JOURNAL OF DIFFERENTIAL EQUATIONS **120**, 30–45 (1995)

Branches of Radial Solutions for Semipositone Problems*

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Received February 8, 1993; revised December 17, 1993

We consider the radially symmetric solutions to the equation

$$-\Delta u(x) = \lambda f(u(x)) \quad \text{for} \quad x \in \Omega,$$
$$u(x) = 0 \quad \text{for} \quad x \in \partial\Omega,$$

where Ω denotes the unit ball in $\mathbb{R}^N$ $(N > 1)$, centered at the origin and $\lambda > 0$. Here $f: \mathbb{R} \to \mathbb{R}$ is assumed to be semipositone $(f(0) < 0)$, monotonically increasing, superlinear with subcritical growth on $[0, \infty)$. We establish the structure of radial solution branches for the above problem. We also prove that if f is convex and $f(t)/(tf'(t) - f(t))$ is a nondecreasing function then for each $\lambda > 0$ there exists at most one positive solution u such that (λ, u) belongs to *the unbounded branch of positive solutions*. Further when $f(t) = t^p - k$, $k > 0$ and $1 < p < (N+2)/(N-2)$, we prove that the set of positive solutions is connected. Our results are motivated by and extend the developments in [4]. © 1995 Academic Press, Inc.

1. INTRODUCTION

Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable, monotone function such that

$$f(0) < 0 \quad \text{(semipositone)}, \tag{1.1}$$

$$\lim_{|d| \to \infty} f(d)/d = \infty \quad \text{(superlinear)}, \tag{1.2}$$

$$F(d) - ((N-2)/2N)\, df(d) \quad \text{is bounded below}, \tag{1.3}$$

* This research was partially supported by NSF Grants DMS-8905936, DMS-9215027 and the Texas Advanced Research Program.

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and for some $k \in (0, 1)$

$$A = \lim_{d \to \infty} (d/f(d))^{N/2} \{F(kd) - ((N-2)/2N)\, df(d)\} = \infty, \tag{1.4}$$

where $F(t) = \int_0^t f(s)\, ds$. In this paper we consider the set of radial solutions to the equation

$$-\Delta u(x) = \lambda f(u(x)) \qquad \text{for} \quad x \in \Omega, \tag{1.5}$$

$$u(x) = 0 \qquad \text{for} \quad x \in \partial\Omega, \tag{1.6}$$

with $u(0) > 0$, where Ω denotes the unit ball in $\mathbb{R}^N$ $(N > 1)$ centered at the origin, $\lambda > 0$, and Δ is the Laplacian operator. The monotonicity of f along with (1.2), implies that there exist unique positive numbers β and θ with $f(\beta) = F(\theta) = 0$. We assume that

$$f'(\beta) > 0. \tag{1.7}$$

Assumption (1.4) implies that f grows subcritically on $[0, \infty)$ (i.e., $|f(t)| \leqslant A(1 + t^p)$ with $p < (N+2)/(N-2)$). On the other hand, (1.3) ensures that f can have even the critical growth ($p = (N+2)/(N-2)$) on $(-\infty, 0]$. These assumptions are needed to prove that, for a given nonnegative integer k, if $u(0)$ is large and u has k interior nodal hypersurfaces then the parameter λ is small. If (1.4) is weakened to allow critical or supercritical growth on $[0, \infty)$, the latter might not hold (see [1]). While assumptions (1.1)–(1.4) and (1.7) can be relaxed we do not pursue in that direction in order to keep the technicalities as simple as possible. We state our main results in the following theorems.

THEOREM 1. *Let $S \subset \mathbb{R} \times \mathscr{C}(\bar{\Omega})$ be a branch (connected component) of radially symmetric solutions to (1.5)–(1.6) with $u(0) > 0$. Here $\mathscr{C}(\bar{\Omega})$ denotes the set of all continuous functions from $\bar{\Omega}$ into $\mathbb{R}$.*

(a) *If S is nonempty then there exists a nonnegative integer k such that if $(\lambda, u) \in S$ then u has $2k$ or $2k+1$ nodal hypersurfaces in Ω.*

(b) *Suppose $(\lambda_0, u_0) \in S$. The function u_0 has $2k$ nodal hypersurfaces in Ω and $\nabla u_0(x) \neq 0$ for $x \in \partial\Omega$ iff there exists $(\lambda_1, u_1) \in S$ satisfying $u_0(0) = u_1(0)$ and u_1 has $2k+1$ nodal hypersurfaces in Ω.*

THEOREM 2. (a) *There exists $\bar{\lambda} > 0$ such that if (λ, u) is a solution to (1.5)–(1.6) with $\lambda < \bar{\lambda}$ then the branch containing (λ, u) is unbounded. Moreover, $\nabla u(x) \neq 0$ for $x \in \partial\Omega$.*

(b) *For any nonnegative integer k, there exists a unique unbounded branch of solutions S_k where $S_k = \{(\lambda, u) : (\lambda, u)$ is a solution to (1.5)–(1.6)*

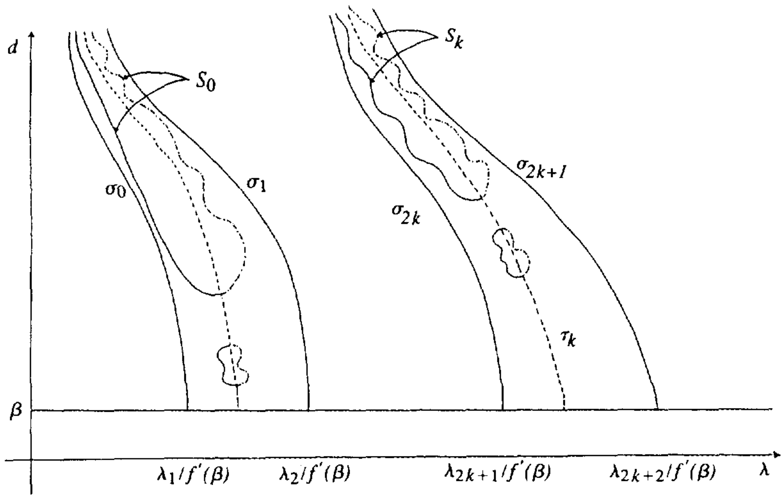

FIGURE 1

with u radial, $u(0) > 0$ *and such that u has either* $2k$ *or* $2k+1$ *nodal hypersurfaces in* $\Omega\}$.

THEOREM 3. *If, in addition,* f *is convex and* $f(t)/(tf'(t) - f(t))$ *is a nondecreasing function then for each* $\lambda > 0$ *there exists at most one positive solution* u *such that* $(\lambda, u) \in S_0$.

