 1$. This implies there is a $s_1 > r$ such that $J(s_1v_1) \leq 0$. By choosing $v = s_1v_1$ we have satisfied the second condition of the Mountain Pass Lemma and Lemma 2.2 is proven. □

Lemma 2.3. *There exist $c_1 > 0$ and $\hat{\lambda} \in (0, \bar{\lambda})$, such that $\|u_\lambda\|_\infty \leq c_1\lambda^{\frac{-1}{q-1}}$ for all $\lambda \in (0, \hat{\lambda})$.*

Proof. Throughout this proof c denotes several positive constants independent of the parameter λ. From (2.2) we have

$$\begin{aligned} J(sv_1) &= \frac{1}{2}\ s^2 - \int_\Omega H_\lambda(sv_1)dx \\ &\leq \frac{1}{2}\ s^2 - \frac{\lambda As^{q+1}}{q+1}\int_\Omega |v_1|^{q+1}dx - \lambda\tilde{A}|\Omega| - \mu F(\beta)|\Omega| \\ &\leq \frac{1}{2}\ s^2 - \frac{\lambda AK_2}{q+1}s^{q+1} - (\mu F(\beta) + \lambda\tilde{A})|\Omega| \text{ where } K_2 = \textstyle\int_\Omega |v_1|^{q+1}dx \\ &\equiv p(s) - (\mu F(\beta) + \lambda\tilde{A})|\Omega|. \end{aligned} \tag{2.9}$$

Since

$$p(s) \leq \Big(\frac{1}{2} - \frac{1}{q+1}\Big)(AK_2)^{-2/(q-1)}\lambda^{-2/(q-1)} \tag{2.10}$$

for $s \in [0,\infty)$, there exists a positive constant c such that for $\lambda > 0$ sufficiently small

$$J(sv_1) \leq c\lambda^{-2/(q-1)} \quad \text{for all } s \in [0,\infty). \tag{2.11}$$

Since $J(u_\lambda) \leq \max\{J(sv_1); s \in [0, s_1]\}$ we have

$$J(u_\lambda) \leq c\lambda^{-2/(q-1)}, \tag{2.12}$$

for $\lambda > 0$ sufficiently small.

From (2.3), for λ small we have

$$\begin{aligned} \|u\|^2_{W_0^{1,2}(\Omega)} &\leq 2c\lambda^{-2/(q-1)} + 2\int_\Omega H_\lambda(u_\lambda)dx \\ &\leq 2c\lambda^{-2/(q-1)} + \frac{2}{\theta_1}\int_\Omega u_\lambda h_\lambda(u_\lambda)dx + 2\theta_2|\Omega| \\ &= 2c\lambda^{-2/(q-1)} + \frac{2}{\theta_1}\|u\|^2_{W_0^{1,2}(\Omega)} + 2\theta_2|\Omega|. \end{aligned} \tag{2.13}$$

Since $\theta_1 > 2$, from (2.13) we see that there exists $c > 0$ such that for λ small

$$\|u_\lambda\|_{W_0^{1,2}(\Omega)} \le c\lambda^{-1/(q-1)}. \tag{2.14}$$

This, (2.3), and the fact that u_λ is a critical point of J also give

$$\int_\Omega u_\lambda h_\lambda(u_\lambda)dx \le c\lambda^{-2/(q-1)} \quad \text{and} \quad \int_\Omega H_\lambda(u_\lambda)dx \le c\lambda^{-2/(q-1)}. \tag{2.15}$$

From (2.14) and the Sobolev imbedding theorem, for $\lambda > 0$ small, $\|u_\lambda\|_{L^{2n/(n-2)}} \le Kc\lambda^{-1/(q-1)}$ where $K > 0$ is the positive constant given in this imbedding. Hence using (2.4) and letting $a_1 = |\Omega|^{\frac{(q-1)(n-2)}{2n}}$, $a_2 = |\Omega|^{\frac{q(n-2)}{(2n)}}$ we have

$$\begin{aligned} \|h_\lambda(u_\lambda)\|_{L^{2^*/q}} &\le \theta_3\Big(\int_\Omega (\lambda|u_\lambda|^q + \mu|u_\lambda| + (\lambda+\mu))^{\frac{2n}{(q(n-2))}}dx\Big)^{\frac{q(n-2)}{(2n)}} \\ &\le \theta_3\left(\lambda\|u_\lambda\|^q_{L^{2^*}} + \mu a_1\|u_\lambda\|_{L^{2^*}} + (\lambda+\mu)a_2\right) \\ &\le \theta_3\left(\lambda K^q\|u_\lambda\|^q_W + \mu a_1 K\|u_\lambda\|_W + (\lambda+\mu)a_2\right), \end{aligned} \tag{2.16}$$

