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# An Inverse Function Theorem via Continuous Newton's Method

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**Abstract**

We prove an inverse function theorem of the Nash-Moser type. The main difference between our method and that of [4] is that we use continuous steepest descent while [4] uses a combination of Newton type iterations and approximate inverses. We bypass the *loss of derivatives problem* by working on finite dimensional subspaces of infinitely differentiable functions.

*Key words:* Inverse Function Theorem Continuous Newton's Method

## 1 Introduction

Inverse function theorems have long had a prominent role in analysis, particularly in the study of differential equations. In [4] there is an inverse function theorem whose proof uses a form of conventional Newton's method. In the present note we use a version of continuous Newton's method to give a new inverse function theorem. Our hypothesis is suggested from [4] but a direct comparison has not yet been established. We believe, however, that the present result covers a substantial portion of the cases covered by the hypothesis in [4].

Suppose each of $H$ and $K$ is a Banach space, $r > 0$ and $F$ is a $C^1$ function from the open ball $B_r(0)$ in $H$, centered at 0, so that $F(0) = 0$. We intend to give a condition on a member $g$ of $K$ so that there exists $\gamma > 0$ such that $tg \in R(F)$ if $0 \leq t \leq \gamma$. In intended applications, $H$ may be the Sobolev space $H^{1,2}(\Omega)$ for some region $\Omega$ in a Euclidean space and $K$ may be $L_2(\Omega)$. The function $F$ then may be a nonlinear differential operator.


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For a motivating example we essentially follow [4] by choosing $F$ defined by

$$F(u) = u_1 \text{ for } u \in H^{1,2}(\Omega) \tag{1}$$

where here $\Omega = [0,1]^2$ and the subscript on $u$ indicates partial differentiation in the first argument of $u$. Some reflection yields that the range of $F$ can not be all of $L_2(\Omega)$ since many members $g$ of that space lack sufficient smoothness to be in that range. For some members $g$ of $L_2(\Omega)$, namely those members of $L_2(\Omega)$ which are also in $H^{1,2}(\Omega)$, there is a solution $u$ to

$$F(u) = g$$

which is in $H^{1,2}(\Omega)$. To be more specific, suppose $r > 0$ and $H'$ is a subset of $H^{1,2}(\Omega)$, uniformly bounded in the norm of that space. Observe that there is $\gamma > 0$ so that if $0 \le t \le \gamma$ and $g \in H'$, then there is

$$u \in H^{1,2}(\Omega) \text{ with } \|u\|_{H^{1,2}(\Omega)} \le r$$

so that

$$F(u) = tg.$$

## 2 An Inverse Function Theorem

Return to the general setting of the introduction, that is suppose $r > 0$,

$$F:\ B_r(0) \subset H \to K,$$

$F$ is so that $F(0) = 0$ and $F$ is $C^1$.

**Theorem 1** *Suppose $g \in K$ and there is a function $h$ with domain $B_r(0) \subset H$ which is Lipschitz continuous so that*

$$F'(x)h(x) = g \text{ for } \|x\|_H \le r.$$

*Then there is $\gamma > 0$ so that if $0 \le t \le \gamma$, there is $u \in B_r(0)$ so that*

$$F(u) = tg.$$

**Proof:** Under the hypothesis of the theorem, denote by $\gamma$ a positive number so that there is a unique solution $z$ on $[0,\gamma]$ to

$$z(0) = 0 \text{ and } z'(t) = h(z(t)) \text{ with } \|z(t)\| \le r \text{ for } t \in [0,\gamma] \tag{2}$$

(that there is such a number $\gamma$ follows from the basic existence and uniqueness theorem for ordinary differential equations). Then

$$F'(z(t))z'(t) = F'(z(t))(h(z(t)) = g,$$

i.e.,

$$(F(z))'(t) = g \text{ for } t \in [0,\gamma].$$

Hence,

$$F(z(t)) = tg \text{ for } \in [0,\gamma],$$

and the argument is complete.

Note that we have not required uniqueness of solution $k \in H$ to

$$F'(x)k = g \tag{3}$$

for any $x \in B_r(0)$, but rather that solutions to 3 for various $x \in B_r(0)$ can be fit together in a smooth enough way in order to provide a function $h$ satisfying the hypothesis of the theorem.