THEOREM 4. *If* $f(t) = t^p - k$, $k > 0$, $1 < p < (N+2)/(N-2)$, (λ, u) *satisfies* (1.5)–(1.6) *and* $u > 0$ *then* $(\lambda, u) \in S_0$.

Figure 1 summarizes the above theorems and Lemma 2.4 given below. Since we concentrate on the radial solutions to (1.5)–(1.6), we use an O.D.E. approach and our techniques include rescaling, variations with respect to parameters, energy analysis and the implicit function theorem. For other studies on the positive solutions to semipositone problems, the reader is referred to [2–4, 6, and 8–11]. However, in the direction of sign changing solutions for semipositone problems we know only of [5], where the one dimensional problem is considered.

2. PRELIMINARIES

We first note that radial solutions to (1.5)–(1.6) correspond to solutions to the singular problem

$$u'' + ((N-1)/r)\,u' + \lambda f(u) = 0 \qquad \text{for} \quad r \in [0, 1], \tag{2.1}$$

$$u'(0) = 0, \tag{2.2}$$

$$u(1) = 0, \tag{2.3}$$

where $'$ denotes the differentiation with respect to $r = \|x\|$. For $d > 0$ we define $u(\cdot, \lambda, d) := u(\cdot)$ as the solution to (2.1), (2.2) and $u(0) = d$. For future reference, we note that $S = \{(\lambda, u) \in \mathbb{R} \times \mathscr{C}(\bar{\Omega}) : (\lambda, u)$ satisfies (2.1)–(2.3)$\}$ is connected iff $\{(\lambda, u(0)) : (\lambda, u) \in S\}$ is connected. This is an immediate consequence of the continuous dependence of solutions to (2.1)–(2.3) on the initial conditions. *To facilitate the proofs of above theorems, we identify S with the latter set in $\mathbb{R}^2$.*

We define $E(r) := E(r, \lambda, d) = (u'(r))^2 + 2\lambda F(u(r))$. From (2.1) it follows that $E'(r) = -2(N-1)(u'(r))^2/r$. Hence, by the uniqueness of solutions to (2.1) subject to initial conditions we see that

$$\text{if } u(s) = 0 \text{ then } E(r) > 0 \text{ for all } r \in [0, s). \tag{2.4}$$

Thus if u is a solution to (2.1)–(2.3) then $(u, u') \neq (0, 0)$ for all $r \in [0, 1)$ (see (2.4)).

For any $\rho > 0$, if w is defined by $w(r) = u(r\rho, \lambda, d)$, then w satisfies

$$w''(r) + ((N-1)/r)\,w'(r) + \lambda\rho^2 f(w(r)) = 0 \qquad \text{for} \quad r > 0,$$

$$w'(0) = 0 \qquad \text{and} \qquad w(0) = d.$$

By the uniqueness of the solution to an initial value problem, this implies that

$$u(r\rho, \lambda, d) = u(r, \lambda\rho^2, d). \tag{2.5}$$

Differentiating with respect to ρ and taking $\rho = 1$, we obtain

$$u_\lambda(r, \lambda, d) = ru'(r, \lambda, d)/2\lambda, \tag{2.6}$$

where u_λ denotes the derivative of u with respect to λ.

An immediate consequence of (2.5) is:

LEMMA 2.1. *Given a nonnegative integer j, for each $d > \beta$ there exists at most one $\lambda > 0$ such that $u(\cdot, \lambda, d)$ has exactly j zeroes in $(0, 1)$ and satisfies (2.1)–(2.3).*

Proof. Suppose, on the contrary that, for some $d > \beta$ we have $u(\cdot, \lambda_1, d)$ and $u(\cdot, \lambda_2, d)$ having exactly j zeroes in $(0, 1)$ and satisfying

(2.1)–(2.3). Without loss of generality we can assume that $\lambda_1 < \lambda_2$. From (2.5) we have

$$u(r\rho, \lambda_1, d) = u(r, \lambda_1 \rho^2, d).$$

Taking $\rho^2 = \lambda_2/\lambda_1$, we get

$$u(r\sqrt{\lambda_2/\lambda_1}, \lambda_1, d) = u(r, \lambda_2, d),$$

from which we get that $u(\cdot, \lambda_2, d)$ has $j+1$ zeroes in $(0, \sqrt{\lambda_1/\lambda_2}] \subset (0, 1)$. This contradicts our assumption that $u(\cdot, \lambda_2, d)$ has exactly j zeroes in $(0, 1)$ and hence proves the lemma. ∎

We let $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty$ to denote the eigenvalues of the problem:

$$\phi'' + ((N-1)/r)\,\phi' + \lambda\phi = 0 \qquad \text{for} \quad r \in [0, 1], \tag{2.7}$$

$$\phi'(0) = 0, \tag{2.8}$$

$$\phi(1) = 0. \tag{2.9}$$

Let ϕ_k denote the eigenfunction corresponding to the eigenvalue λ_k such that $\phi_k(0) = 1$. The following lemma describes the β-levels of $u(1, \lambda, d)$.

LEMMA 2.2. *For each nonnegative integer k there exists a differentiable function* $\sigma_k : (\beta, \infty) \to \mathbb{R}$ *such that* $u(1, \sigma_k(d), d) - \beta = 0$ *and* $u(\cdot, \sigma_k(d), d) - \beta$ *has k zeroes in* $(0, 1)$. *Moreover,* $\lim_{d \to \infty} \sigma_k(d) = 0$ *and* $\lim_{d \to \beta} \sigma_k(d) = \lambda_{k+1}/f'(\beta)$. *Conversely, if* $u(1, \lambda, d) = \beta$ *and* $u(\cdot, \lambda, d) - \beta$ *has k zeroes in* $(0, 1)$ *then* $\lambda = \sigma_k(d)$.

