Journal of Mathematical Analysis and Applications **245**, 282–293 (2000)
doi:10.1006/jmaa.2000.6787, available online at http://www.idealibrary.com on IDEAL®

Evolution of Positive Solution Curves in Semipositone Problems with Concave Nonlinearities[1]

Alfonso Castro

Division of Mathematics and Statistics, University of Texas at San Antonio, San Antonio, Texas 78249-0664

Sudhasree Gadam

Mathematics, "Yashodha," JCR VI Cross, Chitradurga, 577501 India

and

R. Shivaji

Department of Mathematics, Mississippi State University, Mississippi State, Mississippi 39762

Submitted by Jerome A. Goldstein

Received January 4, 2000

We study the existence, multiplicity, and stability of positive solutions to

$$-u''(x) = \lambda f(u(x)) \qquad \text{for } x \in (-1, 1),$$
$$u(-1) = 0 = u(1),$$

where $\lambda > 0$ and f: $[0, \infty) \to \mathbb{R}$ is monotonically increasing and concave with $f(0) < 0$ (semipositone). We establish that f should be appropriately concave (by establishing conditions on f) to allow multiple positive solutions. For any $\lambda > 0$, we obtain the exact number of positive solutions as a function of $f(t)/t$. We follow several families of nonlinearities f for which $f'(\infty) := \lim_{t\to\infty} f'(t) > 0$ and study how the positive solution curves to the above problem evolve. Also, we give

[1] The results in this paper were made public at the Conference on Applied Analysis of April 1996 at Louisiana State University. The corresponding paper was accepted for the proceedings of that conference published in *Contemp. Math.* **221** (1999). Clerical misunderstandings prevented such publication.

0022-247X/00 $35.00

examples where our results apply. This work extends the work of A. Castro and R. Shivaji (1988, *Proc. Roy. Soc. Edinburgh Sect. A* **108,** 291–302) and S.-H. Wang (1994, *Proc. Roy. Soc. Edinburgh* **124,** No. 3, 507–515) by obtaining sharper results and also gives a complete study of positive solutions for concave semipositone nonlinearities.

1. INTRODUCTION

We study the positive solutions to the two point boundary value problem

$$-u''(x) = \lambda f(u(x)) \qquad \text{for } x \in (-1, 1), \tag{1.1}$$

$$u(-1) = 0 = u(1), \tag{1.2}$$

where $\lambda > 0$ and f: $[0, \infty) \to \mathbb{R}$ is monotonically increasing and concave ($f'' < 0$) with

$$f(0) < 0(\text{semipositone}), \qquad f(t) > 0 \text{ for some } t > 0. \tag{1.3}$$

We define F by $F(t) = \int_0^t f(s)\,ds$ and let β and θ denote the unique positive zeros of f and F, respectively. It can be easily seen that for any positive solution u satisfying (1.1), (1.2) we should have $\operatorname{Sup}\{u(x) : x \in [-1, 1]\} \geq \theta$.

The following classification of concave functions plays an important role in our results. It is easy to show that either

(1) $(f(t)/t)' > 0$ for all $t > 0$.

(2) There exists an $\eta > 0$ such that $\eta f'(\eta) = f(\eta)$; $(f(t)/t)' > 0$ for all $t \in (0, \eta)$ and $(f(t)/t)' < 0$ for all $t \in (\eta, \infty)$.

Let $\phi(t) = (f(t)/t)'$. Then differentiating and using $f'' < 0$ we have

$$\phi'(t) < -2\phi(t)/t,$$

so that $\phi' < 0$ whenever $\phi > 0$. Consequently $\phi(t) > 0$ for all t or $\phi(t) \leq 0$ for all t suitably large. In the first case, we have $f'(t) > f(t)/t$ and $f(t)/t$ strictly increasing; it follows that $f'(\infty) > 0$. In case (2), using $\phi'(t) < -2\phi(t)/t$ we conclude that there exists a unique η such that $\phi(\eta) = 0$. Since $f'(\infty) > 0$ in case (1), when $f'(\infty) = 0$ one must therefore be in case (2).

