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A Task Analysis of Algebra Word Problems

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ABSTRACT

Algebra word problems have been a source of consternation to generations of students. Recent cognitive analyses of problem solving have provided a new perspective on the mental processes involved in solving problems. In the study reported in this chapter, a set of algebra word problems was analyzed in terms of the information integration tasks that are required to solve the problems. A comparison of beginning and advanced high school students on their strategies and skills with algebra word problems demonstrated the advantages experts have in recognizing and using the overall structure of problems to guide them to a solution. Instruction that focuses on the structure of the problems was successful in improving performance of a group of novices, although generalization to a related problem was limited. The current study and similar analyses of the cognitive operations involved in problem solving point the way to instructional strategies that can be applied by teachers and designers of instructional materials.

INTRODUCTION

If you have not repressed the memory, you may recall the sense of frustration and feeling of incompetence that probably accompanied your first encounter with algebra word problems. Although some students quickly overcome these feelings as they gain a degree of mastery over word problems, many other students continue to struggle with little success. Even students who are expert at solving algebraic equations are often baffled when the same problems are cloaked in a verbal cover story (Mayer, Larkin, & Kadane, 1984).

In recent years there has been a tremendous surge of interest in problem solving and instruction, providing a growing area of productive interaction between cognitive psychologists and educators (cf. Frederiksen, 1984; Lester, 1982; Lochhead & Clement, 1979; Segal, Chipman, & Glaser, 1985; Silver, 1985; Tuma & Reif, 1980). In this chapter we describe (a) a cognitive analysis of the tasks involved in solving algebra word problems, (b) a comparison of "novice" and "expert" high school problem solvers in terms of their strategies and their competence with these tasks, and (c) the impact of a brief training session focused on these tasks and strategies.

Stages in Problem Solving

Sixty years ago Wallas (1926) described four stages of problem solving: preparation, incubation, illumination, and verification. Over the years there have been many analyses that divide the problem-solving process into stages (e.g., Duncker, 1945; Greeno, 1973; Polya, 1957, 1968). A limitation of many such analyses is that they do not necessarily provide much insight into the mental processes that are involved in crucial stages, such as "illumination." One contribution of cognitive psychologists has been to generate a better understanding of the basic mental mechanisms that are used in the course of problem solving.

In a recent analysis of problem solving, Mayer (1982b) has identified two main stages: (a) forming a representation or understanding of a problem, and (b) searching the problem space in memory for a solution. Within the first stage, problem solvers need to access linguistic, factual, and schematic knowledge so that the problem is correctly encoded along with relationships of the problem to other problems. Within the second stage, problem solvers draw on algorithmic knowledge of how to perform well-defined procedures and strategic knowledge of useful approaches to problems (p. 3).

Gaining Expertise with Algebra Word Problems

Typical classroom instruction in algebra word problems focuses on algorithmic skills, whereas strategies and schemata that can be used to organize principles are not taught directly. Greeno (1980) has suggested that current methods for teaching problem solving are akin to teaching swimming by tossing the student into the water, a method that may be "successful for some students, but it has obvious negative consequences for others" (p. 6). Unfortunately, even if algebra teachers could be persuaded to devote significant instructional effort to teaching organizing principles, we do not have much information on how students learn to use schemata and strategies nor how organizing principles can best be taught.

One useful approach has been to compare the problem-solving performance of *experts* to that of *novices* (e.g., Chi, Feltovich, & Glaser, 1981; Kintsch &

Greeno, 1985; Larkin, McDermott, Simon, & Simon, 1980; Simon & Simon, 1978). A general finding is that persons skilled in a problem domain have a sizable body of domain-specific knowledge, including patterns or schema for types of problems. Recognition of a familiar pattern provides quick access to relevant procedures. Larkin et al. (1980) found that persons expert with physics problems were able to work problems from the *bottom up,* proceeding to combine basic information in a sequence of steps that led to the goal. The experts apparently had a mental model or schema of the structure of the entire problem, which allowed them to take a direct path toward the solution. On the other hand, novices were more likely to apply a *means-ends* strategy, working backwards from the goal by defining subgoals successively farther from the goal and closer to the basic information that was given in the problem. There is recent evidence that when students use means-ends analysis, this strategy may actually interfere with the acquisition of schema and awareness of the structure of the entire problem (Owen & Sweller, 1985) and hence impede the progress from novice to expert status.