Proof. Since $f'(\beta) > 0$, and the eigenvalues to

$$\phi'' + ((N-1)/r)\,\phi' + \mu f'(\beta)\,\phi = 0 \qquad \text{for} \quad r \in [0, 1], \tag{2.10}$$

$$\phi'(0) = 0, \tag{2.11}$$

$$\phi(1) = 0, \tag{2.12}$$

are simple, from standard bifurcation arguments, it follows that $(\lambda_k/f'(\beta), \beta)$ are the points of bifurcation for the equation $u(1, \lambda, d) - \beta = 0$. Note that, because $f(\beta) = 0$, the ray $\{(\lambda, \beta) : \lambda > 0\}$ satisfies $u(1, \lambda, \beta) = \beta$. Also, for $d > \beta$ close to β and λ close to $\lambda_k/f'(\beta)$, well established result on bifurcation from simple eigenvalues implies that, $u(\cdot, \lambda, d)$ is of the form $s\phi_k + o(s)$, which in particular, implies that

$$u(\cdot, \lambda, d) - \beta \text{ has } k-1 \text{ zeroes in } (0, 1). \tag{2.13}$$

By the uniqueness of solutions to the initial value problem (2.1), (2.2), $u(0)=d$, we see that if $d>\beta$ and $u(t,\lambda,d)=\beta$ then $u'(t,\lambda,d)\neq 0$. In particular, from (2.6) we have

$$u_\lambda(t,\lambda,d)\neq 0 \qquad \text{if} \quad u(t,\lambda,d)=\beta. \tag{2.14}$$

This and the implicit function theorem imply that if $J\subset\{(\lambda,d):\lambda>0, d>\beta\}$ is a connected component of solutions to $u(1,\lambda,d)=\beta$, then there exists a nonnegative integer k such that if $u(\cdot,\lambda,d)-\beta$ has k zeroes in $(0,1)$ for each $(\lambda,d)\in J$. Also, from (2.14) we infer that $J=\{(\sigma_k(d),d): d\in(\beta,\infty)$ and $\sigma_k:(\beta,\infty)\to\mathbb{R}$ is a differentiable function$\}$.

Conversely, suppose $u(1,\lambda,d)=\beta$ and $u(\cdot,\lambda,d)-\beta$ has k zeroes in $(0,1)$. If we assume that $\lambda>\sigma_k(d)$, then by taking $\rho^2=\sigma_k(d)/\lambda$ in (2.5) we get that $u(\cdot,\lambda,d)$ has at least $k+1$ zeroes in $(0,1)$ which contradicts our assumption that $u(\cdot,\lambda,d)-\beta$ has exactly k zeroes in $(0,1)$. Similarly the possibility that $\lambda<\sigma_k(d)$ is ruled out. Since (2.13) implies that

$$\lim_{d\to\beta}\sigma_k(d)=\lambda_{k+1}/f'(\beta), \tag{2.15}$$

we have $J_k:=J=\{(\lambda,d):u(1,\lambda,d)=\beta$ and $u(\cdot,\lambda,d)-\beta$ has exactly k zeroes in $(0,1)\}$. Finally, the superlinearity of f along with (1.4) imply that $\lim_{d\to\infty}\sigma_k(d)=0$ (for details, see [7]). Hence the lemma is proven. ∎

LEMMA 2.3. *For any $d>\beta$ and for any nonnegative integer k, one of the following holds.*

(a) *There exists no $\lambda\in(\sigma_{2k}(d),\sigma_{2k+1}(d))$ for which $u(\cdot,\lambda,d)$ satisfies* (2.1)–(2.3).

(b) *There exists exactly one $\lambda_0\in(\sigma_{2k}(d),\sigma_{2k+1}(d))$ for which $u(\cdot,\lambda_0,d)$ satisfies* (2.1)–(2.3). *In this case $u(\cdot,\lambda_0,d)$ has $2k$ zeroes in* $(0,1)$ *and $u'(1,\lambda_0,d)=0$.*

(c) *There exist $\lambda_0,\lambda_1\in(\sigma_{2k}(d),\sigma_{2k+1}(d))$ with $\lambda_0<\lambda_1$ and $u(\cdot,\lambda_i,d)$ satisfying* (2.1)–(2.3) *for $i=0,1$. In this case $u_i:=u(\cdot,\lambda_i,d)$ has exactly $2k+i$ zeroes in* $(0,1)$ *and $u'(1,\lambda_i,d)\neq 0$ for $i=0,1$.*

Proof. Suppose $u(\cdot,\lambda,d)$ satisfies (2.1)–(2.3). First, we observe that

$$u(\cdot,\lambda,d) \text{ has } 2k \text{ or } 2k+1 \text{ interior zeroes iff } \sigma_{2k}(d)<\lambda<\sigma_{2k+1}(d). \tag{2.16}$$

Indeed, suppose u has $2k$ or $2k+1$ zeroes in $(0,1)$ and $\lambda<\sigma_{2k}(d)$. By (2.4) we see that $E(t,\lambda,d)>0$ for all t in $[0,1)$. Hence, since $F<0$ on $[0,\theta]$ if $u'(r,\lambda,d)=0$ then $u(r,\lambda,d)$ is not in $[0,\theta]$. Thus if $s<r$ are two consecutive zeroes of $u(r,\lambda,d)$ with $u'(s,\lambda,d)>0$ then $u(,\lambda,d)-\beta$ has exactly two zeroes in (s, r). Furthermore, because $d > \theta > \beta$ we see that $u(\cdot, \lambda, d) - \beta$ has $2k+1$ zeroes in $(0, 1)$. Since

$$u(r\sqrt{\sigma_{2k}(d)/\lambda}, \lambda, d) - \beta = u(r, \sigma_{2k}(d), d) - \beta,$$

we see that $u(\cdot, \sigma_{2k}(d), d) - \beta$ has $2k+1$ zeroes in $(0, \sqrt{\lambda/\sigma_{2k}(d)})$. This contradicts the fact that $u(\cdot, \sigma_{2k}(d), d) - \beta$ has $2k$ zeroes in $(0, 1)$ and hence proves that $\sigma_{2k}(d) < \lambda$. The proof that $\lambda < \sigma_{2k+1}(d)$ follows along the same lines and we leave it to the reader.

Conversely, suppose $u(\cdot, \lambda, d)$ satisfies (2.1)–(2.3) and $\sigma_{2k}(d) < \lambda < \sigma_{2k+1}(d)$. Since, (2.5) implies,

$$u(r\sqrt{\sigma_{2k}(d)/\lambda}, \lambda, d) = u(r, \sigma_{2k}(d), d),$$

$u(\cdot, \lambda, d)$ has $2k$ zeroes in $(0, \sqrt{\sigma_{2k}(d)/\lambda}) \subset (0, 1)$. Similarly, since

$$u(r\sqrt{\sigma_{2k+1}(d)/\lambda}, \lambda, d) = u(r, \sigma_{2k+1}(d), d),$$

$u(\cdot, \lambda, d)$ has $2k+2$ zeroes in $(0, \sqrt{\sigma_{2k+1}(d)/\lambda}) \supset (0, 1]$. But because between the $(2i+1)$th and the $(2i+2)$th zero of $u(\cdot, \lambda, d) - \beta$ there are two zeroes of $u(\cdot, \lambda, d)$ we see that $u(\cdot, \lambda, d)$ has $2k$ zeroes in $(0, (\sigma_{2k}/\lambda)^{(1/2)})$. Since $u(1, \lambda, d) = 0$ we conclude that $u(\cdot, \lambda, d)$ has at most $2k+1$ zeroes in $(0, 1)$ which completes the proof of (2.16). From Lemma 2.1 and (2.16) we see that for any $d > \beta$ there exists at most one $\lambda_i \in (\sigma_{2k}(d), \sigma_{2k+1}(d))$ for which $u(\cdot, \lambda_i, d)$ satisfies (2.1)–(2.3) and has exactly $2k+i$ zeroes in $(0, 1)$ for $i = 0, 1$.

