8-1-1996

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Nonlinear Analysis, Theory, Methods & Applications, Vol. 27, No. 3, pp. 327–338, 1996

0362-546X/96 $15.00 + 0.00

0362-546X(95)00059-3

POSITIVE SOLUTIONS FOR A SEMILINEAR ELLIPTIC PROBLEM WITH CRITICAL EXPONENT

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(*Received* 1 *June* 1994; *received for publication* 21 *March* 1995)

Key words and phrases: Elliptic boundary value problem, critical exponent, radial solution.

1. INTRODUCTION

The solvability of boundary value problems of the form

$$\Delta u + f(u) = g(x) \text{ in } \Omega, \qquad u(x) = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a smooth bounded region in R^N, $N \geq 3$, and Δ is the Laplacian operator, depends on the *growth of the nonlinearity f*. We say that f grows subcritically if there exists

$$q \in (1, (N + 2)/(N - 2))$$

such that $\limsup_{|u| \to \infty} |f(u)|/|u|^q < \infty$. If $\lim_{|u| \to \infty}(|f(u)|/|u|^{(N+2)/(N-2)}) \in R$ then we say that f grows critically. In order to apply to this problem compactness techniques such as those derived from the imbedding properties of the Sobolev spaces (see [1]) one realizes that f must grow subcritically. Moreover, in [2], Pohozaev showed results for the subcritical case that do not extend to the critical case. Here we show, in particular, the existence of large positive solutions for small values of g when f grows critically, which is not the case when f grows subcritically. For related problems with critical exponents the reader is referred to [3–8].

From now on we consider the boundary value problem (1.1) when Ω is the unit ball in R^N, $f(u) = |u|^p u$ with $p = 4/(N - 2)$, and $g(x) \equiv -\lambda \in R$. Our main result is the following theorem.

THEOREM 1. There exists a continuous function $F: (0, \infty) \to (0, \infty)$ such that u is a positive solution to (1.1) if and only if $\lambda = F(u(0))$. If u_1 and u_2 are positive solutions to (1.1) with $u_1(0) = u_2(0)$ then $u_1 \equiv u_2$. Moreover, $\lim_{d \to 0} F(d) = 0$, $\lim_{d \to \infty} F(d) = 0$, and there exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$ then (1.1) has exactly two solutions. In particular, $\{(\lambda, u); u > 0, u \text{ satisfies } (1.1)\}$ is connected.

The classical work of Gidas *et al.* [7] tells us that positive solutions in Ω are radially symmetric. This allows us to shift our study to the ordinary differential equation

§ Partially supported by Kuwait University grant No. SM089.
|| Partially supported by NSF grant DMS 9215027.

$$u'' + \frac{N-1}{r}u' + |u|^p u + \lambda = 0, \qquad r \in (0, 1), \tag{1.2}$$

$$u'(0) = 0, \tag{1.3}$$

$$u(1) = 0. \tag{1.4}$$

Instead of considering directly this boundary value problem we study $u(r, \lambda, d)$, the solution to the initial value problem (1.2), (1.3), and

$$u(0) = d. \tag{1.5}$$

In order to prove theorem 1 we establish that $u_\lambda(1, \lambda, d) < 0$ and that for $d > 0$ large, $u_d(1, \lambda, d) < 0$ when $u(\cdot, \lambda, d)$ satisfies (1.2), (1.3), (1.4).

2. PRELIMINARIES

First we establish "Pohazaev identity" (see [2, 9]) for the initial value problem (1.2), (1.3), (1.5). Given $d \in R$, and $\lambda \in R$, define

$$E(r, \lambda, d) = \frac{(u'(r, \lambda, d))^2}{2} + \frac{(u(r, \lambda, d))^{p+2}}{p+2} + \lambda u(r, \lambda, d). \tag{2.1}$$

LEMMA 2. Let $u(r, \lambda, d)$ be a solution to (1.2), (1.3) and (1.5). If $0 \le \bar{r} \le r$, then

$$r^{N-1}H(r) - (\bar{r})^{N-1}H(\bar{r}) = \frac{N+2}{2}\int_{\bar{r}}^{r} \lambda s^{N-1}u(s)\,ds, \tag{2.2}$$

where

$$H(r) = rE(r) + \frac{N-2}{2}u(r, \lambda, d)u'(r, \lambda, d).$$

Proof. Multiplying (1.2) by $r^N u'(r)$ and integrating over $[\bar{r}, r]$, we obtain,

$$r^N E(r) = \int_{\bar{r}}^{r} s^{N-1}\left\{N\left(\frac{u^{p+2}(s)}{p+2} + \lambda u\right) - \left(\frac{N-2}{2}\right)(u'(s))^2\right\} ds + (\bar{r})^N E(\bar{r}). \tag{2.3}$$

Similarly, multiplying (1.2) by $r^{N-1}u(r)$ and integrating over $[\bar{r}, r]$ we infer,

$$\int_{\bar{r}}^{r} s^{N-1}(u'(s))^2\,ds = u'(r)u(r)r^{N-1} - u'(\bar{r})u(\bar{r})(\bar{r})^{N-1} + \int_{\bar{r}}^{r} s^{N-1}(u^{p+2}(s) + \lambda u)\,ds. \tag{2.4}$$

Replacing (2.4) in (2.3), we obtain (2.2). This completes the proof.