## 3 Discussion

Here we attempt to justify calling the above process a version of continuous Newton's method. Suppose here that $F$ is a function from $H$ to $K$ so that $F(0) = 0$ and $(F'(x))^{-1}$ exists and is in $L(K,H)$ for each $x \in H$. Suppose furthermore that $(F'(\cdot))^{-1}$ is locally Lipschitz. Given $g \in K$, conventional continuous Newton's method for finding $u \in H$ such that $F(u) = g$ might consist of first finding $z : [0,\infty) \to H$ so that

$$z(0) = 0 \text{ and } z'(t) = -(F'(z(t)))^{-1}(F(z(t)) - g) \text{ for } t \geq 0 \tag{4}$$

and then seeking

$$u = \lim_{t\to\infty} z(t)$$

so that $F(u) = g$.

Assuming we have $z$ satisfying (4), we then have

$$F'(z(t))z'(t) = -(F(z(t)) - g) \text{ for } t \geq 0,$$

and consequently

$$(F(z) - g)'(t) = -(F(z(t)) - g) \text{ for } t \geq 0.$$

Hence we have

$$F(z(t)) - g \;=\; e^{-t}(F(z(0)) - g) \text{ for } t \geq 0. \tag{5}$$

Using 5 and the fact that $F(z(0)) = 0$ we substitute in 4 to obtain an alternate expression

$$z(0) = 0,\ z'(t) \;=\; e^{-t}(F'(z(t))^{-1}g \text{ for } t \geq 0.$$

Deleting the exponential factor in the above just changes the time scale from $[0, \infty)$ to $[0, 1)$, which leaves us with

$$z(0) = 0,\ z'(t) \;=\; (F'(z(t)))^{-1}g \text{ for } t \in [0, 1).$$

This provided motivation for the process 2 in which it is not assumed that $(F'(x))^{-1}$ for $x \in B_r(0)$, exist, but rather that for a fixed $g \in K$ and any $x \in B_r(0)$, there is $k \in H$ such that

$$F'(x)k \;=\; g$$

(and these solutions $k$ for $x \in B_r(0)$ can be fit together to make a function $h$ as in the hypothesis of the theorem).

## 4 Application: range of the sum of two maximal monotone operators

Finally we consider the semilinear boundary value problem

$$-\Delta u + f(u) = g \;\text{ in }\; \Omega,\;\; u(x) = 0 \;\text{ for }\; x \in \partial\Omega, \tag{6}$$

where $\Omega$ is a smooth bounded region in $R^n$ with $n > 2$, and $f : R \to R$ is a monotonically increasing function with supercritical growth (i.e.,

$$\liminf_{|t|\to\infty} f(t)/t^{\rho} > 0$$

for some $\rho > (n+2)/(n-2)$) and has a locally Lipschitzian derivative. For the sake of simplicity in the calculations we assume that $f(0) = 0$. Due to the growth of $f$, the sum of the maximal monotone operators defined by $-\Delta$ and $f$ is not maximal monotone (see [1]) and general theory does not provide adequate information on its range. However letting $p > \min\{1, n/2\}$ and taking $H$ to be the Sobolev space $H_0^{2,p}(\Omega)$ of functions having second order derivatives in $L^p(\Omega)$ and vanishing on the boundary of $\Omega$, and $K$ the space $L^p(\Omega)$ we see that for each $u \in H, g \in K$ the equation

$$-\Delta v + f'(u)v = g \;\text{ in }\; \Omega,\;\; u(x) = 0 \;\text{ for }\; x \in \partial\Omega, \tag{7}$$

has a unique solution $v = h(u)$ (see Thoerem 9.15 of [3]). In order to apply Theorem 1 we prove the following result.

**Lemma 1** *The function $h : H \to H$ given by $h(u) = v$, where $u, v$ are as in (7), is bounded on bounded sets and locally Lipzchitzian.*