Suppose there exists a λ^* with $u(1, \lambda^*, d) = 0$ and $u'(1, \lambda^*, d) \neq 0$. If $u'(1, \lambda^*, d) < 0$, then by the intermediate value theorem, there exists $T \in (1, \sqrt{\sigma_{2k+1}(d)/\lambda^*})$ such that $u(T, \lambda^*, d) = 0$. Thus $u(1, \lambda^* T^2, d) = 0$ and $\lambda^* < \lambda^* T^2 < \sigma_{2k+1}(d)$. Taking $\lambda_0 = \lambda^*$ and $\lambda_1 = \lambda^* T^2$, we see that (c) holds. On the other hand, if $u'(1, \lambda^*, d) > 0$, there exists $T \in (\sqrt{\sigma_{2k}(d)/\lambda^*}, 1)$ such that $u(T, \lambda^*, d) = 0$. Thus $u(1, \lambda^* T^2, d) = 0$ and $\sigma_{2k}(d) < \lambda^* T^2 < \lambda^*$. Taking $\lambda_0 = \lambda^* T^2$ and $\lambda_1 = \lambda^*$ we see that (c) holds. On the other hand, if such a λ^* does not exist (i.e., if (c) does not hold), then either (a) holds, or else for every $\lambda \in (\sigma_{2k}(d), \sigma_{2k+1}(d))$ that $u(1, \lambda, d) = 0$ we have $u'(1, \lambda, d) = 0$. Since in the latter case (see (2.4)) there can be at most one such λ, we see that (b) holds. ∎

LEMMA 2.4. *For any nonnegative integer k, there exists a differentiable function τ_k such that* $\{(\lambda, d) : \sigma_{2k}(d) < \lambda < \sigma_{2k+1}(d)$ *and* $u'(1, \lambda, d) = 0\} = \{(\tau_k(d), d) : d > \beta\}$. *If S is a connected component of solutions to* (2.1)–(2.3) *then $S = U \cup V$ where* $U = \{(\lambda, d) \in S : u(\cdot, \lambda, d)$ *has $2k$ zeroes in* $(0, 1)\}$ *and* $V = \{(\lambda, d) \in S : u(\cdot, \lambda, d)$ *has $2k+1$ zeroes in* $(0, 1)\}$. *Moreover, there exists* *a $\sigma: I \to \mathbb{R}$ such that $U = \{(\sigma(d), d) : d \in I\}$, with $I = [a, b]$ if $b < \infty$ and $I = [a, b)$ if $b = \infty$ where $a = \min\{d: (\lambda, d) \in S\}$ and $b = \sup\{d: (\lambda, d) \in S\}$.*

Proof. For each $d > \beta$, we have $u'(1, \sigma_{2k}(d), d) < 0$ and $u'(1, \sigma_{2k+1}(d), d) > 0$. Hence there exists a $\lambda \in (\sigma_{2k}(d), \sigma_{2k+1}(d))$ such that $u'(1, \lambda, d) = 0$. Since, from (2.6)

$$u_{\lambda\lambda}(1, \lambda, d) = u''(1, \lambda, d)/2\lambda, \tag{2.17}$$

we get that $u_{\lambda\lambda}(1, \lambda, d) > 0$ because $f(t) < 0$ if $t < \beta$. Hence such a λ is unique and we write $\lambda = \tau_k(d)$. By the implicit function theorem (see (2.17)) τ_k is a differentiable function. Moreover, if $\lambda < \tau_k(d)$ then $u'(1, \lambda, d) < 0$ and $u'(1, \lambda, d) > 0$ for $\lambda > \tau_k(d)$. Thus if u is a solution to (2.1)–(2.3) with $2k+1$ interior zeroes then $\lambda > \tau_k(d)$ whereas if u is a solution to (2.1)–(2.3) with $2k$ interior zeroes then $\lambda \leqslant \tau_k(d)$. Observe that if S is unbounded then $b = \infty$. We claim that for each $d \in [a, b]$ there exists a unique $\sigma(d) \in (\sigma_{2k}(d), \tau_k(d)]$ such that $(\sigma(d), d) \in S$ and σ is a continuous function. The uniqueness follows from Lemma 2.3. Suppose that $S \cap \{(\lambda, d) : \sigma_{2k}(d) < \lambda \leqslant \tau_k(d)\} = \phi$ for some $d \in (a, b)$. From Lemma 2.3 again, this implies that, $S \cap \{(\lambda, d) : \sigma_{2k}(d) < \lambda \leqslant \sigma_{2k+1}(d)\} = \phi$, which contradicts that S is connected. That σ is also defined at a, or b if $b < \infty$, and that σ is continuous follow from the continuous dependence of u on the parameters.

For future reference we note that assumptions (1.3) and (1.4) imply the functions $u(, \lambda, d)$ do not degenerate in $[0, 1]$ if λ is small and d is large. Indeed we have

$$t^N E(t) + ((N-2)/2)\, t^{N-1} u(t)\, u'(t) \geqslant \lambda\{C(d)\, \lambda^{-N/2} + B/N\}, \tag{2.18}$$

where $B < 0$ is a lower bound for the expression in (1.3) and $C(d)$ tends to infinity as d tends to infinity. For the details of the proof of inequality (2.18) we refer the reader to Lemma 3.2 of [9]. From (2.18) we have

$$\begin{gathered}\text{There exists } D \text{ and } \Lambda \text{ such that if } d > D \text{ and } \lambda > \Lambda \\ \text{then } E(r, \lambda, d) > 0 \text{ for all } r \text{ in } [0, 1]. \quad \blacksquare\end{gathered} \tag{2.19}$$

3. Proof of Theorem 1

Part (a)

Let Γ be a connected compenent of $\{(\lambda, d) : u(\cdot, \lambda, d)$ is a solution to (2.1)–(2.3)$\}$. Let $(\lambda_0, d_0) \in \Gamma$. By (2.4) $u_0 := u(\cdot, \lambda_0, d_0)$ has (say) j zeroes in $(0, 1)$. Let $\Sigma = \{(\lambda, d) \in \Gamma : u(\cdot, \lambda, d)$ has j or $j+1$ zeroes in $(0, 1)\}$ if j is even, and $\Sigma = \{(\lambda, d) \in \Gamma : u(\cdot, \lambda, d)$ has $j-1$ or j zeroes in $(0, 1)\}$ if j is odd. We claim that in either case $\Gamma = \Sigma$.