Semipositone problems are not only of mathematical interest but also occur in applications such as population models with constant harvesting effort (see [6]). In [1, 4, 5] semipositone problems with concave nonlineari-

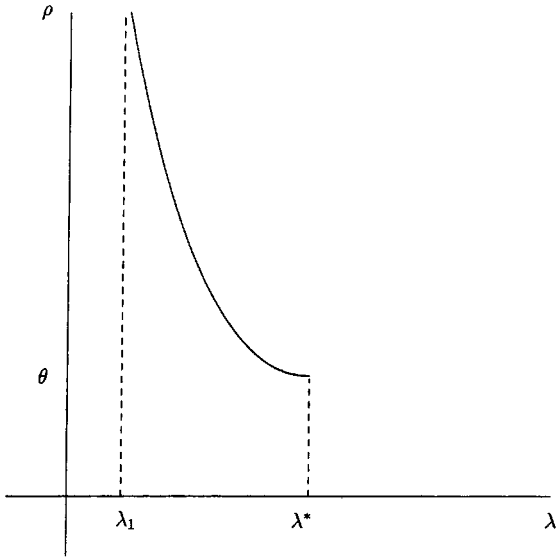

FIG. 1. When $(1.5)_1$ holds.

ties have been extensively studied with an additional condition that

$$f'(\infty) := \lim_{t \to \infty} f'(t) = 0. \tag{1.4}$$

We note that this hypothesis is necessary for the existence of positive solutions for large values of λ. It also implies that the supremum norm of positive solutions tends to ∞ as $\lambda \to \infty$ (see [4]). Here we consider the case when

$$f'(\infty) > 0 \tag{1.5}$$

and study how the existence, multiplicity, and stability of positive solutions can drastically differ from case (1) to case (2). In fact, we establish the exact geometry of the positive solution curves and hence the exact number of positive solutions for any $\lambda > 0$. As a by-product we relax the hypotheses on f in [5, Theorem 1.1 (B)] and also establish the exact number of positive solutions for any $\lambda > 0$. Also, we rule out a few cases that were considered in [7]. Hence, our results completely classify semipositone problems with monotonically increasing concave nonlinearities.

As remarked above, we have two cases to consider. In the rest of the paper we use the following notation. If f satisfies (1.5) and is in case (1) then we write f satisfies $(1.5)_1$; if f satisfies (1.5) and is in case (2) then we write f satisfies $(1.5)_2$.

Our main results are

THEOREM 1. (1) *If f satisfies* $(1.5)_1$, *then there exist* λ_1, λ^* *with* $0 < \lambda_1 < \lambda^* < \infty$ *such that* (1.1), (1.2) *has no positive solutions for* $\lambda \notin (\lambda_1, \lambda^*]$ *and for* $\lambda \in (\lambda_1, \lambda^*]$ (1.1), (1.2) *has a unique positive solution. These positive solutions are unstable. If* ρ_λ *denotes the supremum norm of the positive solution, then* ρ_λ *is a decreasing function of* λ, *and in particular* $\rho_{\lambda^*} = \theta$ *and* $\lim_{\lambda \to \lambda_1} \rho_\lambda = +\infty$.

(2) *If f satisfies* $(1.5)_2$ *then there exist* $0 < \lambda_1 < \lambda_2 < \infty$ *and* $\lambda_1 < \lambda^* < \infty$ *such that the problem* (1.1), (1.2) *has no positive solutions for* $\lambda < \lambda_1$. *For* $\lambda = \lambda_1$ (1.1), (1.2) *has exactly one positive solution; it is unstable. If* $\lambda^* \geq \lambda_2$ *then for* $\lambda \in (\lambda_1, \lambda_2)$ *the problem* (1.1), (1.2) *has exactly one stable and one unstable positive solution. For* $\lambda \in [\lambda_2, \lambda^*]$ *the above problem has exactly one positive solution and it is unstable (see Fig.* 2). *If* $\lambda^* < \lambda_2$ *then for* $\lambda \in (\lambda_1, \lambda^*]$ *the problem* (1.1), (1.2) *has exactly two positive solutions, one stable and one unstable. For* $\lambda \in (\lambda^*, \lambda_2)$ *the problem* (1.1), (1.2) *has exactly one positive solution and it is stable (see Fig.* 2). *Also,* $\rho_{\lambda^*} = \theta$ *and* $\lim_{\lambda \to \lambda_2} \rho_\lambda = +\infty$.