Analysis of Algebra Word Problems

Our understanding of the domains within algebra word problems was greatly advanced by Mayer (1981) when he compiled and categorized 1,097 word problems from 10 major algebra texts used in California public schools. On the basis of underlying source formulas (e.g., rate × time = distance), he identified 25 families, such as time-rate and unit-cost problems. Families were divided into categories that share variables, formulas, and methods of derivations. For example, the time-rate family was divided into 13 categories, such as motion, current, and work problems. Each category was further divided into templates, defined by the propositional structure of the problems. Motion problems, for example, have at least 12 templates, including vehicles converging, one overtaking another, and one vehicle making a round trip. Problems within a template differ only in the values that are used, and in details of the wording of the problem.

Mayer (1981) found that the relevant information in nearly all problems could be described using four types of propositions:

1. assignments (e.g., A cup holds 8 ounces.)
2. relations (e.g., A bottle holds four times as much as a cup.)
3. relevant facts (e.g., John will sell cups of soda.)
4. questions (e.g., How much money will John earn?)

In a series of studies with college freshmen, Mayer (1982a) found that recall was poorer for relational propositions than for assignment propositions, and when students were asked to construct their own word problems they rarely made use of relational propositions. Reed, Dempster, and Ettinger (1985) found that college students also had special difficulty expressing relationships correctly in word problems.

These findings suggest that an analysis of the ways in which information is collected and combined in solving algebra word problems might provide a useful tool for characterizing problem structure and relationships between problems. We conducted such an analysis and used a coding system based on this analysis to examine the approach used by first-year algebra students (novices) compared to the approach taken by more experienced problem solvers (experts). We also examined the effects special training with problem tasks and structure had on task performance.

Information Integration Tasks in Algebra Word Problems

We analyzed the structure of 50 common word problems selected from Mayer's (1981) collection, and organized a list of the information integration tasks involved in solving these problems. The tasks were sorted into three levels of information integration: value assignment, value derivation, and equation construction. Each level contained three types of tasks. An example of each of these nine tasks is shown in Table 10.1.

The first level, Value Assignment, requires little more than direct translation of text into an equation. Value Assignments are established by equivalence assignment when a noun phrase from the problem is set equal to a numerical value, by unknown assignment when a noun phrase is set equal to a symbol representing an unknown value, or by relation assignment when a noun phrase is set equal to a simple relationship with another noun phrase, as Rate 2 = 2x (see Table 10.1).

The second level, Value Derivation, involves operating on assigned values to produce new values. Values may be derived through transformation by operating on a Level 1 value assignment, such as adding a constant to both sides of an equation. A second mode of value derivation is by construction, where Level 1 value assignments are combined, as by addition or subtraction, to produce a value for a noun phrase. Third, a value may be derived by using a source formula such as "rate × time = distance," where knowledge of any two values permits derivation of the third.

The third level of information integration is Equation Construction, which requires creation of a computational representation of the structural relationship between the variables in the problem. One way this may be accomplished is by applying a function rule provided in the statement of the problem (See Table 10.1). A second way is by applying a source formula not presented directly in the problem, such as "area = length × width." A third type of equation is formed by combining components, such as "area of garden = total area − area of walkway."