**Proof:** Let $B$ be a bounded subset of $H$. By the Sobolev imbedding theorem (see Corollary 7.11 of [3]), without loss of generality we may assume that $\|u\|_\infty \leq 1$ for $u \in B$. Let $t > ((n-2)p - n)/n$ be an odd positive integer. Multiplying (7) by $v^t$, and using that $f' \geq 0$ and Holder's inequality we have

$$\begin{aligned} \int_\Omega \nabla v^{(t+1)/2} \cdot \nabla v^{(t+1)/2}\, dx &\leq \frac{(t+1)^2}{4\,t} \int_\Omega g v^t\, dx \\ &\leq \frac{(t+1)^2}{4\,t} \|g\|_s \|v\|^t_{(t+1)n/(n-2)}, \end{aligned} \tag{8}$$

where $s = (t+1)n/(n+2t) \leq n/2$. Let $C > 0$ (see again Corollary 7.11 of [3]) be a constant such that

$$\Big(\int_\Omega |w|^{2n/(n-2)} dx\Big)^{(n-2)/(2n)} \leq C\Big(\int_\Omega \nabla w \cdot \nabla w dx\Big)^{1/2}, \tag{9}$$

for all $w \in H_0^{1,2}$. Taking $w = v^{(t+1)/2}$ and using (8) we see that

$$\|v\|_{n(t+1)/(n-2)} \leq \frac{C^2(t+1)^2}{4\,t} \|g\|_s. \tag{10}$$

Since $s \leq n/2 < p$, we see that there exists a constant $M$ such that

$$\|v\|_p \leq M \|g\|_{n/2}. \tag{11}$$

Thus $\|g - f'(u)v\|_p \leq \|g\|_p + M_1 \|v\|_p$, where $M_1 =$ $\leq \|u\|_\infty + 1\}$. Therefore by *a priori* estimes for elliptic bounaary vaiue problems (see Theorem 9.15 of [3]) we infer $\|v\|_H \leq M_2$, with $M_2$ depending on $M_1, \Omega$, and $p$. Thus $h$ is bounded in bounded sets.

Let $u_1, u_2, v_1, v_2 \in H$ be such that

$$-\Delta v_i + f'(u_i) v_i = g \tag{12}$$

for $i = 1, 2$, with $\|u_1 - u_2\|_H \leq 1$. An elementary algebraic manipulation shows that $-\Delta(v_1 - v_2) + f'(u_1)(v_1 - v_2) = (f'(u_2) - f'(u_1))v_2$. From Lemma 9.17

of [3], there exists a positive constant $C_q$ for each $q \in (1, \infty)$ such that if

$$-\Delta w + f'(u_1)w = y \tag{13}$$

with $y \in L^q(\Omega)$ then $w \in H_0^{2,q}(\Omega)$ and

$$\|w\|_{H_0^{2,q}(\Omega)} \leq C_q \|y\|_q. \tag{14}$$

Also by the Sobolev imbedding theorem there exists a constant $\bar{C}$ such that

$$\|z\|_\infty \leq \bar{C}\|z\|_H, \tag{15}$$

for all $z \in H$. Therefore

$$\begin{aligned} \|v_1 - v_2\|_H &\leq C_p \|(f'(u_1) - f'(u_2))v_2\|_K \leq AC_p \|u_1 - u_2\|_\infty \|v_2\|_K \\ &\leq AC_p M_2 \|u_1 - u_2\|_\infty \leq AC_p M_2 \bar{C} \|u_1 - u_2\|_H, \end{aligned} \tag{16}$$

where $A$ is a Lipschitz constant for $f'$ on $[-\bar{C}(\|u\|_\infty + 1), \bar{C}(\|u\|_\infty + 1)]$. This proves that $h$ is a locally Lipchitzian function, which proves the lemma.

Now combining Lemma 1 and Theorem 1 we prove that for any $g \in K$ the equation (6) has a solution.

**Theorem 2** *Under the above assumptions, for each $g \in K$ the equation (6) has a solution.*

**Proof:** Let $F : H \to K$ be the operator defined by $F(u) = -\Delta u + f \circ u$. Let $z(t)$ be as in (2). Multiplying the equation $F(z(t)) = tg$ by $|f(z(t))|^{p-2} f(z(t))$ we see that

$$\int_\Omega |f(z(t))|^p dx \leq t \int_\Omega |g||f(z(t))|^{p-1} dx.$$

Hence, by Holder's inequality, we see that $\|f(z(t))\|_K$ is bounded on bounded intervals of $[0, \infty)$. Thus, by Theorem 9.15 of [3], $\|z(t)\|_H$ is bounded on bounded intervals. Since $h$ is defined on all of $H$ it follows that $z$ is defined on $[0, \infty)$. In particular $F(z(1)) = g$, which proves the theorem.

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