THEOREM 2. *If f satisfies* (1.4) *then there exist* $0 < \lambda_1 < \lambda^* < \infty$ *such that for* $\lambda < \lambda_1$ *the problem* (1.1), (1.2) *has no positive solutions. For* $\lambda = \lambda_1$ *the problem* (1.1), (1.2) *has exactly one positive solution and it is unstable. For* $\lambda \in (\lambda_1, \lambda^*]$ *the problem* (1.1), (1.2) *has exactly two positive solutions, one stable and one unstable. For* $\lambda > \lambda^*$ *the problem* (1.1), (1.2) *has exactly one*

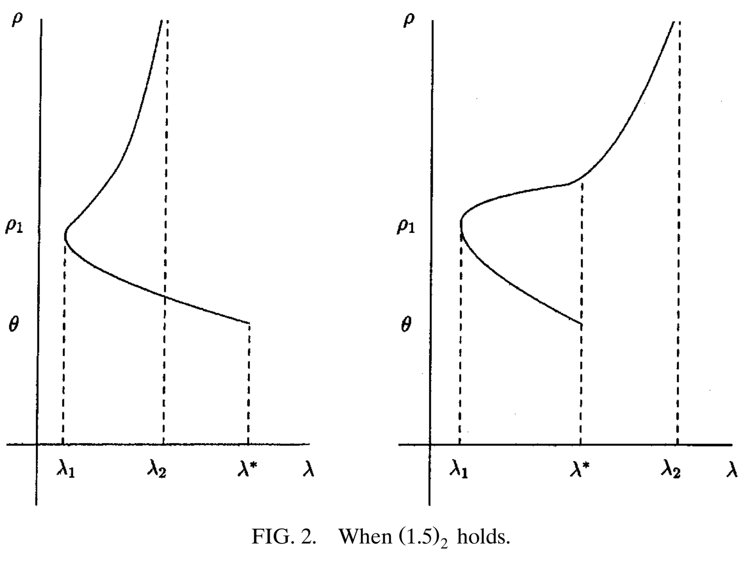

FIG. 2. When $(1.5)_2$ holds.

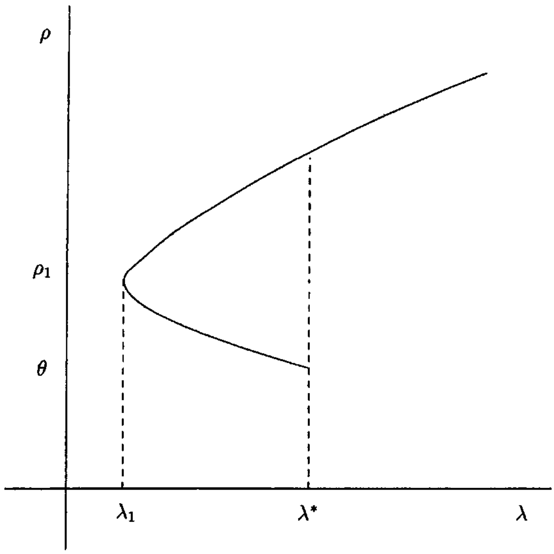

FIG. 3. $f'(\infty) = 0$.

positive solution and it is stable (*see Fig*. 3). *Also*, $\rho_{\lambda^*} = \theta$ *and* $\lim_{\lambda \to \infty} \rho_\lambda = +\infty$.

Theorems 1 and 2 establish that [7, Theorem 1.2, case (B), Theorem 1.3, case (D)] do not occur at all. Also, note that Theorem 2, in addition to relaxing the hypotheses on f, establishes the exact number of positive solutions for any $\lambda > 0$ (see [5, Theorem 1.1 (B)]).