Since the constants $\theta_3, K, \mu, a_1, a_2$ in (2.16) are independent of λ, from (2.14) we see that there exists a positive constant c such that for λ small enough

$$\|h_\lambda(u_\lambda)\|_{L^{2^*/q}} \le c\lambda^{-1/(q-1)}. \tag{2.17}$$

By a priori estimates for elliptic boundary-value problems (see [1]) $\|u_\lambda\|_2 \le c\lambda^{-1/(q-1)}$, where $\| \ \|_2$ denotes the norm in the Sobolev space $W^{2,2}(\Omega)$ and c is a constant independent of λ. Since $W^{2,2}(\Omega)$ may be imbedded into $L^{2n/(n-4)}$ repeating the argument in (2.16) and (2.17) we see that

$$\|h_\lambda(u_\lambda)\|_{L^{2n/(q(n-4))}} \le c\lambda^{-1/(q-1)} \quad \text{and} \quad \|u_\lambda\|_{2,\frac{2n}{q(n-2)}} \le c\lambda^{-1/(q-1)}, \tag{2.18}$$

where $\|\cdot\|_{2,\frac{2n}{q(n-2)}}$ denotes the norm in the Sobolev space $W^{2,\frac{2n}{q(n-2)}}(\Omega)$. Iterating this argument we conclude that

$$\|u_\lambda\|_{2,r} \le c\lambda^{-1/(q-1)}, \tag{2.19}$$

with $r > n/2$. Since for such $r's$, $W^{2,r}$ is continuously imbedded in L^∞, we have $\|u_\lambda\| \le c\lambda^{-1/(q-1)}$, which proves the lemma. □

Proof of Theorem 1.1. From the definition of g we see that G is bounded from below. We let $\hat{G} = \inf\{G(s); s \in \mathbb{R}\}$. This, Lemma 2.2, and (2.7) give

$$\begin{aligned} \int_\Omega h_\lambda(u_\lambda)u_\lambda dx &= \|u_\lambda\|^2_W \\ &\ge 2J(u_\lambda) + 2(\hat{G} + F(\beta))|\Omega| \\ &\ge \frac{C^2}{4}\lambda^{-2/(q-1)} + 2(\hat{G} + F(\beta))|\Omega| \\ &\ge \frac{C^2}{8}\lambda^{-2/(q-1)}, \end{aligned} \tag{2.20}$$

for $\lambda > 0$ small. Let $\gamma > 0$ be such that $|\Omega|\theta_3\gamma[(\gamma^q + \gamma\mu) = C^2/(32|\Omega|)$ with C as in (2.20), and $\Omega_\lambda = \{x; u_\lambda(x) \ge \gamma\lambda^{-1/(q-1)}\}$. From Lemma 2.3, (2.20), and (2.4) we have

$$\begin{aligned} \frac{C^2}{8}\lambda^{-2/(q-1)} &\le \int_\Omega h_\lambda(u_\lambda)u_\lambda dx \\ &= \int_{\Omega_\lambda} h_\lambda(u_\lambda)u_\lambda dx + \int_{\Omega-\Omega_\lambda} h_\lambda(u_\lambda)u_\lambda dx \\ &\le |\Omega_\lambda|\theta_3 c_1\lambda^{-1/(q-1)}[(c_1^q+c_1\mu)\lambda^{-1/(q-1)}+\lambda+\mu] \\ &\quad + |\Omega|\theta_3\gamma\lambda^{-1/(q-1)}[(\gamma^q+\gamma\mu)\lambda^{-1/(q-1)}+\lambda+\mu] \\ &\le 2\theta_3\lambda^{-2/(q-1)}(|\Omega_\lambda|c_1(c_1^q+c_1\mu)+|\Omega|\gamma(\gamma^q+\gamma\mu)), \end{aligned} \tag{2.21}$$

for $\lambda > 0$ small. Now by the definition of γ we conclude

$$|\Omega_\lambda| \ge \frac{C^2}{32\theta_3 c_1(c_1^q+c_1\mu)} \equiv k_1. \tag{2.22}$$