We prove that Σ is both open and closed. In fact, let $(\lambda_1, d_1) \in \Sigma$. If $u'(1, \lambda_1, d_1) \neq 0$, by continuous dependence on parameters we see that if (λ, d) is close to (λ_1, d_1) then $u(\cdot, \lambda, d)$ has as many zeroes as $u(\cdot, \lambda_1, d_1)$. On the other hand, if $u'(1, \lambda_1, d_1) = 0$, from (2.1) we see that $u(\cdot, \lambda_1, d_1)$ has a local minimum at 1. However, by (2.4), all zeroes of $u(, \lambda, d)$ in $(0, 1)$ are nondegenerate. Hence, since $u(0, \lambda_1, d_1) > 0$ we see that $u(\cdot, \lambda_1, d_1)$ has an even number of zeroes in $(0, 1)$. That is, $u(\cdot, \lambda_1, d_1)$ has $2k \in \{j-1, j\}$ zeroes in $(0, 1)$. Let $a_1 < a_2 < \cdots < a_{2k} \in (0, 1)$ be the critical points of $u(\cdot, \lambda_1, d_1)$. Since $d > 0$, again by (2.4), we have that $u(a_i, \lambda_1, d_1) < 0$ if i is odd and $u(a_i, \lambda_1, d_1) > 0$ if i even. By the continuous dependence of u on parameters we see that if (λ, d) is close to (λ_1, d_1) then $u(\cdot, \lambda, d)$ has a zero in each interval of the form (a_i, a_{i+1}), $i = 0, ..., 2k$ where $a_0 = 0$ and $a_{2k+1} = 1$. In addition, $u(a_i, \lambda_1, d_1) \cdot u(a_i, \lambda, d) > 0$ for $i = 0, ..., 2k$. Thus $u(\cdot, \lambda, d)$ has at least $2k$ zeroes in $(0, 1)$. Let us see that $u(\cdot, \lambda, d)$ can not have more than $2k+1$ zeroes in $(0, 1)$. Suppose there is a sequence (λ_n, d_n) converging to (λ_1, d_1) and such that each $u(\cdot, \lambda_n, d_n)$ has at least $2k+2$ zeroes in $(0, 1)$. Hence, by taking a subsequence if necessary, we can assume that there exists $l \in \{0, 1, ..., 2k+1\}$ such that $u(\cdot, \lambda_n, d_n)$ has two zeroes $\alpha_n, \beta_n \in (a_l, a_{l+1})$. Since $u(a_l, \lambda_n, d_n) \cdot u(a_{l+1}, \lambda_n, d_n) \leqslant 0$, without loss of generality we can assume that

$$u'(\alpha_n, \lambda_n, d_n)[u(a_{l+1}, \lambda_1, d_1) - u(a_l, \lambda_1, d_1)] < 0. \tag{3.1}$$

Since $\{\alpha_n\} \subset (a_l, a_{l+1})$, without loss of generality, we can assume that $\alpha_n \to \alpha \in [a_l, a_{l+1}]$. By the continuous dependence on parameters, we have

$$u(\alpha, \lambda_1, d_1) = \lim_{n \to \infty} u(\alpha_n, \lambda_n, d_n) = 0.$$

Hence $\alpha \in (a_l, a_{l+1})$. From (3.1) we see that

$$u'(\alpha, \lambda_1, d_1)[u(a_{l+1}, \lambda_1, d_1) - u(a_l, \lambda_1, d_1)] < 0, \tag{3.2}$$

which contradicts the fact that $u(\cdot, \lambda_1, d_1)$ has exactly one zero in (a_l, a_{l+1}). Thus, for (λ, d) close to (λ_1, d_1), we conclude that $u(\cdot, \lambda, d)$ has either $2k$ or $2k+1$ zeroes in $(0, 1)$ and hence Σ is an open subset of Γ.

Also, if $\{\lambda_n, d_n\} \subset \Sigma$ converges to (λ^*, d^*), then by the continuous dependence of u on parameters, $u(\cdot, \lambda^*, d^*)$ is a solution to (2.1)–(2.3). By the definition of Σ, we can assume that for all n, $u(\cdot, \lambda_n, d_n)$ has the same number of zeroes (say j) in $(0, 1)$. If j is even, then $u(\cdot, \lambda^*, d^*)$ has j zeroes in $(0, 1)$ and hence $(\lambda^*, d^*) \in \Sigma$. If j is odd, then $u(\cdot, \lambda^*, d^*)$ has j zeroes or $j-1$ zeroes in $(0, 1)$ and hence $(\lambda^*, d^*) \in \Sigma$. This proves that Σ is closed and hence $\Gamma = \Sigma$ as claimed, which concludes the proof of part (a).

Part (b)

Let $(\lambda_i, u_i) \in S$ with u_i having $2k+i$ zeroes in $(0, 1)$ and such that $u'(1, \lambda_i, d_0) \neq 0$. Here $d_0 = u_i(0)$, for $i = 0, 1$; that is $u_i := u(\cdot, \lambda_i, d_0)$. Suppose $S \cap \{(\lambda, d_0) : \lambda \in \mathbb{R}\} = \{(\lambda_i, d_0)\}$. Since $u_\lambda(1, \lambda_i, d_0) \neq 0$ (see (2.6)), by the implicit function theorem, there exists a differentiable function $\chi: (d_0 - \varepsilon, d_0 + \varepsilon) \to \mathbb{R}$ such that $u(1, \chi(d), d) = 0$ with $\chi(d_0) = \lambda_i$. Then, by the definition of χ, $S - \{(\lambda_i, d_0)\}$ is disconnected. Let us see that this is not possible.