This paper is organised as follows. In Section 1, we study the variations of the positive solutions with respect to the parameters λ and $\rho = \sup\{u(x) : x \in (-1, 1)\}$. We prove Theorems 1 and 2 in Section 3. In Section 4 we give a family of examples which satisfies all the hypotheses of Theorem 1 for case (2). For case (1) it is much simpler to construct an example. In Section 5, we follow a sequence of nonlinearities f_n as in case (2) of Theorem 1 which converge to an f with $f'(\infty) = 0$ to illustrate how the positive solution curve to (1.1), (1.2) would evolve from the positive solution curve to the corresponding problem for f_n. Also, we discuss an example where the positive solution curves for case (1) evolve from that of case (2). In a forthcoming paper [3] we report similar results for the higher dimensional analogue of the above problem,

$$-\Delta u(x) = \lambda f(u(x)) \qquad \text{for } x \in \Omega,$$

$$u(x) = 0 \qquad \text{for } x \in \partial\Omega,$$

where Ω is the unit ball in $\mathbb{R}^N (N > 1)$.

2. FIRST AND SECOND VARIATIONS WITH RESPECT TO PARAMETERS

For any $\rho > 0$, we define $u(x, \lambda, \rho)$ to be the solution to the initial value problem

$$u''(x) + \lambda f(u(x)) = 0 \tag{2.1}$$

$$u'(0) = 0, \qquad u(0) = \rho. \tag{2.2}$$

Since solutions to (1.1), (1.2) are symmetric with respect to $x = 0$, $u(x, \lambda, \rho)$ gives a positive solution to (1.1), (1.2) if, in addition, it satisfies

$$u(x, \lambda, \rho) > 0 \qquad \text{for } x \in (0,1) \quad \text{and} \quad u(1, \lambda, \rho) = 0. \tag{2.3}$$

We shall frequently use u rather than $u(\cdot, \lambda, \rho)$. It can be easily shown that if u is a positive solution to (2.1)–(2.3) then $u(0) \geq \theta$. Let $S = \{(\lambda, u) \in \mathbb{R} \times \mathscr{C}(\overline{\Omega}) : (\lambda, u)$ satisfies (2.1)–(2.3)$\}$. We note that studying the behaviour of S is equivalent to the study of $\{(\lambda, \rho) : u(1, \lambda, \rho) = 0\}$. This follows from the continuous dependence of solutions to (2.1)–(2.3) on the initial conditions. *We identify S with the latter subset of* $\mathbb{R}^2$. Using a rescaling (see [1]) and the uniqueness of the solution to the initial value problem (2.1), (2.2) we obtain

$$u(xd, \lambda, \rho) = u(x, \lambda d^2, \rho).$$

This, after differentiating with respect to d, gives

$$u_\lambda(x, \lambda, \rho) = xu'(x, \lambda, \rho)/2\lambda. \tag{2.4}$$

Let v denote the solution to the corresponding linearized problem

$$v''(x) + \lambda f'(u(x))v(x) = 0 \tag{2.5}$$

$$v(0) = 1, \qquad v'(0) = 0. \tag{2.6}$$

Let w denote the solution to the problem

$$w''(x) + \lambda f'(u(x))w(x) = -\lambda f''(u(x))v^2(x) \tag{2.7}$$

$$w(0) = 0, \qquad w'(0) = 0. \tag{2.8}$$

Thus, v is the derivative of $u(x, \lambda, \rho)$ with respect to ρ and w is the second derivative of $u(x, \lambda, \rho)$ with respect to ρ.

LEMMA 2.1. *If* $u(x, \lambda, \rho)$ *is a positive solution to* (1.1), (1.2) *then* $u_\rho(x, \lambda, \rho)$ *has at most one zero in* $(0, 1]$.

Proof. Since u is a positive solution to (1.1), (1.2) ρ has to be $\geq \theta$. Let $x_0 \in (0, 1)$ be such that $u(x_0) = \beta$. We show that u_ρ cannot have a zero in $(0, x_0)$. Suppose, on the contrary, that there exists an $s \in (0, x_0)$ such that $u_\rho(s) = 0$. Then by setting $\psi(x) = u_\rho(x)/f(u(x))$ we see that ψ satisfies

$$\psi''(x) + \frac{2f'(u(x))u'(x)}{f(u(x))}\psi'(x) + \frac{f''(u(x))[u'(x)]^2}{f(u(x))}\psi(x) = 0 \quad (2.9)$$

for $x \in (0, x_0)$ with $\psi'(0) = 0$. Since the coefficient of ψ is negative in (2.9), by the maximum principle ψ attains its maximum at $x = 0$ and $\psi'(0) < 0$. This is a contradiction since we have $\psi'(0) = 0$.