Taking $\bar{r} = 0$ in (2.2) we get

$$\frac{N+2}{2}\int_0^r \lambda s^{N-1}u(s)\,ds = \frac{r^N(u'(r))^2}{2} + \frac{r^N(u^{p+2}(r))}{p+2} + r^N\lambda u(r) + \frac{N-2}{2}r^{N-1}u'(r)u(r). \tag{2.5}$$

COROLLARY 3. The problem (1.1) has no nonnegative solutions for $\lambda \le 0$.

Proof. Taking $\bar{r} = 0$, $r = 1$ in (2.2) we obtain

$$\frac{(u'(1))^2}{2} = \frac{N+2}{2}\int_0^r \lambda s^{N-1}u(s)\,ds. \tag{2.6}$$

Since u is positive, (2.6) yields $(u'(1))^2 \le 0$ for $\lambda \le 0$. Hence, $u \equiv 0$. This completes the proof.

Now, for a positive solution u of (1.1), we define the function

$$h(r) = -\frac{ru'(r)}{u(r)}, \qquad r \in [0, 1). \tag{2.7}$$

Clearly, h is continuous, and $h(0) = 0$. Since $u(1) = 0$, we see that $\lim_{r\to 1^-} h(r) = \infty$. Furthermore, h is an increasing function. Indeed,

$$h'(r) = \frac{-u'(r)u(r) - ru(r)u''(r) + r(u'(r))^2}{(u(r))^2}. \tag{2.8}$$

Substituting (1.2) in (2.8) we have,

$$h'(r) = \frac{(N-2)u'(r)u(r) - ru^{p+2}(r) + r\lambda u(r) + r(u'(r))^2}{(u(r))^2}. \tag{2.9}$$

Combining (2.5) and (2.9) we have,

$$h'(r) = \frac{\lambda(N+2)r^{1-N}\int_0^r s^{N-1}u(s)\,ds - r\lambda u(r) + (1 - 2/(p+2))ru^{p+2}(r)}{(u(r))^2}. \tag{2.10}$$

Since u is a decreasing function, it follows from (2.10) that

$$\begin{aligned} h'(r) &\ge \frac{(1 - 2/(p+2))ru^{p+2}(r) + (2/N)\lambda ru(r)}{(u(r))^2} \\ &\ge \left(1 - \frac{2}{p+2}\right)ru^p(r) + \frac{2}{N}\lambda ru(r) = \frac{2}{N}ru^p(r) > 0. \end{aligned} \tag{2.11}$$

LEMMA 4. If u is a positive solution to (1.1), then there exists $M_0 > 0$ and a unique $\bar{r} \in (0, 1)$ such that $u(\bar{r}) = M_0\bar{r}^{-2/p}$. Moreover, if $0 < M < M_0$ then there exists exactly two numbers $r_1, r_2 \in (0, 1)$ such that $u(r_i) = M(r_i)^{-2/p}$, $i = 1, 2$.

Proof. Let $\bar{r} \in [0, 1]$ be such that $M_0 = \max\{u(r)r^{2/p} : r \in [0, 1]\} = u(\bar{r})\bar{r}^{2/p}$. Thus, the graph of u is tangent to the graph of $M_0 r^{-2/p}$, at $\bar{r}$, and $u(r) \le M_0 r^{-2/p}$ for all $r \in [0, 1]$.

Now for $M < M_0$ we show that the graph of u intersects the graph of $Mr^{-2/p}$ at exactly two points. Suppose $0 < r_1 < r_2 < r_3 < 1$ are the first three numbers such that $u(r_i) = Mr_i^{-2/p}$, $i = 1, 2, 3$. Since u is a decreasing function, $u(r_1) < u(r_2) < u(r_3)$. Let $Z = Mr^{-2/p}$, then, we have $Z(r_2) = u(r_2)$, $Z'(r_2) > u'(r_2)$, $Z(r_3) = u(r_3)$, and $Z'(r_3) < u'(r_3)$. Hence,

$$h(r_3) = \frac{-r_3u'(r_3)}{u(r_3)} < \frac{-r_3Z'(r_3)}{Z(r_3)} = \frac{2}{p},$$

and,

$$h(r_2) = \frac{-r_2 u'(r_2)}{u(r_2)} > \frac{-r_2 Z'(r_2)}{Z(r_2)} = \frac{2}{p}.$$

However, then $h(r_3) < h(r_2)$ with $r_2 < r_3$, which contradicts that h is an increasing function (see (2.11)). Assuming that $u(\bar{r})\bar{r}^{2/p} = u(\hat{r})\hat{r}^{2/p} = M_0$ we see that $h(\bar{r}) = h(\hat{r}) = 2/p$ which contradicts that h is an increasing function. Hence, $\bar{r}$ is unique.