TABLE 10.1
Taxonomy of Information Integration Tasks

Sample Problems	Level of Information Integration: Level 1: Value Assignment	Level 2: Value Derivation	Level 3: Equation Construction
A man is now 40 years old and his son is 14. How many years will it be until the man is twice as old as his son?	Equivalence Assignment Man's age today = 40 Son's age today = 14	Transformation Man's age x years from now = 40 + x Son's age x years from now = 14 = x	Function Rule (40+x) = 2(14+x)
A framed mirror is 45 by 55 cm. 1911 square cm of the mirror shows. How wide is the frame?	Unknown Assignment Width of the frame = x	Construction L = 55 - 2x W = 45 - 2x	Source Formula A = (L)(W) A = (55-2x)(45-2x)
Two hikers start at the same time from towns 36 miles apart, and meet in 3 hours. One hiker walks twice as fast as the other. What is the rate of each hiker?	Relation Assignment [Rate 1 = x] Rate 2 = 2x	Source Formula (R)(T) = D (x)(3) = D_1 (2x)(3) = D_2	Combination D = D_1 + D_2 36 = 3x + 6x

There is a hierarchical relationship among the three levels of information integration. The value assignments from the first level are often operated upon in the second level to derive new values, which are then used in the third level for the construction of the equations. However, problem solving can begin at any of the three levels. Activities at each level place constraints on activities to be completed at each of the other levels. For example, if one can determine the form of the final equation, the range of possibly appropriate value assignments and derivations may be reduced.

THE CURRENT RESEARCH

The present research was designed to compare novice and expert problem solvers in terms of their facility with information integration tasks at each level, and in terms of their awareness and use of the structure of the problem. We expected the novice problem solvers to have relatively more difficulty with aspects of problem solving that require an appreciation for the structure of the problem. This would be reflected in greater difficulty with the information integration tasks at the second and third levels than for tasks at the first level, which require only assignment of values and unknowns. We also examined the effects of instruction on problem structure for novice problem solvers. We were hopeful that the training would improve the performance of novices on the higher levels of information integration.

Method

Three groups of high school students were recruited for this study. Volunteers from first-semester algebra classes were assigned to one of two groups of novice problem solvers: an instruction group and a control group. A third group of experienced problem solvers was composed of volunteers from analytic geometry classes. Each student was paid $2.00 for participating. A detailed description of the study is available as an unpublished dissertation (Wilde, 1984).

The initial task for all groups was a set of six word problems, followed by a vocabulary test. About a month later half of the novices were given special instruction on word problems (described in a later section) while the other half served as a control group. A posttest followed for both groups. A test of ability to identify the structure of word problems was given to the control group of novices at the end of the second session and to the experts at the end of their first and only session.

The six word problems in the initial test were two problems each from Mayer's (1981) motion, age, and rectangle families. All students had been exposed to these three categories of problems in their classrooms. Students were tested individually. The problems were presented in a booklet, with each problem written on the top of a separate page, leaving space for calculations below. Students were asked to write down each step, and to report their thoughts as they worked. The experimenter recorded all comments. Students who were unable to get started on a problem were prompted with the hint that they should first determine what the problem's unknown was. If this failed, they were told to go on to the next problem. No problem could be returned to once the page was turned.

Results and Discussion

On the six problems, the combined novice groups solved only 9% of the problems compared to 85% for the experts. The problem protocols were analyzed for each group to determine the proportion of tasks completed at each level of information integration. The results of this analysis are shown in Table 10.2.

Here, and in other analyses where the dependent variable was a proportion, we used an arc sine transformation to reduce potential effects of skew in the data prior to conducting an analysis of variance. The two main effects and the interaction were all highly significant in this table, all in the expected direction. Experts

TABLE 10.2
Mean Proportion of Information Integration Tasks Completed

		Level of Information Integration		
Group	n	1	2	3
Novice	35	.73	.27	.09
Expert	13	1.00	.92	.85

outperformed the novices, and performance for both groups was progressively poorer as the integration tasks required more structure-specific integrations. Value assignments (Level 1) were easiest whereas formula constructions (Level 3) were the most difficult. The interaction indicated that the effect of integration level was greater for novices than for experts. Novices were reasonably competent at setting up givens but very poor at applying procedures that depended on the problem structure. Details of the results, including statistical tests of significance, are available in Wilde (1984).