Now we rule out the possibility of u_ρ having more than one zero in $[x_0, 1]$. Suppose, on the contrary, that s_1 and s_2 are the first two zeros of u_ρ in $[x_0, 1]$. Then there exists a $t \in (s_1, s_2)$ such that $u'_\rho(t) = 0$. From (2.1) we have

$$u'''(x) + \lambda f'(u)u'(x) = 0. \quad (2.10)$$

And since u_ρ satisfies (2.5) we have

$$u''_\rho(x) + \lambda f'(u)u_\rho(x) = 0. \quad (2.11)$$

Now multiplying (2.10) by u_ρ and (2.11) by u', subtracting one from the other, and integrating over (t, s_2) we obtain

$$u'_\rho(s_2)u'(s_2) + u''(t)u_\rho(t) = 0,$$

which is a contradiction. Hence the lemma is proven. ∎

LEMMA 2.2. *If* $u(x, \lambda_0, \rho_0)$ *is a positive solution to* (1.1), (1.2) *with* $u_\rho(1, \lambda_0, \rho_0) = 0$ *then* $u'(1, \lambda_0, \rho_0) \neq 0$ *and* $u_{\rho\rho}(1, \lambda_0, \rho_0) > 0$.

Proof. Let us assume, on the contrary, that $u'(1, \lambda_0, \rho_0) = 0$. Multiplying (2.10) by u_ρ and (2.11) by u', subtracting one from the other, and integrating over $(0, 1)$ we obtain $u''(0)u_\rho(0) = 0$, which is a contradiction to the fact that $u''(0) < 0$. Note that u_ρ satisfies (2.5), (2.6) and $u_{\rho\rho}$ satisfies (2.7), (2.8). Multiplying (2.5) by w and (2.7) by v, subtracting one from the other, and integrating over $(0, 1)$ we obtain

$$w(1)v'(1) = \int_0^1 \lambda f''(u)v^3. \quad (2.12)$$

From Lemma 2.1, we have $v > 0$ in $(0,1)$. With this (2.12) implies that $w(1) \equiv u_{\rho\rho}(1) > 0$ since $v'(1) < 0$ and $f'' < 0$. This proves the lemma. ∎

3. PROOFS OF THE THEOREMS

Multiplying (1.1) by $u'(x)$ and integrating, we obtain

$$-[u'(x)]^2/2 = \lambda F(u(x)) + C. \tag{3.1}$$

Since positive solutions are known to be symmetric with respect to $x = 0$ and $u'(x) > 0$ for $x \in (-1,0)$ we have $\rho := \operatorname{Sup}\{u(x) : x \in (-1,1)\} = u(0)$ and $\rho \geq \theta$. Taking $x = 0$ in (3.1) implies that

$$u'(x) = \sqrt{2\lambda[F(\rho) - F(u)]} \qquad \text{for } x \in [-1,0]. \tag{3.2}$$

Now integrating (3.2) over $[-1, x]$, we obtain

$$\int_0^{u(x)} \frac{du}{\sqrt{F(\rho) - F(u)}} = \sqrt{2\lambda}\,(x+1) \qquad \text{for } x \in [-1,0], \tag{3.3}$$

which in turn implies that

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^{\rho} \frac{du}{\sqrt{F(\rho) - F(u)}} := G(\rho) \tag{3.4}$$

by taking $x = 0$ in (3.3). Hence for any λ if there exists a $\rho \in [\theta, \infty)$ with $G(\rho) = \sqrt{\lambda}$, then (1.1), (1.2) has a positive solution $u(x)$ given by (3.3) satisfying $\operatorname{Sup}\{u(x) : x \in (-1,1)\} = u(0) = \rho$. In fact, $G(\rho)$ is a continuous function which is differentiable over (θ, ∞) with

$$\frac{d}{d\rho} G(\rho) = \frac{1}{\sqrt{2}} \int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{3/2}}\, dv, \tag{3.5}$$

where

$$H(t) = F(t) - (t/2) f(t). \tag{3.6}$$

For $\rho \in (\theta, \infty)$, we recall from (3.4) that

$$G(\rho) = \frac{1}{\sqrt{2}} \frac{\rho}{\sqrt{F(\rho)}} \int_0^1 \frac{dv}{\sqrt{1 - [F(\rho v)/F(\rho)]}}. \tag{3.7}$$