If 0 is a regular value of $u(1, \lambda, d)$, then S is homeomorphic to either the unit circle in $\mathbb{R}^2$ or the open interval $(-1, 1)$. If S is homeomorphic to the unit circle S^1 and $h: S^1 \to S$ denotes a homeomorphism, then $S - \{(\lambda_i, d_0)\}$ is homeomorphic to $S^1 - \{h^{-1}(\lambda_i, d_0)\}$. This is a contradiction since $S - \{(\lambda_i, d_0)\}$ is not connected whereas $S^1 - \{h^{-1}(\lambda_i, d_0)\}$ is connected. On the other hand, if S is homeomorphic to $(-1, 1)$, and $S \cap \{(\lambda, d_0): \lambda > 0\} = \{(\lambda_i, d_0)\}$, we let $h: (-1, 1) \to S$ to denote a homeomorphism with $h(0) = (\lambda_i, d_0)$. Without loss of generality, we can assume that $S \cap \{(\lambda, d_0) : \lambda \leqslant \lambda_i\} = h([0, 1))$. Since $\{(\lambda, d) : \beta \leqslant d \leqslant d_0, \ \sigma_{2k}(d) \leqslant \lambda \leqslant \sigma_{2k+1}(d)\}$ is compact, we see that $\{h(n/(n+1)) : n \in \mathbb{N}\}$ has a limit point $(\hat{\lambda}, \hat{d})$. By the continuity of u we see that $(\hat{\lambda}, \hat{d}) \in S$. Hence $(\hat{\lambda}, \hat{d}) = h(x)$ for some $x \in [0, 1)$. By the implicit function theorem $(\hat{\lambda}, \hat{d}) \neq (\lambda_i, d_0)$. Thus, $(\hat{\lambda}, \hat{d}) = h(x)$ for some $x \in (0, 1)$ and hence $h([x, 1))$ is a closed loop, which contradicts the fact that S is homeomorphic to $(-1, 1)$. Thus there exists a λ_j ($j \neq i$, and $i, j \in \{0, 1\}$) with $u(1, \lambda_j, d_0) = 0$ and by Lemma 2.3 (c) we have $u'(1, \lambda_j, d_0) \neq 0$.

If 0 is not a regular value, we let $\{\varepsilon_n : n \geqslant 1\}$ to denote regular values of $u(1, \lambda, d)$ converging to 0. We let G_n to denote the connected component of $\{(\lambda, d) : u(1, \lambda, d) - \varepsilon_n = 0\}$ containing (λ_n, d_0) with λ_n converging to λ_i. Arguing as above, we see that for each n there exists $(\bar{\lambda}_n, d_0) \in G_n$ with $\bar{\lambda}_n \neq \lambda_n$ and $u(1, \bar{\lambda}_n, d_0) - \varepsilon_n$ has $2k+j$ zeroes in $(0, 1)$; $j \neq i$ and $i, j \in \{0, 1\}$. Taking $\lambda_j = \lim_{n \to \infty} \bar{\lambda}_n$, we see that $(\lambda_j, d_0) \in S$ and u_j has $2k+j$ zeroes in $(0, 1)$. This completes the proof of the theorem. ∎

4. Proof of Theorem 2

Part (a)

Since the basic idea of this part of the proof can be found in Lemma 3.2 of [9], we omit the details given in [9]. By (1.3) and (1.4) we see that

$$t^N E(t) + ((N-2)/2)\, t^{N-1} u(t)\, u'(t) \geqslant \lambda \{ C(d)\, \lambda^{-N/2} + B/N \}, \tag{4.1}$$

where $B \leqslant 0$ is a lower bound for the expression in (1.3) and $C(d) \to \infty$ as $d \to \infty$ (see (1.4)). From (4.1) we see that there exists $\bar{\lambda} > 0$ and $\bar{d} > 0$ such that if $\lambda < \bar{\lambda}$ and $d > \bar{d}$ then $u_\lambda(1, \lambda, d) \neq 0$ (see (2.5)). Thus, by the implicit function theorem, if $u(1, \lambda, d) = 0$ then there exists a differentiable function $\tau\colon [d - \varepsilon, \infty) \to \mathbb{R}$ such that $u(1, \tau(d), d) = 0$. In particular, the component containing (λ, u) is unbounded, which proves part (a).

Part (b)

Now we prove that for any nonnegative integer k there exists an unbounded component of solutions having $2k$ or $2k+1$ interior zeroes. Let $D_1 > D$ (see 2.19) be such that for $d > D_1 \sigma_{(2k+1)}(d) < \bar{\lambda}$. By part (a) we know that $E(1, \sigma_{(2k+1)}(d), d) > 0$. Hence there exists $\rho < 1$ such that $u(\rho, \sigma_{(2k+1)}(d), d) = 0$. Hence $u(\cdot, (\rho)^2 \sigma_{(2k+1)}(d), d)$ is a solution to (2.1)–(2.3) having $2k+1$ interior zeroes. Since $(\rho)^2 \sigma_{(2k+1)}(d) < \bar{\lambda}$ and $d > D_1$ by part (a) we see that $u'(1, (\rho)^2 \sigma_{(2k+1)}(d), d) > 0$. This (see 2.6) and the implicit function theorem imply the existence of a decresing function $s\colon [d, +\infty) \to (0, \bar{\lambda})$ such that $u(\cdot, s(d), d)$ is a solution to (2.1)–(2.3) is a solution having $2k+1$ interior zeroes. Thus the connected component containing $u(\cdot, (\rho)^2 \sigma_{(2k+1)}(d), d)$ is unbounded which proves the claim.

Suppose now that S_k and S'_k are two unbounded components of solutions containing $2k$ and $2k+1$ zeroes in $(0, 1)$. By (3.3) we know that S_k and S'_k lie between J_{2k} and J_{2k+1} and hence are bounded in the λ direction (see Lemma 2.2). Thus S_k and S'_k are unbounded in the d direction and hence we can choose $\lambda_1 \neq \lambda_2$ and d such that $(\lambda_1, u_1) \in S_k$ and $(\lambda_2, u_2) \in S'_k$, where $u_i = u(\cdot, \lambda_i, d)$ for $i = 1, 2$. If both u_1 and u_2 have $2k$ zeroes in $(0, 1)$, then it is a contradiction to Lemma 2.1. So, we may assume that u_1 has $2k$ interior zeroes and u_2 has $2k+1$ interior zeroes. From Lemma 2.3, we have $u'_1(1) \neq 0$. Then, by part (b) of Theorem 1, there exists a $\bar{\lambda}_1 > \lambda_1$ such that $(\bar{\lambda}_1, \bar{u}_1) \in S_k$ with $\bar{u}_1(0) = u_2(0) = d$ and such that $\bar{u}_1 := u(\cdot, \bar{\lambda}_1, d)$ has $2k+1$ zeroes. By virtue of (2.5) $\bar{\lambda}_1 \neq \lambda_2$. This is a contradiction to Lemma 2.1 since both $(\bar{\lambda}_1, \bar{u}_1)$ and (λ_2, u_2) satisfy (2.1)–(2.3) with $\bar{u}_1(0) = u_2(0) = d$ and $\bar{u}_1, u_2$ have $2k+1$ interior zeroes. This proves the uniqueness of the unbounded branch S_k. ∎

5. Proof of Theorem 3

To prove that for any $\lambda > 0$ there is at most one positive solution $u(r, \lambda, d)$ such that $(\lambda, d) \in S_0$, we concentrate our study on the variations of $u(r, \lambda, d)$ with respect to the parameters λ and d. The following lemma on the zeroes of u_d, the derivative of u with respect to d, is crucial and is an extension of Lemma 3.1 of [4].