Let $z : \bar{\Omega} \to \mathbb{R}$ be the solution to

$$\begin{gathered} -\Delta z = 1 \quad \text{in } \Omega \\ z = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.23}$$

Since Ω is assumed to be of class C^2, from regularity theory for elliptic boundary-value problems it is well know (see [18]) that there exist a positive constants σ_1, σ_2 such that

$$\sigma_1 d(x,\partial\Omega) \le z(x) \le \sigma_2 d(x,\partial\Omega), \tag{2.24}$$

where $d(x,\partial\Omega)$ denotes the distance from x to the boundary of Ω.

Let $\eta(x)$ denote the inward unit normal to Ω at $x \in \partial\Omega$. Since Ω is a smooth region, there exist an $\varepsilon > 0$ such that

$$N_\varepsilon(\partial\Omega) = \{x + \beta\eta(x) : \beta \in [0,\varepsilon), x \in \partial\Omega\}$$

is an open neighborhood of $\partial\Omega$ relative to $\overline{\Omega}$. Also (see [19]), this ε can be chosen small enough so that if $y = x + \beta\eta(x)$ then $d(y,\partial\Omega) = |\beta|$. Since $|N_\varepsilon(\partial\Omega)| = O(\varepsilon) \to 0$ as $\varepsilon \to 0$, we can without loss of generality assume that

$$|N_\varepsilon(\partial\Omega)| \le \frac{k_1}{2}.$$

Letting $K_\lambda = \Omega_\lambda - N_\varepsilon(\partial\Omega)$, we have that

$$|K_\lambda| \ge \frac{k_1}{2}.$$

Let G denote the Green's function of the Laplacian operator, $-\Delta$, in Ω, with Dirichlet boundary condition. For $x \in K_\lambda$ and $\xi \in \partial\Omega$ we have, by Hopf's maximum principle,

$$\frac{\partial G}{\partial\eta}(x,\xi) > 0.$$

Since $K_\lambda \times \partial\Omega$ is compact there exists $\varepsilon_1 \in (0,\varepsilon)$ and $b > 0$ such that if $x \in K_\lambda$ and $\xi \in N_{\varepsilon_1}(\partial\Omega)$ then

$$\frac{\partial G}{\partial\eta}(x,\xi) \ge b.$$

In particular, for $x \in K_\lambda$ and $d(\xi,\partial\Omega) < \varepsilon_1$ we have $G(x,\xi) \ge bd(\xi,\partial\Omega)$. For ξ such that $d(\xi,\partial\Omega) < \varepsilon_1$ we have

$$u_\lambda(\xi) = \int_\Omega G(x,\xi)h_\lambda(u_\lambda)dx = \int_\Omega G(x,\xi)\lambda g(u_\lambda)dx + \int_\Omega G(x,\xi)\mu f(u_\lambda)dx.$$

Since $g(u_\lambda) > 0$ for all u_λ

$$\begin{aligned} u_\lambda(\xi) &\geq \int_{K_\lambda} G(x,\xi)\lambda g(u_\lambda)dx + \int_\Omega G(x,\xi)\mu f(u_\lambda)dx \\ &\geq \int_{K_\lambda} G(x,\xi)\lambda g(u_\lambda)dx + \mu f(0)z(\xi). \end{aligned}$$

Therefore, for λ small enough by (1.2) and (2.24),

$$\begin{aligned} u_\lambda(\xi) &\geq \int_{K_\lambda} bd(\xi,\partial\Omega)\lambda\ Au_\lambda^q dx + \mu f(0)z(\xi) \\ &\geq bd(\xi,\partial\Omega)A\gamma^q\lambda^{\frac{-1}{q-1}}|K_\lambda| + \mu f(0)\sigma_2 d(\xi,\partial\Omega) \\ &\geq \tilde{c}d(\xi,\partial\Omega)\lambda^{\frac{-1}{q-1}}, \end{aligned} \tag{2.25}$$

where $\tilde{c} > 0$ is independent of λ.