To prove the nonexistence of positive solutions for large values of λ we proceed as follows. Let $L(v) := F(\rho v)/F(\rho)$. Then $L(0) = 0$, $L'(v) =$

$\rho f(\rho v)/F(\rho)$, and $L''(v) = \rho^2 f'(\rho v)/F(\rho)$. In view of (1.3), for any $\rho \in (\theta, \infty)$, we have $L(v) \le v$ for $v \in [0,1]$. With this (3.7) would yield

$$G(\rho) \le \frac{1}{\sqrt{2}} \frac{\rho}{\sqrt{F(\rho)}} \int_0^1 \frac{dv}{\sqrt{1-v}} = \rho \sqrt{\frac{2}{F(\rho)}}. \tag{3.8}$$

Since $\lim_{\rho \to \infty} \rho^2/F(\rho) = 2/f'(\infty)$ it follows from (3.8) that if $\lambda > 2/f'(\infty)$ then (1.1), (1.2) has no positive solutions.

To prove the nonexistence of positive solutions for small λ, we use an eigenvalue comparison argument. Let μ_1 denote the smallest eigenvalue of

$$-\phi''(x) = \mu\phi(x) \qquad \text{for } x \in (-1,1),$$
$$\phi(-1) = 0 = \phi(1),$$

and let $\phi_1 > 0$ be an eigenfunction corresponding to μ_1. Multiplying (1.1) by ϕ_1 and integrating by parts we get

$$\mu_1 \int_{-1}^1 \phi_1 u = \lambda \int_{-1}^1 f(u)\phi_1 \le \lambda \int_{-1}^1 f'(0)\phi_1 u \tag{3.9}$$

and hence for $\lambda \le \mu_1/f'(0)$ (1.1), (1.2) has no positive solutions.

Case 1. It remains to prove that $G'(\rho) < 0$ for $\rho \in (\theta, \infty)$. We have $H'(t) = \frac{1}{2}[f(t) - tf'(t)]$ and $H''(t) = -(\frac{1}{2})tf''(t)$. Since $(1.5)_1$ holds we infer that $H'(t) < 0$ and $H''(t) > 0$ for all $t > 0$ and hence $G'(\rho) < 0$ follows from (3.5). From (3.8) $\lim_{\rho \to \infty} G(\rho)$ exists and we define $\lambda_1 = \lim_{\rho \to \infty} G(\rho)$. Now, multiplying (2.1) by $v(x)$ and (2.5) by $u(x)$, subtracting one from the other, and integrating by parts we obtain

$$\int_0^1 [f'(u(x))u(x) - f(u(x))]v(x) = 0, \tag{3.10}$$

which, in view of $(1.5)_1$, implies that $v(x)$ changes sign in $(0,1)$. Since, $v(x) \equiv u_\rho(x, \lambda, \rho)$ this implies from the theory of the linearized stability that u is unstable.

Case 2. In view of $(1.5)_2$ we have $H'(t) < 0$ for $t < \eta$, $H'(\eta) = 0$ and $H'(t) > 0$ for $t > \eta$. Since $H(0) = 0$ we have $H(t) < 0$ for $t \in (0, \eta]$ which, in turn, implies that $G'(\rho) < 0$ for $\rho \le \eta$. Since $\lim_{t \to \infty} H(t) = +\infty$ we have $H(t) > 0$ for t large and hence $G'(\rho) > 0$ for ρ large. Now we make use of the developments in Section 2 to complete the proof. For $\rho \in (\theta, \eta]$ positive solutions $u(\cdot, \lambda, \rho)$ are unstable (see (3.10)). Thus from Lemma 2.1 for $\rho \in (\theta, \eta]$, $u_\rho(x, \lambda, \rho)$ has exactly one zero in $(0,1)$ and $u_\rho(1) < 0$. Let $\rho_0 \in (\theta, \eta]$ be fixed. Let λ_0 be such that $G(\rho_0) = \sqrt{\lambda_0}$. Since $u_\rho(1, \lambda_0, \rho_0) < 0$, by the implicit function theorem there exists an