From (2.5) and the quadratic formula we obtain,

$$ru'(r) = -\frac{N-2}{2}u(r) \pm \frac{1}{2}A(r), \tag{2.12}$$

where

$$A(r) = \left\{(N-2)^2u^2(r) - \frac{8}{p+2}r^2u^{p+2}(r) - 8r^2\lambda u(r) + 4(N+2)\lambda\frac{1}{r^{N-2}}\int_0^r s^{N-1}u(s)\,ds\right\}^{1/2} \tag{2.13}$$

From (2.12) we have

$$\frac{2}{N-2}h(r) = 1 \pm \frac{1}{N-2}\frac{A(r)}{u(r)}. \tag{2.14}$$

Since $h(0) = 0$ and $\lim_{r\to 1^-} h(r) = \infty$ we see from (2.14) that for r near zero

$$\frac{2}{N-2}h(r) = 1 - \frac{1}{N-2}\frac{A(r)}{u(r)}, \tag{2.15}$$

and for r near 1,

$$\frac{2}{N-2}h(r) = 1 + \frac{1}{N-2}\frac{A(r)}{u(r)}. \tag{2.16}$$

The fact that h is an increasing function together with (2.15), and (2.16) imply the existence of a unique $\hat{r}$ such that $(2/(N-2))h(\hat{r}) = 1$, that is, $A(\hat{r}) = 0$. Since $A(\hat{r}) = 0$ implies $h(\hat{r}) = 2/p$ and $\bar{r}$ is the only element in $[0, 1]$ for which $h(r) = 2/p$ we see that $\bar{r} = \hat{r}$. Using that $u(\hat{r}) = M_0\hat{r}^{-2/p}$ and integrating (2.11) on $[0, \bar{r}]$ we obtain

$$M_0 \le \left(\frac{N(N-2)}{2}\right)^{1/p}. \tag{2.17}$$

Lemma 5. If $\hat{r}$ is as above, then $\hat{r} \le O(d^{-p/2})$.

Proof. Let $r_0 = d^{-p/2}$, and put $K_0 = r_0^2u^p(r_0)$. We claim that $K_0 \ge (1 - 1/4N)^p$. Indeed, since

$$r^{N-1}u'(r) = -\int_0^r s^{N-1}(\lambda + u^{p+1}(s))\,ds \ge -\frac{d^{p+1}+\lambda}{N}r^N,$$

it follows that $u'(r) \ge -((d^{p+1} + \lambda)/N)r$. Integration over $[0, r_0]$ yields

$$u(r_0) \ge u(0) - \frac{d^{p+1}+\lambda}{2N}r_0^2 = d - \frac{d^{p+1}+\lambda}{2N}d^{-p} = \frac{2N-1}{N}d - \frac{\lambda}{2N}d^{-p}.$$

Thus, for $\lambda \in [0, 1]$ and d large, $K_0^{1/p} r_0^{-2/p} = u(r_0) \geq ((2N - 1)/3N)d$. Hence, $K_0^{1/p} \geq (2N - 1)/3N$ and the claim is established.

If $\hat{r} \geq d^{-p/2}$, then integrating $(2/(N - 2))h'(r)$ over $[r_0, \hat{r}]$ and using (2.11) we obtain

$$1 = \frac{2}{N-2} h(\hat{r}) \geq \frac{2}{N-2} h(r_0) + \frac{4}{N(N-2)} \int_{r_0}^{\hat{r}} K_0 r^{-2} r \, dr.$$

Hence

$$\ln\left(\frac{\hat{r}}{r_0}\right) \leq \frac{N(N-2)}{4K_0}, \qquad \text{or equivalently, } \hat{r} \leq d^{-p/2} \exp\left(\frac{N(N-2)}{4K_0}\right),$$

which proves lemma 5.