LEMMA 5.1. *If* $u(1, \lambda, d) = 0$, $u > 0$ *on* $[0, 1)$, *and* $u_d(r, \lambda, d)$ *has exactly one zero in* $(0, 1)$ *then* $u_d(1) < 0$.

Proof. Let $r_0 \in (0, 1)$ be such that $u(r_0) = \beta$. Such an r_0 exists since $u(0, \lambda, d) > \theta$. Let s denote the zero of u_d in $(0, 1)$. Let us see that $s < r_0$. For any $\gamma \in (0, \beta)$, let $r_\gamma \in (0, 1)$ be such that $u(r_\gamma) = \gamma$. We define $g(t) = f(t + \gamma)$ and $w(r) = u(r) - \gamma$. Thus w satisfies (see (2.1))

$$(r^{N-1}w')' + \lambda r^{N-1} g(w(r)) = 0. \tag{5.1}$$

Differentiating with respect to d we have

$$(r^{N-1}w_d')' + \lambda r^{N-1} g'(w)\, w_d = 0, \tag{5.2}$$

$$w_d(0) = 1 \qquad \text{and} \qquad w_d'(0) = 0. \tag{5.3}$$

Multiplying (5.1) by w_d, (5.2) by w, subtracting and integrating by parts, we get

$$r_\gamma^{N-1} w'(r_\gamma)\, w_d(r_\gamma) = \lambda \int_0^{r_\gamma} r^{N-1}(wg'(w) - g(w))\, w_d\, dr. \tag{5.4}$$

Since $w'(r_\gamma) < 0$ and g is convex with $g(0) < 0$, w_d has a zero in $(0, r_\gamma]$. Since this holds for any $\gamma \in (0, \beta)$ and since $w_d = u_d$, we conclude that $s \leqslant r_0$.

Since ψ is nondecreasing, for each $k \in [0, 2\psi(u(0)))$ the function $\varphi(u(r), k) := kf'(u(r))\, u(r) - (k + 2)\, f(u(r))$ has exactly one zero $r^*(k) \in [0, 1]$. Indeed, r^* is a continuous function of k and is given by $k = 2\psi(u(r^*))$. Moreover,

$$\varphi(u(r), k) < 0 \text{ for } r < r^*(k), \text{ and } \varphi(u(r), k) > 0 \text{ for } r > r^*(k). \tag{5.5}$$

Since $r^*(0) = r_0$ and $r^*(2\psi(d)) = 0$, by the continuity of r^*, there exists a $k^* \in [0, 2\psi(d))$ such that $r^*(k^*) = s$. Thus we have

$$\varphi(u(r), k^*)\, u_d(r) \leqslant 0 \qquad \text{in } [0, 1]. \tag{5.6}$$

Now, defining $v(r) = ru'(r) + k^* u(r)$, it can be seen that

$$(r^{N-1}v')' + \lambda r^{N-1} f'(u)\, v = \lambda r^{N-1} \varphi(u(r), k^*). \tag{5.7}$$

Differentiating (2.1) with respect to d we get

$$(r^{N-1}u_d')' + \lambda r^{N-1} f'(u)\, u_d = 0, \tag{5.8}$$

$$u_d(0) = 1 \qquad \text{and} \qquad u_d'(0) = 0. \tag{5.9}$$

Now, multiplying (5.7) by u_d, (5.9) by v, subtracting and integrating by parts, we get

$$v'(1)\, u_d(1) - u'_d(1)\, v(1) = \lambda \int_0^1 r^{N-1} \varphi(u(r), k^*)\, u_d(r)\, dr. \tag{5.10}$$

If we assume that $u_d(1) = 0$, then the integral in (5.10) is nonnegative, which is a contradiction to (5.6). This proves the lemma. ∎

Proof of Theorem 3. By Lemma 2.4 we know that $S_0 = U_0 \cup V_0$, with $U_0 = \{(\sigma(d), d) : d \geqslant a\}$. From Lemma 3.1 of [4] we know that if d is large (i.e., λ small) then $u_d(\cdot, \sigma(d), d)$ vanishes exactly once in $[0, 1]$ and $u_d(1, \sigma(d), d) < 0$. We claim that this holds for all $d \geqslant a$. If not, taking $d_1 = \inf\{d : u_d(\cdot, \sigma(d), d)$ has exactly one zero in $[0, 1]\}$, we see that $u_d(1, \sigma(d_1), d_1) = 0$. Then by [6] and the definition of d_1, $u_d(\cdot, \sigma(d_1), d_1)$ has exactly one zero in $(0, 1)$. Since by Lemma 5.1 this is impossible, we have $u_d(1, \sigma(d), d) < 0$ for all $d \geqslant a$, which proves that σ is invertible and that for any λ there exists a unique d with $(\lambda, d) \in S_0$ and $u(\cdot, \lambda, d) > 0$ on $(0, 1)$. ∎

Remark 1. The assumption that $\psi(t)$ is nondecreasing which played a significant role in Lemma 5.1 is not very restrictive. It covers the model family of superlinear nonlinearities $f(t) = t^p - k$, $p > 1$, and $k > 0$. Another example which satisfies the requirement is

$$f(t) = \begin{cases} t - 1 & \text{for} \quad t \leqslant 1, \\ (t^2 - t)\, e^{1/t} & \text{for} \quad t \geqslant 1. \end{cases}$$

In fact, many more functions which satisfy the requirement can be constructed.

6. Proof of Theorem 4

In this section $f(t) = t^p - k$; $k > 0$, $1 < p < (N+2)/(N-2)$.