$\epsilon > 0$ and a continuous decreasing function $\sigma: (\lambda_0 - \epsilon, \lambda_0 + \epsilon) \to \mathbb{R}$ with $\sigma(\lambda_0) = \rho_0$ and such that $u(\cdot, \lambda, \sigma(\lambda))$ satisfies (2.1)–(2.3). Let $\Gamma \subset S$ denote the connected component of solutions to (2.1)–(2.3) containing $\{(\lambda, \sigma(\lambda)): \lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)\}$. Let $\lambda_1 \equiv \inf\{c : \sigma$ can be extended as a continuous function from $(c, \lambda_0 + \epsilon) \to \mathbb{R}$ such that $(\lambda, \sigma(\lambda)) \subset \Gamma\}$ and $\lambda^* \equiv \sup\{c : \sigma$ can be extended as a continuous function from $(\lambda_0 - \epsilon, c) \to \mathbb{R}$ such that $(\lambda, \sigma(\lambda)) \subset \Gamma\}$. Note that $\lambda_1 > 0$ (see (3.9)). We define $\rho_1 \equiv \sup\{\sigma(\lambda): \lambda \in (\lambda_1, \lambda^*)\}$ and $\rho^* \equiv \inf\{\sigma(\lambda): \lambda \in (\lambda_1, \lambda^*)\}$. Note that $\rho_1 > \eta$ and $\rho^* = \theta$. Also $u_\rho(1, \lambda_1, \rho_1) = 0$, for otherwise $u_\rho(1, \lambda_1, \rho_1) < 0$ which implies that σ can be extended to the left of λ_1, which contradicts the definition of λ_1. By Lemma 2.2, $u_\lambda(1, \lambda_1, \rho_1) < 0$ and $u_{\rho\rho}(1, \lambda_1, \rho_1) > 0$. These imply that there is a differentiable function $\Lambda: (\rho_1 - \epsilon, \rho_1 + \epsilon) \to \mathbb{R}$ such that $u(\cdot, \Lambda(\rho), \rho)$ is a solution to (2.1)–(2.3) for any $\rho \in (\rho_1 - \epsilon, \rho_1 + \epsilon)$. In addition, $\Lambda'(\rho_1) = 0$ with $\Lambda''(\rho_1) > 0$. We define $\rho_2 = \sup\{c : \Lambda$ can be extended as a continuous function from $(\rho_1, c) \to \mathbb{R}$ such that $(\Lambda(\rho), \rho) \subset \Gamma\}$. Note that $\rho_2 = \infty$. We define $\lambda_2 \equiv \sup\{\Lambda(\rho): \rho > \rho_1\}$. By (3.8) $\lambda_2 < \infty$. From the uniqueness of degenerate positive solution (see [5]) we conclude that $\Gamma \equiv S$. This completes the proof of (ii).

Proof of Theorem 2. The proof follows from arguments similar to the ones in case (2) above.

4. EXAMPLES

Consider $f(t) = \frac{1}{2} + t - e^{-t}$. Then $f(0) < 0$, $f'(t) > 0$ for all $t \geq 0$ and $f''(t) < 0$ for all $t \geq 0$. Then

$$\phi(t) := (f(t)/t)' = \frac{te^{-t} - (1/2) + e^{-t}}{t^2}.$$

Note that $\phi(t) > 0$ near $t = 0$. Also note that $\phi(1) > 0$ and $\phi(10) < 0$ and hence there exists an $\eta \in (1, 10)$ with $\phi(\eta) = 0$. Now, $F(t) = \int_0^t f(s)\,ds = \frac{1}{2}(t + t^2) + e^{-t} - 1$ and $F(1) > 0$. Thus $\theta \in (0, 1)$ and hence $\eta > \theta$. Moreover, $f'(\infty) = \lim_{t \to \infty} f'(t) = 1 > 0$. Thus the above f satisfies all the hypotheses of case (2) in Theorem 1. In fact, for any positive real numbers a, b, c with $a < c$ we get an $f(t) = a + bt - ce^{-t}$ satisfying all the hypotheses of case (2) in Theorem 1. Constructing examples for case (1) is very simple and hence we leave it to the reader.