LEMMA 6. If u is a solution to (1.2), (1.3) and (1.5), then

$$\frac{2}{p} d u_d(r) = -\frac{N+2}{2} \lambda u_\lambda(r) + \frac{N-2}{2} u(r) + r u'(r). \tag{2.18}$$

Proof. Let $v(r) = \beta^{-2/p} u(r/\beta, \lambda, d)$. Thus,

$$\begin{aligned} v'' + \frac{N-1}{r} v' + v^{p+1} &= \frac{1}{\beta^{2+2/p}} u''\left(\frac{r}{\beta}\right) + \frac{1}{\beta^{1+2/p}} \cdot \frac{N-1}{r} u'\left(\frac{r}{\beta}\right) + \frac{1}{\beta^{2+2/p}} u^{p+1}\left(\frac{r}{\beta}\right) \\ &= \frac{1}{\beta^{2+2/p}} u''\left(\frac{r}{\beta}\right) + \frac{1}{\beta^{2+2/p}} \cdot \frac{\beta(N-1)}{r} u'\left(\frac{r}{\beta}\right) + \frac{1}{\beta^{2+2/p}} u^{p+1}\left(\frac{r}{\beta}\right) \\ &= \frac{1}{\beta^{2+2/p}} (-\lambda), \end{aligned}$$

which implies that $v(r) = u(r, \lambda/\beta^{2+2/p}, d/\beta^{2/p})$. Thus, for all $\beta > 0$,

$$u(r/\beta, \lambda, d) = \beta^{2/p} u(r, \lambda/\beta^{2+2/p}, d/\beta). \tag{2.19}$$

Differentiating (2.19) with respect to β we obtain, for all $r > 0$, $\lambda > 0$, $d > 0$, $\beta > 0$,

$$\begin{aligned} -\frac{r}{\beta^2} u'\left(\frac{r}{\beta}, \lambda, d\right) &= \frac{2}{p} \beta^{2/p-1} u\left(r, \frac{\lambda}{\beta^{2+2/p}}, \frac{d}{\beta^{2/p}}\right) - \left(2 + \frac{2}{p}\right) \beta^{-3} u_\lambda\left(r, \frac{\lambda}{\beta^{2+2/p}}, \frac{d}{\beta^{2/p}}\right) \\ &\quad - \left(\frac{2d}{p}\right) \beta^{-1} u_d\left(r, \frac{\lambda}{\beta^{2+2/p}}, \frac{d}{\beta^{2/p}}\right). \end{aligned}$$

In particular, taking $\beta = 1$, we obtain (2.18). This completes the proof.

Substituting (2.12) in (2.18) we have

$$\frac{N-2}{2} d u_d(r) = -\frac{N-2}{2} \lambda u_\lambda(r) \pm \frac{1}{2} A(r). \tag{2.20}$$

In (2.20) the "+" sign is to be used on the interval $[0, \hat{r}]$, and the "−" sign on the interval $(\hat{r}, 1]$ (see (2.15) and (2.16)). The existence and uniqueness of the number $\hat{r}$ are due to the fact that $h(r) = -ru'/u$ is an increasing function.

LEMMA 7. If u is a positive solution to (1.2), (1.3) and (1.5) then

$$-\frac{1}{2N} \leq -\frac{r^2}{2N} \leq u_\lambda(r, \lambda, d) \leq 0 \qquad \text{for } r \in [0, 1].$$

Proof. Let $z(r) = u_\lambda(r, \lambda, d)$. Thus,

$$z'' + \frac{N-1}{r} z' + (p+1)|u|^p z + 1 = 0, \qquad z(0) = 0, \qquad \text{and} \qquad z'(0) = 0. \tag{2.21}$$

Let G be defined by

$$G(r) = \tfrac{1}{2}(z')^2 + \tfrac{1}{2}(p+1)|u|^p z^2 + z. \tag{2.22}$$

Differentiating (2.22) we obtain

$$\begin{aligned} G'(r) &= z'z'' + (p+1)|u|^{p-1} z\left[z'|u| + \frac{p}{2} u' z\right] + z' \\ &= -\frac{(N-1)}{r}(z')^2 + \frac{1}{2}(p+1)|u|^p u' z^2 \leq 0, \end{aligned} \tag{2.23}$$

where we have used that $u' \leq 0$.

Suppose, there is an $\tilde{r} > 0$ such that $z(\tilde{r}) = 0$. Then substituting in (2.22) yields $G(\tilde{r}) = \frac{1}{2}(z'(\tilde{r}))^2 \geq 0$, contradicting the fact that G is a decreasing function. Therefore, $z \leq 0$ on $[0, 1]$. However, then

$$r^{N-1} z'(r) = -\int_0^r s^{N-1}(1 + (p+1)|u|^p(u) z)\, ds \geq \int_0^r s^{N-1}\, ds$$

and, hence, $z'(r) \geq -r/N$. Integrating over $[0, r]$ we obtain

$$z(r) \geq z(0) - \frac{r^2}{2N} = -\frac{r^2}{2N} \geq -\frac{1}{2N}.$$

This completes the proof.