We define $w_\lambda(x)$ and $z_\lambda(x)$ such that

$$\begin{gathered} -\Delta w_\lambda = \lambda g(u_\lambda) + \mu f^+(u_\lambda) \quad \text{in } \Omega \\ w_\lambda = 0 \quad \text{on } \partial\Omega \end{gathered}$$

and

$$\begin{gathered} -\Delta z_\lambda = \mu f^-(u_\lambda) \quad \text{in } \Omega \\ z_\lambda = 0 \quad \text{in } \partial\Omega \end{gathered}$$

where

$$f^+(x) = \begin{cases} f(x) & x \geq \beta \\ 0 & x < \beta \end{cases} \quad \text{and} \quad f^-(x) = \begin{cases} f(x) & x \leq \beta \\ 0 & x > \beta\,. \end{cases}$$

It is clear that $u_\lambda = w_\lambda + z_\lambda$. Also, note that

$$z_\lambda(x) = \int_\Omega G(x,y)\mu f^-(u_\lambda(y))dy$$

so clearly $z_\lambda \leq 0$ and since $f^-(u_\lambda(y)) \geq f(0)$ we have

$$z_\lambda(x) \geq \int_\Omega G(x,y)\mu f(0)dy = \mu f(0)\int_\Omega G(x,y)dy.$$

So we have $-M_1 \leq z(x) \leq 0$ where $M_1 = -\mu f(0)\max_{x\in\overline{\Omega}}\int_\Omega G(x,y)dy > 0$. For x such that $d(x,\partial\Omega) = \varepsilon_1$ we have

$$w_\lambda(\xi) = u_\lambda(\xi) - z_\lambda(\xi) \geq u_\lambda(\xi) \geq \epsilon_1\tilde{c}\lambda^{\frac{-1}{q-1}},$$

and by the maximum principle we have $w_\lambda(x) \geq \epsilon_1\tilde{c}\lambda^{\frac{-1}{q-1}}$ for all $x \in \Omega - N_{\varepsilon_1}(\partial\Omega)$. This implies that $u_\lambda(x) = w_\lambda(x)+z_\lambda(x) \geq \epsilon_1\tilde{c}\lambda^{\frac{-1}{q-1}} - M_1$ and so $u_\lambda(x) \geq (\epsilon_1\tilde{c}/2)\lambda^{\frac{-1}{q-1}}$ for all $x \in \Omega\backslash N_{\varepsilon_1}(\partial\Omega)$ for small λ. This and (2.25) imply that for λ small enough $u_\lambda(x) > 0$ on Ω, which proves Theorem 1.1. □

3. Proof of Theorem 1.2

In this section we prove a multiplicity result for $\mu > \mu_0$ and λ small using a sub and super solution method. According to [11] there exists a $\mu_0 > 0$ such that for $\mu \geq \mu_0$ there exists a w such that

$$\begin{gathered} -\Delta w = \mu f(w) \quad \text{in } \Omega \\ w = 0 \quad \text{on } \partial\Omega \end{gathered}$$

where $w > 0$ on Ω. Since $\lambda > 0$ and $g > 0$ it follows that

$$\begin{gathered} -\Delta w \leq \lambda g(w) + \mu f(w) \quad \text{in } \Omega \\ w \leq 0 \quad \text{on } \partial\Omega, \end{gathered}$$

which implies that w is a sub solution of (1.1).

Let z be as in (2.23). Define $\phi = \sigma z$ where $\sigma > 0$, independent of λ, is large enough so $\phi > w$ in Ω and

$$\mu \frac{f(\sigma z)}{\sigma} < \frac{1}{2}.$$

This is possible since f is a sublinear function (see (1.4)). Next let $\lambda > 0$ be so small that

$$\lambda \frac{g(\sigma z)}{\sigma} < \frac{1}{2}.$$

Thus

$$-\Delta\phi = \sigma \geq \lambda g(\sigma z) + \mu f(\sigma z) = \lambda g(\phi) + \mu f(\phi) \quad \text{in } \Omega.$$

Hence ϕ is a supersolution of (1.1) and there exists a solution $\tilde{u}_\lambda$ (say) of (1.1) such that $w \leq \tilde{u}_\lambda \leq \phi$ for $\mu \geq \mu_0$ and $\lambda > 0$ small. However, from Theorem 1.1, for λ small, we have the existence of a positive solution, u_λ, such that $\|u_\lambda\|_\infty \geq c_0 \lambda^{-\frac{1}{q-1}}$. Hence λ. small $\widetilde{u}_\lambda$ and u_λ are two distinct positive solutions of (1.1).