5. EVOLUTION OF POSITIVE SOLUTION CURVES

Let f be a monotonically increasing, concave function which is semipositone ($f(0) < 0$) satisfying $f'(\infty) = 0$. Then, as was remarked earlier, f is in case (2). Setting $f_n(t) := (t/n) + f(t)$, for $n > 1$ we see that f_n satisfies $(1.5)_2$. Hence, we can apply Theorem 1 to study the positive solutions to the problem

$$-u''(x) = \lambda f_n(u(x)) \qquad \text{for } x \in (-1,1), \tag{5.1}$$

$$u(-1) = 0 = u(1). \tag{5.2}$$

Also, since we have $\lim_{\rho\to\infty} \rho/F_n(\rho) = \lim_{\rho\to\infty} 1/f_n(\rho) = 0$ and $\lim_{\rho\to\infty} \rho^2/F_n(\rho) = \lim_{\rho\to\infty} 2\rho/f_n(\rho) = 2/f_n'(\infty)$, (3.7) implies that that $\lim_{\rho\to\infty} G_n(\rho) \geq 1/\sqrt{n}$ using the fact that $f_n'(\infty) = 1/n$ (see [5] for details). Since $f_n \to f$, the positive solution curve to the problem

$$-u''(x) = \lambda f(u(x)) \qquad \text{for } x \in (-1,1), \tag{5.3}$$

$$u(-1) = 0 = u(1) \tag{5.4}$$

evolves from the positive solution curve to (5.1), (5.2) as $n \to \infty$. The bifurcation diagrams in Figs. 1 and 2 illustrate this very clearly.

To illustrate the evolution of positive solution curves from $(1.5)_1$ to $(1.5)_2$, we consider the sequence of nonlinearities $f_n(t) := (1/n) + t - e^{-t}$ for $n > 1$. Then f_n satisfies $(1.5)_2$. As $n \to \infty$ we have $f_n \to f := t - e^{-t}$ and f satisfies $(1.5)_1$. Let η_n be such that $tf_n'(t) - f_n(t) = 0$ at $t = \eta_n$ and let θ_n and θ denote the positive zeros of $F_n(t) := \int_0^t f_n(t)\,dt$ and $F(t) := \int_0^t f(t)\,dt$, respectively. Note that $\eta_n \to \infty$ and $\theta_n \to \theta$ as $n \to \infty$. Figures 1 and 2 illustrate the evolution of positive solution curves for this example.

We can summarize the evolution of positive solutions by considering $f(s,t) = (1-s)f_1(t) + sf_2(t)$ with f_1 satisfying $(1.5)_1$ and f_2 satisfying $(1.5)_2$. If $f_2'(\infty) > 0$ then the positive solution curves evolve from Fig. 1 to Fig. 2 as s varies from 0 to 1. Moreover, if $f_2'(\infty) = 0$ then the curves evolve from Fig. 1 to Fig. 3 passing through Fig. 2.

REFERENCES

1. A. Castro and S. Gadam, Uniqueness of stable and unstable positive solutions for semipositone problems, *Nonlinear Anal.* **22**, No. 4, 425–429.
2. A. Castro, S. Gadam, and R. Shivaji, Branches of radial solutions for semipositone problems, *J. Differential Equations* **120**, No. 1, 30–45.
3. A. Castro, S. Gadam, and R. Shivaji, Positive solution curves of semipositone problems with concave nonlinearities, *Proc. Roy. Soc. Edinburgh Sect. A* **127A** (1997), 921–934.

4. A. Castro, J. Garner, and R. Shivaji, Existence results for classes of sublinear semipositone problems, *Results. Math.* **23** (1993), 214–220.
5. A. Castro and R. Shivaji, Non-negative solutions for a class of non-positone problems, *Proc. Roy. Soc. Edinburgh Sect. A* **108** (1988), 291–302.
6. M. R. Myerscough, B. F. Gray, W. L. Hogarth, and J. Norbury, An analysis of an ordinary differential equations model for a two species predator-prey system with harvesting and stocking, *J. Math. Biol.* **30** (1992), 389–411.
7. S.-H. Wang, Positive solutions for a class of nonpositone problems with concave nonlinearities, *Proc. Roy. Soc. Edinburgh* **124**, No. 3 (1994), 507–515.