From (2.20), lemma 7, and the definition of $\hat{r}$ we obtain that

$$w(r) = u_d(r, \lambda, d) > 0, \qquad \text{for } r \in [0, \hat{r}]. \tag{2.24}$$

LEMMA 8. If $u(r_2) = ((N-2)/4(N+2))^{1/p} r_2^{-2/p}$, with $0 < \hat{r} \leq r_2 < 1$ then $r_2 \leq O(d^{-p/2})$.

Proof. Since

$$\frac{2}{N-2}(h(r_2) - h(\hat{r})) = \sqrt{\frac{6N-4}{N(N+2)}} + O(r^{2+2/p}),$$

and $(u(r))^p \geq \frac{1}{4}((N-2)^3/(N+2))(1/r^2)$, for $r \in (\hat{r}, r_2)$; from (2.11) we have

$$\sqrt{\frac{6N-4}{N(N+2)}} + O(r^{2+2/p}) \geq \frac{(N-2)^2}{N(N+2)} \int_{\hat{r}}^{r_2} r^{-1}\, dr.$$

Therefore,

$$\ln\left(\frac{r_2}{\hat{r}}\right) \le \frac{\sqrt{N(N+2)(6N-4)}}{(N-2)^2} = \delta(N),$$

that is, $r_2 \le \hat{r}\exp(\delta(N))$. Since from lemma 5, $\hat{r} = O(d^{-p/2})$, we conclude that $r_2 = O(d^{-p/2})$. This completes the proof.

Now we show that for d sufficiently large $u_d(r_2) < 0$. Indeed,

$$\left(\frac{2}{p}\right) \mathrm{d}u_d(r_2) = -\lambda\frac{N+2}{2}u_\lambda(r_2) - \frac{1}{2}A(r_2),$$

and since $u_\lambda(r) \ge -r^2/2N$ (see lemma 7) we have

$$\left(\frac{2}{p}\right) \mathrm{d}u_d(r_2) \le \lambda\frac{N+2}{2N}r_2^2$$
$$- \sqrt{r_2^{-4/p}\left(\frac{(N-2)^3}{4(N+2)}\right)^{2/p}\frac{6N-4}{N(N+2)} - \lambda\left(8 - \frac{4(N+2)}{N}\right)\left(\frac{(N-2)}{4(N+2)}\right)^{1/p} r_2^{2-2/p}}.$$

Simplifying the expression on the right we get

$$\left(\frac{2}{p}\right) \mathrm{d}u_d(r_2) \le \lambda\frac{N+2}{4N}r_2^2 - \frac{N-2}{2}r_2^{-2/p}\left(\frac{(N-2)^3}{4(N+2)}\right)^{1/p}\sqrt{\frac{6N-4}{N(N+2)}} + O(r_2^{2-2/p}). \quad (2.25)$$

From lemma 8 we obtain

$$\left(\frac{2}{p}\right) \mathrm{d}u_d(r_2) \le \lambda\frac{N+2}{2N}r_2^2 - O(d) \quad (2.26)$$

and, hence,

$$u_d(r_2) \le -O(1) + O(d^{-1}) < 0, \qquad \text{for } d \text{ sufficiently large.} \quad (2.27)$$

From (2.24) and (2.27) we see that there exists an $\tilde{r} \in (\hat{r}, r_2)$ such that

$$u_d(\tilde{r}) = 0. \quad (2.28)$$

Let $\gamma(r) = r^{-(N-2)/2}$. A straightforward calculation shows that $\gamma'' + ((N-1)/r)\gamma' + ((N-2)/2r)^2\gamma = 0$. Since u_d satisfies a linear differential equation, its zeros are nondegenerate. Since $(p+1)u^p(r) \le ((N-2)^2/4r^2)$ for $r \in [r_2, 1)$, and γ is positive on $(0, \infty)$, by the Sturm comparison theorem (see [10]) we see that $u_d(\cdot, \lambda, d)$ cannot have two zeros in $[r_2, 1]$. The next three lemmas are devoted to proving that $u_d(\cdot, \lambda, d) < 0$ in $(r_2, 1]$.