4. Proof of Theorem 1.3

Let u be a positive solution to (1.1). There exist $\sigma > 0$ and $\varepsilon > 0$ such that $g(u) \geq (\sigma u + \varepsilon)$ for all $u \geq 0$. So for $\lambda > 0$, it follows that

$$\lambda g(u) + \mu f(u) \geq \begin{cases} \lambda(\sigma u + \varepsilon) & \text{for } u \geq \beta \\ \lambda(\sigma u + \varepsilon) + \mu f(0) & \text{for } u \leq \beta\,. \end{cases}$$

Choosing λ large enough so that $\lambda\varepsilon + \mu f(0) \geq \frac{\lambda\varepsilon}{2}$, we have

$$\lambda g(u) + \mu f(u) \geq \lambda\sigma u + \frac{\lambda\varepsilon}{2}$$

for $u \geq 0$ and λ large. Now let λ_1 be the first eigenvalue and $\phi > 0$ be a corresponding eigenfunction of $-\Delta$ with Dirichlet boundary condition. Multiplying both sides of (1.1) by ϕ and integrating we get

$$\int_\Omega (-\Delta u)\phi dx = \int_\Omega (\lambda g(u) + \mu f(u))\phi dx$$

which implies

$$\int_\Omega u\lambda_1\phi dx = \int_\Omega (\lambda g(u) + \mu f(u))\phi dx,$$
$$\int_\Omega u\lambda_1\phi dx \geq \int_\Omega (\lambda\sigma u + \frac{\lambda\varepsilon}{2})\phi dx,$$
$$\int_\Omega [\lambda_1 - \lambda\sigma]u\phi dx \geq \int_\Omega \frac{\lambda\varepsilon}{2}\phi dx.$$

For $\lambda > \frac{\lambda_1}{\sigma}$ we obtain a contradiction. So for a given $\mu > 0$, (1.1) has no positive solution for large λ.

Appendix A. (see also [9] and [25]) Let $1 < q < \frac{n+2}{n-2}$ and $\alpha_0 = 2n/(n-2)$. If $\{\alpha_j\}$ is the sequence defined by

$$\alpha_j = \frac{\alpha_{j-1}n}{qn - 2\alpha_{j-1}}$$

then there exists an integer $k \geq 0$ such that $qn - 2\alpha_k \leq 0$.

Proof. Assume $2\alpha_j < qn$ for $j = 0, 1, 2, \dots, p$, for all $p \geq 0$. Then

$$\begin{aligned}\alpha_j - \alpha_{j-1} &= \frac{\alpha_{j-1}n}{qn - 2\alpha_{j-1}} - \alpha_{j-1}\\ &= \frac{\alpha_{j-1}n - \alpha_{j-1}qn + 2(\alpha_{j-1})^2}{qn - 2\alpha_{j-1}}\\ &= \alpha_{j-1}[\frac{n - qn + 2\alpha_{j-1}}{qn - 2\alpha_{j-1}}]\end{aligned}$$

for $j = 0, 1, 2, \dots, p$, for all $p \geq 0$. Hence

$$\alpha_1 - \alpha_0 = \alpha_0[\frac{n}{qn - 2\alpha_0} - 1] = A(q, n) > 0$$

since $1 < q < \frac{n+2}{n-2}$, and $\alpha_1 > \alpha_0$. Similarly,

$$\alpha_2 - \alpha_1 = \alpha_1[\frac{n}{qn - 2\alpha_1} - 1] > \alpha_0[\frac{n}{qn - 2\alpha_0} - 1],$$

so $\alpha_2 > \alpha_1$ and $\alpha_2 \geq \alpha_0 + 2A(q, n)$. Repeating this argument p times we have $\alpha_p \geq \alpha_0 + pA(q, n)$ and (α_j) to be increasing in constant increments, which contradicts $2\alpha_p < qn$ for all $p \geq 0$. □

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