Problem Solving and Verbal Abilities. One might expect verbal comprehension to be a good predictor of algebra word problem solving success, since translation of problems depends on verbal comprehension. To evaluate this notion, we gave all students the first part of Vocabulary Test II from the ETS Kit of Factor Referenced Cognitive Tests (Ekstrom, French, Harman, & Derman, 1976). The correlation between the proportion of information integration tasks completed and verbal comprehension for the entire sample was a highly significant .75. This high correlation was the result of large differences between the groups on both measures. The average score on the 18-item vocabulary test was 14.3 for the experts, and only 8.2 for the novices. The correlation between problem solving and vocabulary for the experts alone was .17 and for the notives .01, both nonsignificant. A high level of verbal ability may be required to become established in the high-math-performance group, but verbal ability does not account for the variability of math performance found within a group. These data also indicate that the experts were a select group. It seems unlikely that all of the novices could be expected eventually to make the transition to expert status.

Comparison of Novices and Experts on Information Integration Tasks. We next examined performance of the experts and novices on specific tasks. Table 10.3 shows the mean proportion of success for each type of task at each of the three levels. Split-plot analyses of variance were computed on the proportion of tasks successfully completed at each level by the two groups. ANOVAs were followed by a posteriori Tukey's HSD tests for individual comparisons.

Following Mayer (1982a), we expected performance on Value Assignments (Level 1) to be poorer for relation assignments than for equivalence and unknown assignments. The ANOVA showed both of the main effects and the interaction to be highly significant. The experts performed better than the novices on each task. For the novices, performance on unknown assignments and relation assignments did not differ significantly, but both were easier than equivalence assignments. This was surprising since equivalence assignments seem so straightforward. A closer look at equivalence assignments uncovered an important distinction: Some assignments describe individual objects or actors (e.g., in the Hikers problem in Table 10.1, Rate 1 = x) and some assignments are used to connect parts of the problem (e.g., in the Hikers problem, the distance between two towns = 36 miles). The mean proportion of correct Equivalence Assignments was .77 for the individual objects, but only .45 for the connections.

At Level 2 were the Value Derivation tasks using transformations, construction, and source formulas. Transformations were completely specified by the problem in that the initial value, the transforming value, and the transforming operation were all stated explicitly. Constructions and source formulas, however, involve combining information based on ideas about the problem structure that were not stated explicitly in the problem. This led to the prediction that for novices transformations would be easier than constructions and source formulas.

The analyses showed that novices found the construction tasks significantly more difficult than either the source formula or the transformation tasks, which did not differ significantly from each other. No differences between tasks were reliable for the experts.

One might suspect that the poor performance of the novices in using source formulas might be because they do not know the formulas. However, when they were asked to recall the formulas at the end of the experiment, 83% of the novices correctly recalled the area formula and 71% recalled the rate formula. It is the application of the formulas that is not well understood.

Performance of the novices on value derivation by construction was abysmal. An inability to construct variables by combining components of the problem is consistent with the hypothesis that novices do not have a good understanding of the structure of the word problems.

The Level 3 integration tasks of Equation Construction involved the use of a function rule, source formula, or combination of variables to produce a summary equation reflecting the structure of the problem. We expected the performance of novices to be especially poor at Level 3 because these tasks are most dependent on a good understanding of the structure of the problem. (In the six problems used here, no source formulas were needed at Level 3.)

Expectations were confirmed in that novices had little success with the Level 3 tasks (see Table 10.3). Both groups were more likely to obtain the final equation when a function rule was required than when a combination of variables was needed.

It is important to note that although novices were usually successful at translating the problem text into equations, this skill did not guarantee that they would be able to solve the problem. The inability of novices to combine information at the higher levels resulted in very low solution rates.