Lemma 9. Let u be a positive solution to (1.2), (1.3) and (1.5). Then

$$\lim_{d \to +\infty} \int_0^1 r^{N-1}u(r, \lambda, d)\, \mathrm{d}r = 0.$$

Proof. Let $\varepsilon > 0$ be such that for $r > \hat{r} + \delta$, $(2/(N-2))h(r) \geq 1 + \varepsilon$ (see (2.16)). Thus,

$$-\frac{u'}{u} \geq \frac{1}{2}(1+\varepsilon)(N-2)\frac{1}{r}.$$

Integrating this over $[r_2, r]$ we get

$$u(r) \leq U(r_2) r_2^{(1/2)(1+\varepsilon)(N-2)} r^{-(1/2)(1+\varepsilon)(N-2)},$$

which in turn gives that

$$u(r) \leq \left(\frac{(N-2)^3}{4(N+2)}\right)^{1/p} r_2^{((N-2)/2)\varepsilon} r^{-((N-2)/2)(1+\varepsilon)}.$$

Therefore,

$$\int_{r_2}^{1} r^{N-1} u(r)\,\mathrm{d}r \leq \left(\frac{(N-2)^3}{4(N+2)}\right)^{1/p} r_2^{N-\varepsilon((N-2)/2)\varepsilon} \int_{r_2}^{1} r^{(N-\varepsilon(N-2))/2}\,\mathrm{d}r$$

and, hence,

$$\int_{r_2}^{1} r^{N-1} u(r)\,\mathrm{d}r \leq \left(\frac{(N-2)^3}{4(N+2)}\right)^{1/p} \frac{2r_2^{((N-2)/2)\varepsilon}}{2N-(1+\varepsilon)(N-2)}, \tag{2.29}$$

which tends to zero as d tends to $+\infty$, since $r_2 \to 0$ as $d \to \infty$. Also, since

$$\int_{0}^{r_2} r^{N-1} u(r)\,\mathrm{d}r \leq \frac{d}{N} r_2^N = \frac{d}{N}(d^{-p/2})^N = \frac{1}{N} d^{-(2+N)/(N-2)},$$

which tends to zero as $d \to +\infty$. We conclude that

$$\int_0^1 r^{N-1} u(r)\,\mathrm{d}r \to 0 \qquad \text{as } d \to +\infty.$$

This completes the proof.

LEMMA 10. If u is a positive solution to (1.2), (1.3) and (1.5) then

$$\int_0^1 r^{N-1} u^{p+1}\,\mathrm{d}r \leq o(\sqrt{\lambda}).$$

Proof. Taking $r = 1$ in Pohozaev's identity (2.5) we obtain

$$(u'(1))^2 = \lambda(N+2)\int_0^1 r^{N-1} u(r)\,\mathrm{d}r.$$

Hence,

$$\left(\int_0^1 r^{N-1} u^{p+1}\,\mathrm{d}r\right)^2 \leq \left(\int_0^1 r^{N-1}(\lambda + u^{p+1})\,\mathrm{d}r\right)^2 = (u'(1))^2.$$

This and lemma 9 yield

$$\left(\int_0^1 r^{N-1} u^{p+1}\,\mathrm{d}r\right)^2 \leq \lambda(N+2)\int_0^1 r^{N-1} u(r)\,\mathrm{d}r = o(\lambda).$$

Thus we obtain

$$\int_0^1 r^{N-1}u^{p+1}\,\mathrm{d}r \le o(\sqrt{\lambda}),$$

which proves the lemma.

LEMMA 11. If $u(r, \lambda, d)$ is a positive solution to (1.2), (1.3) and (1.5) then for $\lambda \in [0, 1]$ we have

$$\left\| u_\lambda(r, \lambda, d) + \frac{r^2}{2N} \right\|_\infty \le o(\lambda^{p/(2(p+1))})$$

and, hence,

$$\lim_{d\to+\infty} \left\| u_\lambda(r, \lambda, d) + \frac{r^2}{2N} \right\|_\infty = 0.$$

Proof. From the definition of u_λ we have

$$\begin{aligned} u_\lambda(r, \lambda, d) &= -\int_0^r s^{-N+1} \int_0^s r^{N-1}(1 + (p+1)|u|^p u_\lambda)\,\mathrm{d}r\,\mathrm{d}s \\ &= -\frac{r^2}{2N} - \int_0^r s^{-N+1}\int_0^s r^{N-1}(p+1)|u|^p u_\lambda\,\mathrm{d}r\,\mathrm{d}s. \end{aligned} \tag{2.30}$$

Since $-u_\lambda \le 1/2N$ (see lemma 7), from (2.30) we obtain that

$$u_\lambda(t, \lambda, d) \le -\frac{r^2}{2N} + \frac{p+1}{2N}\int_0^1 s^{-N+1}\int_0^s r^{N-1}u^p\,\mathrm{d}r\,\mathrm{d}s.$$

Hence, it suffices to show that $\int_0^s r^{N-1}u^p\,\mathrm{d}r \to 0$ as $d \to \infty$. Now by Hölders inequality we have

$$\int_0^s r^{N-1}u^p\,\mathrm{d}r \le \int_0^1 r^{N-1}u^p\,\mathrm{d}r \le M\left(\int_0^1 r^{N-1}u^{p+1}\right)^{p/(p+1)}.$$