LEMMA 6.1. *If* (λ, u) *satisfies* (2.1)–(2.3) *with* $u > 0$ *in* $[0, 1)$, *then* $u_d(1) \neq 0$.

Proof. Rewriting (2.1) as

$$(r^{N-1} u')' + \lambda r^{N-1} f(u(r)) = 0, \tag{6.1}$$

a differentiation with respect to d gives that u_d satisfies

$$(r^{N-1} u'_d)' + \lambda r^{N-1} f'(u)\, u_d = 0, \tag{6.2}$$

$$u_d(0) = 1 \quad \text{and} \quad u'_d(0) = 0. \tag{6.3}$$

Multiplying (6.1) by u_d and (6.2) by u, subtracting and integrating by parts, we get

$$u'(1)\, u_d(1) = \lambda \int_0^1 r^{N-1}(uf'(u) - f(u))\, u_d\, dr. \tag{6.4}$$

Since $u'(1) \leqslant 0$ and $uf'(u) - f(u) > 0$, we infer that u_d has at least one zero in $(0, 1)$. Defining $z(r) = \int_0^r s^{N-1} u_d(s)\, ds$, a simple computation yields

$$(r^{N-1}z')' = r^{2(N-1)}u'_d + 2(N-1)\, r^{2N-3} u_d. \tag{6.5}$$

Multiplying (6.2) by z and (6.5) by u_d, subtracing and integrating by parts, we obtain

$$2z'(1)\, u_d(1) - 2u'_d(1)\, z(1) = 2(N-1) \int_0^1 r^{2N-3} u_d^2(r)\, dr$$

$$- \lambda \int_0^1 f''(u(r))\, z^2(r)\, u'(r)\, dr + u'_d(1)^2.$$

The convexity of f along with the fact that $u' < 0$ in $(0, 1)$ implies that the right side of the above equation is positive. Hence

$$2z'(1)\, u_d(1) - 2u'_d(1)\, z(1) > 0. \tag{6.6}$$

If we let $v(r) = ru'(r) + k_p u(r)$ with $k_p = 2/(p-1)$, then

$$(r^{N-1}v')' + \lambda r^{N-1} f'(u)\, v = \lambda r^{N-1}(k_p + 2). \tag{6.7}$$

Multiplying (6.7) by u_d and (6.2) by v, subtracting and integrating by parts, we get

$$v'(1)\, u_d(1) - u'_d(1)\, v(1) = \lambda(k_p + 2)\, z(1). \tag{6.8}$$

Suppose $u_d(1) = 0$. If u_d has an even number of zeros in $(0, 1)$, then from (6.6) we infer that $z(1) > 0$ while (6.8) implies that $z(1) \leqslant 0$. A similar argument leads to a contradiction if we assume that u_d has an odd number of zeros in $(0, 1)$. Hence $u_d(1) \neq 0$ and the lemma is proven.

Proof of Theorem 4. Let U be a connected component of $S = \{(\lambda, u) \in (0, \infty) \times C(\bar{\Omega});\ u > 0 \text{ in } \Omega \text{ and } (\lambda, u) \text{ satisfies (1.1)–(1.2)}\}$. From Lemma 2.4, there exists a continuous function $\sigma\colon I \to \mathbb{R}$ such that $U = \{(\sigma(d), d) : d \in I\}$, with $I = [\tilde{a}, \tilde{b}]$ if $\tilde{b} < \infty$ and $I = [\tilde{a}, \tilde{b})$ if $\tilde{b} = \infty$ where $\tilde{a} := \min\{d \colon (\lambda, d) \in U\}$ and $\tilde{b} := \sup\{d \colon (\lambda, d) \in U\}$. Let us suppose that $\tilde{b} < \infty$. We observe that

$$u_\lambda(1, \sigma(d), d) = 0 \qquad \text{for} \quad d \in \{\tilde{a}, \tilde{b}\}. \tag{6.9}$$

For if $u_\lambda(1, \sigma(\tilde{a}), \tilde{a}) \neq 0$, then by the implicity function theorem there exists $\varepsilon > 0$ such that σ can be extended to $[\tilde{a} - \varepsilon, \tilde{b}]$ such that $(\sigma(d), d) \in U$ for $d \in [\tilde{a} - \varepsilon, \tilde{b}]$. This contradicts our choice of $\tilde{a}$. A similar argument shows that $u_\lambda(1, \sigma(\tilde{b}), \tilde{b}) = 0$. From (2.1), (2.6) and (6.9) we obtain that

$$u_{\lambda\lambda}(1, \sigma(d), d) > 0 \qquad \text{for} \quad d \in \{\tilde{a}, \tilde{b}\}. \tag{6.10}$$

For $d \in [\tilde{a}, \tilde{b}]$ we consider $g(d) := u_\lambda(1, \sigma(d), d)$. Since $g(d) < 0$ on $(\tilde{a}, \tilde{b})$ and $g(\tilde{a}) = g(\tilde{b}) = 0$, we conclude that for each $n \in \mathbb{N}$ there exists $a_n \in [\tilde{a}, \tilde{a} + 1/n)$ and $b_n \in (\tilde{b} - 1/n, \tilde{b}]$ with $g'(a_n) < 0$ and $g'(b_n) > 0$. Thus, without loss of generality, we can assume that for values of d close to $\tilde{a}$

$$g'(d) = u_{\lambda\lambda}(1, \sigma(d), d) \cdot \sigma'(d) + u_{\lambda d}(1, \sigma(d), d) < 0 \tag{6.11}$$

and for values of d close to $\tilde{b}$

$$g'(d) = u_{\lambda\lambda}(1, \sigma(d), d) \cdot \sigma'(d) + u_{\lambda d}(1, \sigma(d), d) > 0. \tag{6.12}$$

Differentiating $u(1, \sigma(d), d) = 0$ we get $u_\lambda(1, \sigma(d), d) \cdot \sigma'(d) + u_d(1, \sigma(d), d) = 0$. Since $u_d(1, \sigma(d), d) \neq 0$ for $d \in [\tilde{a}, \tilde{b}]$ (see Lemma 6.1) and since $u_\lambda(1, \sigma(d), d) < 0$ we see that $\lim_{d \to \tilde{a}} \sigma'(d) = \lim_{d \to \tilde{b}} \sigma'(d) = \pm\infty$. This contradicts (6.11)–(6.12) in view of (6.10). Hence $\tilde{b} = \infty$ and this proves that U is unbounded. Therefore, by Theorem 2, $U = S_0$ which the theorem.

Remark 2. The question of whether or not there are bounded branches of solutions to (2.1)–(2.3) for other types of nonlinearities f remains open.

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