TABLE 10.3
Mean Proportion of Information Integration Tasks Completed

Level of Information Integration	Novices (n=35)	Experts (n=13)
1. Value Assignment Tasks:		
Unknown Assignment	.81	1.00
Relation Assignment	.75	1.00
Equivalence Assignment	.64	.99
2. Value Derivation Tasks:		
Transformation	.45	.98
Source Formula	.20	.89
Construction	.03	.83
3. Formula Construction:		
Function	.19	.92
Combination	.04	.81

These results should be interpreted with some caution since there were only six problems in the test set. Generalization to a wider range of problems has not been established.

Perceptions of Problem Structure. To assess more directly the students' ability to detect and compare the structure of algebra word problems, we developed a simple test for this purpose. The test consisted of five triads of problems, where each triad was constructed of three problems from the same category, with two from one template and the third from a different template. Students were asked to determine, for each triad, which two problems were most alike. An example of a problem triad is shown in Fig. 10.1. The first two problems here are isomorphs that differ only in values of the variables. The third problem presents the second proposition in a form different from the first two problems.

STRUCTURE TASK

Which two of the following three problems are most alike?

(a) Problem 1 and Problem 2. ( )
(b) Problem 1 and Problem 3. ( )
(c) Problem 2 and Problem 3. ( )

1. Dana is five times as old as his dog, Texas. In 9 years Dana will be twice as old as Texas. What are their ages now?
2. Roger is four times as old as his sister. In 6 years he will be twice as old as she. How old are they now?
3. Pam is twice as old as her brother. In 5 years their ages will total 22 years. How old are they now?

FIG. 10.1. Sample Problem from the Structure Task.

Chance performance on the structure task was 33% correct. The novices performed right at chance, 33% correct, whereas the experts were correct on 88% of the triads. This is convincing evidence that the novices had little appreciation for the structure of the problems, in contrast to the experts who were able to identify the structure quite consistently.

Analysis of Problem Protocols. A third source of data was the problem protocols. All students were asked to write down each step of their solution attempt, and to "think out loud" as they proceeded. These protocols showed striking differences between the novices and the experts in terms of their use of the problem structure.

Strategies that led to solution are listed first in Table 10.4. Just over half of the problems solved by experts showed a "working-down" strategy that started with the Level 3 integration, demonstrating an understanding of the structure of the problem, which in turn provided the solver with a relatively clear indication of a path to the goal. The novices never started with the Level 3 integration. On the 18 problems solved by novices, 17 showed a working-up strategy where the protocols gave all Level 1 integrations first, then the Level 2 integrations, and

TABLE 10.4
Strategies Used by Novice and Expert Problem Solvers

Strategy Type	Number of Problems: Novices	Number of Problems: Experts
1. Successful Strategies	18 (9%)	66 (85%)
(a) work up	17	25
(b) work from the middle	0	4
(c) work down	0	31
(d) diagram a familiar procedure	0	5
(e) ideosyncratic arithmetic model	1	1
2. List Variables	108 (51%)	4 (5%)
3. Memory for a Similar Problem	20 (10%)	6 (8%)
(a) generate a formula table	20	1
(b) diagram a familiar procedure	0	5
4. Oversimplify Structure	36 (17%)	2 (3%)
(a) simplify formula	4	0
(b) simplify diagram	32	2
5. Structure Insensitive	28 (13%)	0 (0%)
(a) incorrect direct translation	10	0
(b) incorrect arithmetic relations	14	0
(c) no apparent strategy	4	0
Total number of problems	210	78

Note. This table is based on six problems given to 35 novices and 13 experts. The numbers indicate a count of instances where each strategy or type of strategy was used.

finally the Level 3 equation. These data are consistent with the conclusions of Larkin et al. (1980) that experts make greater use of knowledge of the problem structure in their approach to problems.

The most common strategy for the novices, shown on 51% of the problems, was a simple listing of some or all of the Level 1 value assignments, with little else.