Now using lemma 10 we obtain that

$$\int_0^1 r^{N-1}u^p\,\mathrm{d}r \le o(\lambda^{p/(2(p+1))}).$$

Therefore,

$$\int_0^r s^{-N+1}\int_0^s r^{N-1}(p+1)u^p(-u_\lambda) \le \left(\frac{1+p}{2N}\right)\int_0^r s^{-N+1}\int_0^s r^{N-1}u^p\,\mathrm{d}r = o(\lambda^{p/(2(p+1))})$$

and, hence, $u_\lambda(r) \le -r^2/2N + o(\lambda^{p/(2(p+1))})$ which, in turn, implies

$$\left\| u_\lambda(r, \lambda, d) + \frac{r^2}{2N} \right\|_\infty \le o(\lambda^{p/(2(p+1))}),$$

thus, completing the proof.

Now we show that $u_d(1, \lambda, d) \neq 0$. Consider the initial value problem (1.2), (1.3) and (1.5) for $\lambda = 0$. That is

$$v'' + \frac{N-1}{r} v' + |v|^p v = 0, \tag{2.31}$$

$$v(0) = d \quad \text{and} \quad v'(0) = 0. \tag{2.32}$$

That is we denote $u(r, d, 0)$ by $v(r, d)$. From Pohozaev's identity (2.5) and the quadratic formula we obtain

$$v' = -\frac{N-2}{2r} v - \frac{N-2}{2r} v \sqrt{1 - \frac{4r^2}{N(N-2)} v^p} \tag{2.33}$$

for all $r > \hat{r} = K_1 d^{-p/2}$, where K_1 is a constant independent of d (see lemma 5). Also, given an $\varepsilon > 0$, there exists a constant K_2 independent of d such that for $r \geq k_2 d^{-p/2}$ we have

$$\frac{2}{N(N-2)} r^2 v^p \leq \varepsilon. \tag{2.34}$$

Substituting (2.34) in (2.33) we get

$$v'(r) \leq -\sigma \frac{v}{r}, \tag{2.35}$$

where $\sigma = \frac{1}{2}(N-2)(1 + \sqrt{1-\varepsilon})$. Integrating this over $[r, 1]$ we obtain

$$v(r) \geq v(1) r^{-\sigma}. \tag{2.36}$$

From this we infer

$$\int_{kd^{-p/2}}^{1} r^{N-1} v(r) \, \mathrm{d}r \geq (N-\sigma)^{-1} v(1)[1 - (kd^{-p/2})^{N-\sigma}]. \tag{2.37}$$

Thus, by choosing ε small enough and d large enough we get

$$\int_0^1 r^{N-1} v(r) \, \mathrm{d}r \geq v(1)\tau, \tag{2.38}$$

with $\tau < \frac{1}{2}$ but arbitrarily close to $\frac{1}{2}$. Suppose now that $u_d(1, \lambda, d) = 0$. Then using the mean value theorem we obtain

$$v(1) = u(1, 0, d) = u(1, \lambda, d) - \lambda u_\lambda(1, \hat{\lambda}, d)$$

for some $\hat{\lambda} \in [0, \lambda]$. From lemma 11 and the fact that $u(1, \lambda, d) = 0$, we obtain

$$v(1) = \lambda\left(\frac{1}{2N} - o(\lambda^{p/(2(1+p))})\right). \tag{2.39}$$

From lemma 11 and the rescaling equation in (2.18), we obtain

$$u'(1, d, \lambda) = \lambda\left(\frac{N+2}{2}\right)\left(-\frac{1}{2N} + o(\lambda^{p/(2(1+p))})\right). \tag{2.40}$$

From Pohozaev's identity and (2.5) we have

$$u'(1, d, \lambda) = -\sqrt{\lambda(N+2)\int_0^1 r^{N-1}u(r)\,dr}, \tag{2.41}$$

and by the mean value theorem and lemma 3 we get

$$u(r, \lambda, d) \geq v(r, d) - \frac{\lambda}{2N}r^2. \tag{2.42}$$

Combining (2.40) and (2.41) we obtain

$$\lambda(N+2)\int_0^1 r^{N-1}u(r)\,dr = \left(\frac{N+2}{2}\lambda\right)^2\left(\frac{1}{4N^2} + o(\lambda^{p/(2(1+p))})\right).$$

That is,

$$\int_0^1 r^{N-1}u(r)\,dr = \left(\frac{N+2}{4}\lambda\right)\left(\frac{1}{4N^2} + o(\lambda^{p/(2(1+p))})\right). \tag{2.43}$$