Another common strategy was to draw on memory for similar problems. This strategy was reflected in errors where a familiar formula or procedure was applied inappropriately. Novices were likely to show formulas, whereas experts tended to show diagrams. The novices also were likely to oversimplify the structure of the problems, usually resulting in simple diagrams that did not reflect the structure of the problem. Finally, a small proportion of the protocols from novices showed inappropriate direct translation of variables (e.g., for the Hikers problem in Table 10.1, the student might simply write: Hiker = 3) or inappropriate combinations of the values presented in the problem, such as simply adding values together.

Consistent with other information, the protocol data show that the novices generally made little use of the structure of the problems in determining their approach to the problems. The experts, on the other hand, made extensive use of their knowledge of the problem structure to find their path to the solution.

These data suggest that if we wish to train novices to approach problems more like experts, it may help to teach students how to make value assignments and derivations, but it is likely to be more effective to concentrate on helping students learn how to generate a representation of the structure of the problems. Specifically, we would expect training on the three levels of information integration and diagram construction to facilitate problem solving. We designed a short training program to test this notion.

Effects of Instruction. The novice problem solvers were paired on the basis of their performance on the six word problems, and then were randomly split into two groups, instruction ($n = 16$) and control ($n = 17$). Four weeks after the initial testing, the instruction group received about 30 minutes of individual training.

Students were first given the Hikers problem (third example in Table 10.1) and were asked to list the variables, defined as things named in the problem that have a numerical value. Examples were given, and the students were helped to produce a list of the rates and times for the two hikers and the initial distance between them. Students were next asked to determine the values for each variable on the list. Particular note was made of the facts that x can represent an unknown value and that the value of one variable may be defined in terms of another. Next, the students were asked to find the equality and find the values that must be derived to complete the equality. Figure 10.2 was provided to aid students.

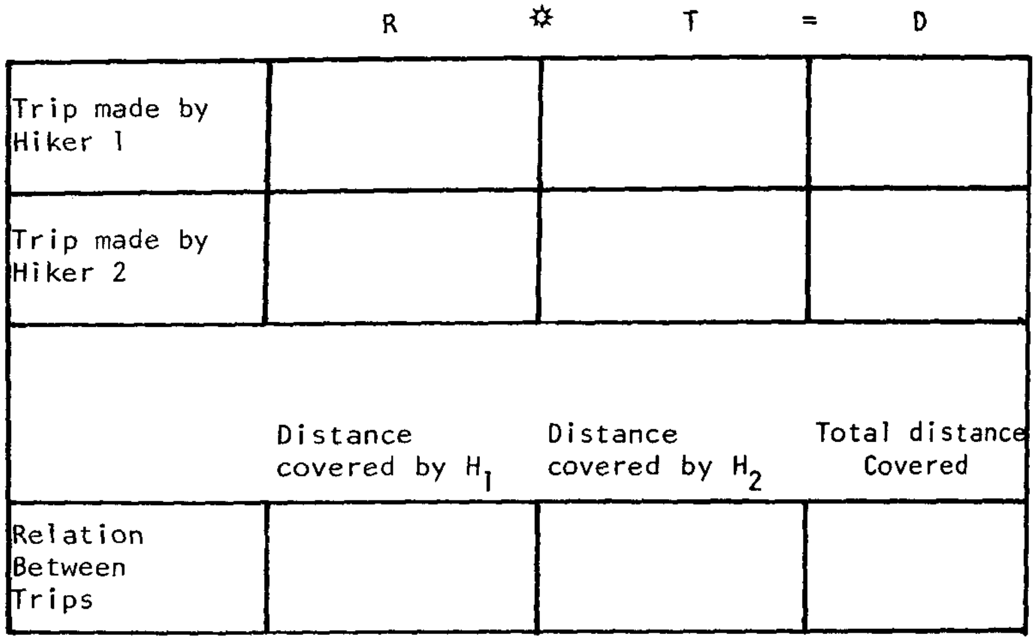

FIG. 10.2. Diagram used to demonstrate the structure of the rate problems used for instruction.