On the other hand, from (2.38), (2.39) and (2.42) we obtain

$$\begin{aligned}\int_0^1 r^{N-1}u(r)\,dr &\geq \int_0^1 r^{N-1}\left(v(r) - \frac{\lambda}{2N}r^2\right)dr > v(1)\tau - \frac{\lambda}{2N(N+2)}\\ &= \lambda\left[\frac{1}{2N} - o(\lambda^{p/(2(1+p))})\right]\tau - \frac{\lambda}{2N(N+2)}.\end{aligned} \tag{2.44}$$

Combining (2.43) and (2.44) we obtain

$$o(\lambda^{p/(2(1+p))}) \geq \frac{(8\tau - 1)N^2 + (16\tau - 12)N - 4}{16N^2(N+2)}. \tag{2.45}$$

Since τ can be chosen arbitrarily close to $\frac{1}{2}$, we see that the numerator of the right-hand side can be made arbitrarily close to $3N^2 - 4N - 4$, which is positive for $N \geq 3$. Hence, (2.45) cannot hold for small values of λ which is a contradiction. Thus, there exist $D > 0$ and $\Lambda > 0$ such that if $u(\cdot, \lambda, d)$ is a positive solution to (1.2), (1.3), (1.5), $\lambda \in (0, \Lambda)$ and $d > D$ then

$$u_d(1, \lambda, d) < 0. \tag{2.46}$$

3. PROOF OF THEOREM 1

Since $u_\lambda(1, \lambda, d) < 0$ (see lemma 7), the implicit function theorem implies that if S is a connected component of $\{(\lambda, d); u(1, \lambda, d) = 0, u(r, \lambda, d) > 0 \text{ for all } r \in [0, 1]\}$ then there exists a differentiable function $F\colon (0, \infty) \to (0, \infty)$ such that $S = \{(F(d), d); d \in (0, \infty)\}$. Integrating (1.2) on [0, 1] we see that $-d \leq -F(d)/2N$. Hence,

$$\lim_{d\to 0} F(d) = 0. \tag{3.1}$$

Let us see now that

$$\lim_{d\to\infty} F(d) = 0. \tag{3.2}$$

By lemma 9 we have $\lim_{d\to\infty} u(1/4, F(d), d) = 0$. Thus, if $\limsup_{d\to\infty} F(d) > 0$ then for some sequence $\{d_n\} \to \infty$ we have $\{F(d_n)/u(1/4, F(d), d)\} \to \infty$. Hence, because $u(\cdot, F(d_n), d_n)$ satisfies

$$u'' + \frac{N-1}{r}u' + \left(|u|^p + \frac{F(d_n)}{u}\right)u = 0,$$

by the Sturm comparison theorem we see that $u(\cdot, F(d_n), d_n)$ must have a zero in $[1/4, 1)$. This contradicts that $u(\cdot, F(d_n), d_n)$ is positive in $(0, 1)$. Thus, (3.2) is proven.

Since $u(r, F(d), d) \le d$ for all $r \in [0, 1]$, by the Sturm comparison theorem we see that for $d > 0$ small $u_d(r, F(d), d) > 0$ for all $r \in [0, 1]$. hence, by the implicit function theorem there exists $\delta > 0$ and an increasing differentiable function $\phi: (0, \delta) \to (0, \infty)$ such that $u(\cdot, \lambda, d)$ is a solution to (1.2), (1.3), (1.5) if and only if $d = \phi(\lambda)$. Thus, if $S_1 = \{(F(d), d); d \in (0, \infty)\}$ and $S_2 = \{(F_1(d), d); d \in (0, \infty)\}$ are connected components of positive solutions to (1.2), (1.3), (1.5), by (3.1) we see that $d = \phi(F(d)) = \phi(F_1(d))$. Hence, $F(d) = F_1(d)$ for d close to 0. Therefore, $S_1 = S_2$, which proves that the set of positive solutions to (1.1) is connected.

Differentiating $u(1, \lambda(d), d)) = 0$ with respect to d we obtain

$$u_d(1, \lambda(d), d)) + u_\lambda(1, \lambda(d), d)) \cdot \lambda'(d) = 0.$$

This, (2.46) and (3.2) imply that F is a decreasing function in (D, ∞), which proves that for $\lambda < F(D)$ the problem (1.1) has exactly one solution with $u(0) > D$. Since ϕ is an increasing function, so is F. Thus, if $\lambda \in (0, \phi(\delta))$ then (1.1) has exactly one small solution. Thus, if $0 < \lambda < \min\{\delta/2, F(D), \min\{F(d); d \in [\delta/2, F(D]\}\}$ then the problem (1.1) has exactly two positive solutions.

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