The same procedure was repeated for a second motion problem, two cyclists riding toward each other. The final step was to have the students compare the two problems using the figure. The similarity of variables, value derivation, and equation construction was pointed out.

The control group of novices were given the same two motion problems and asked to set them up, but no special instruction was provided. A posttest for both groups consisted of four problems, three of which were isomorphs of the training problems sharing category and template features. The fourth was a generalization problem which was a motion problem from a different category. The isomorphs involved combining two subdistances to equal a known total, whereas the generalization problem involved comparing two subdistances to find an unknown total.

We expected performance of the Instruction group to be improved for the problems isomorphic to the training problems. Expectations for the generalization problem were not as clear, since others have found little improvement on generalization problems following training (e.g., Gick & Holyoak, 1980; Reed et al., 1985).

The Instruction group outperformed the Control group on both the isomorphs and on the generalization problem. The Instruction group solved .87 of the isomorphic problems, and the Control group solved .47. For the generalization problem the corresponding proportions were .79 and .40, respectively.

An examination of performance at each of the three levels of information integration showed that the Instruction group did better than the Control group at all three levels on the isomorphic problems, and also at the first level (Value

Assignment) on the generalization problem. Differences between the groups on the second and third levels of information integration were in the direction of an advantage for the training group, although they did not attain statistical significance.

These results are encouraging in that they demonstrate an apparently effective approach to teaching students how to solve algebra word problems. Obviously our special instruction involved training on numerous skills, including the assignment of variables, the use of diagrams, and comparisons of the structure of similar problems. Although it may be interesting to attempt to separate the contribution of factors such as these, we believe it will be more productive to study how these components interact. For example, value assignment skills are crucial, and students must be able to translate equivalencies expressed in words into equations; yet training on value assignment skills alone will produce no more than marginal effects for many students, since the relationships among the variables must also be understood if the problem is to be solved. On the other hand, it may be futile to teach students about the structure of problems if they have not mastered the skill of translating basic equivalencies into algebraic expressions.

CONCLUSIONS

Overall, the clearest lesson to be drawn from our study is that an appreciation of problem structure is a crucial part of expertise in problem solving. Experts are quickly able to identify the form of the equation to be solved and they use this information to guide them on the path to solution. Novices are much more likely to stop after they have generated a list of value assignments, unable to see relationships inherent in the structure of the problem. A key advantage that experts hold is that they are familiar with a large number of problem forms. If the structure of a problem is recognized, the problem becomes a mere exercise in applying familiar algorithms. A likely reason that algebra word problems are so difficult for many people is because there are so many different patterns of word problems to be learned.

We still do not know much about how expertise with word problems is developed. Cross-sectional studies that compare good and poor solvers provide a somewhat distorted view of the development of expertise, because novices and experts may not be from comparable populations. Most novices will never reach expert status, and we have no way of knowing which novices will.

There is little information on how some novices do acquire expertise, and how instruction can facilitate the process. A common educational strategy is simply to present students with a large number of problems to solve, presumably in the hopes that expertise with a domain of problems will develop through an inductive process. There is ample evidence that solvers do not necessarily see the connection between problems unless these relationships are made explicit (e.g., Gick & Holyoak, 1980; Mawer & Sweller, 1982; Reed, Ernst, & Banerji, 1974).

An implication of the current study is that instruction on word problems should give explicit attention to helping students build schemata for the general structure of word problems and the specific structures found within problem categories. Our data on the use of diagrams by novices and experts indicate that diagrams can play an important role in helping students to organize information about a problem and to generate a structural representation of the problem. Our small training study suggests that detailed side-by-side comparisons of the structure of problems from the same category may be a useful technique for teaching students how to draw figures and create schematic representations of problems.

We are encouraged by the potential for cognitive analyses to provide new insights into the mental processes that are involved in problem solving, and we are hopeful that teachers and designers of instructional materials will be able to put the new information to good use.

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