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ABSTRACT. In [4] J. Neuberger gave various sufficient conditions for the solvability of nonvariational operator equations via a variant of the steepest descent method. In this note we give versions of these conditions under weaker assumptions on the smoothness of the operators.

1. **Introduction.** In [4] J. Neuberger studied the solvability of the equation

(1) $F(x) = 0$

where F is an operator of class $C^{(2)}$ defined on a real Hilbert space H with values in another real Hilbert space K. The method used in [4] consists of considering the problem

(2) $\phi(x) = 0$

where $\phi: H \to \mathbf{R}$ is the function defined by $\phi(x) = \|F(x)\|^2/2$. This converts (1) into a variational problem which can be considered via the steepest descent method. We refer the reader to [5] and [6] for the numerical relevance of this approach.

The steepest descent method is based on the study of the equation $z'(t) = -\nabla\phi(z(t))$. In order to make sense out of this equation is that in [4] F is assumed to be of class $C^{(2)}$. Here we extend the results of [4] to the case F locally Lipschitzian. We do this by introducing a vector field $g: H \to H$ which plays the role of $\nabla\phi$ in the former equation. The construction of g is based on the notion of generalized gradients defined by F. Clark in [2]. A brief account of this technique is given in Section 2.

Our main results are Theorems 1, 2 and 3 below. Their proofs follow the line of the proofs given in [4].

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2. **Generalized gradients.** In this section we summarize the properties of

generalized gradients to be used later on. For details and proofs we refer the reader to [1] and [2].

Given a Hilbert space H and a locally Lipschitzian function $f: H \to \mathbf{R}$ the generalized gradient of f at $x \in H$ is the set of all elements $w \in H$ such that

$$(3)\quad \langle w,v\rangle \leqslant \limsup_{\substack{h\to 0\\ t\to 0, t>0}} [f(x+h+tv) - f(x+h)]/t$$

for all $v \in H$, where $\langle\ ,\ \rangle$ denotes the inner product in H. We will denote, the generalized gradient of f at x by $\partial f(x)$. The set $\partial f(x)$ is nonempty closed and convex. Let $\partial^o f(x)$ denote the element of minimal norm in $\partial f(x)$. It can be proved that $\|\partial^o f(x)\|$ is a lower semi-continuous function (see [1, Proposition 7]). In addition we have:

PROPOSITION 1. *Let* $x_0 \in H$ *be such that* $\|\partial^o f(x_0)\| > 0$. *There exists* $\epsilon \equiv \epsilon(x_0) > 0$ *such that if* $\|x - x_0\| < \epsilon$, $w \in \partial f(x)$ *then*

$$(4)\quad \langle w, a(x_0)\partial^o f(x_0)\rangle \geqslant \|\partial^o f(x)\|^2/2,$$

where $a(x_0) = \limsup_{x\to x_0}(\|\partial^o f(x)\|/\|\partial^o f(x_0)\|)^2$.

PROOF. If it were false, then there would be a sequence $\{x_n\}$ converging to x_0 and a sequence $\xi_n \in \partial f(x_n)$ such that

$$(5)\quad \langle \xi_n, \partial^o f(x_0)\rangle < (\|\partial^o f(x_n)\|^2/2a(x_0)).$$

Let $M > 0$ denote a Lipschitz constant for f in a neighborhood of x_0. From the definition of f it follows then that $\|\xi_n\| \leqslant M$ for n large enough. Hence, we can assume that $\{\xi_n\}$ converges weakly to some $\xi_0 \in H$. Arguing as in the proof of Proposition 6 of [1] it follows that $\xi_0 \in \partial f(x_0)$. Therefore $\langle \xi_0 - \partial^o f(x_0), \partial^o f(x_0)\rangle \geqslant 0$ (see [3, Theorem 2.3]). On the other hand we have from (5) $\langle \xi_0, \partial^o f(x_0)\rangle \leqslant \|\partial^o f(x_0)\|^2/2$. Clearly the last two inequalities contradict each other, which proves the proposition.

3. **Main results.** Now we are ready to prove:

THEOREM 1. *Suppose* ϕ *is a locally Lipschitzian function from* H *into* $[0,\infty)$ *and there is a unique* $u \in H$ *such that* $\phi(u) = 0$. *In addition suppose that:*

(i) *If* $r > 0$, *there exists* $c > 0$ *such that if* $\|x - u\| \leqslant r$, *then* $\|\partial^o\phi(x)\| \geqslant c\phi(x)$.

(ii) *If* $\epsilon > 0$, *there is* $\delta > 0$ *such that if* $\phi(x) < \delta$, *then* $\|x - u\| < \epsilon$.

Assertion. *There exists* $g: H \to H$ *locally Lipschitzian on* $H - \{u\}$ *such that if*

$z'(t) = -g(z(t))$, $t \in [0,\infty)$, *and the range of* z *is bounded then*

$$u = \lim_{t\to\infty} z(t).$$

PROOF. From (i) we see that $\|\partial^o\phi(x)\| > 0$ iff $x \neq u$. For $x_0 \in H - \{u\}$ let $\epsilon(x_0)$ and $a(x_0)$ be as in Proposition 1. Since $H - \{u\}$ is paracompact, there exists a locally finite open cover $\{V_\alpha;\ \alpha \in A\}$ of $H - \{u\}$ with each V_α contained some ball $B(x_0,\epsilon(x_0))$. Let $\{\psi_\beta;\ \beta \in B\}$ be a partition of the unity subordinate to $\{V_\alpha; \alpha \in A\}$. For each $\beta \in B$ let $h(\beta) \in H - \{u\}$ be such that the support of ψ_β is contained in $B(h(\beta),\epsilon(h(\beta)))$. We define $g(u) = 0$ and

$$g(x) = \Sigma_\beta \psi_\beta(x) a(h(\beta)) \partial^o\phi(h(\beta))$$

for $x \neq u$. From (4) it follows that

(6) $\langle \xi, g(x)\rangle \geqslant \|\partial^o\phi(x)\|^2/2$

for all $x \in H$.

Let now $z'(t) = -g(z(t))$, $t \in [0,\infty)$, have bounded range. From Proposition 9 of [1] and (6) we have

$$\begin{aligned}(7)\ (\phi(z(t)))' &\leqslant \max\{\langle \xi,-g(z(t))\rangle; \xi \in \partial\phi(z(t))\}\\ &= -\min\{\langle \xi,g(z(t))\rangle; \xi \in \partial\phi(z(t))\}\\ &\leqslant -\|\partial^o\phi(z(t))\|^2 \leqslant -c^2(\phi(z(t)))^2,\ t \geqslant 0.\end{aligned}$$

If $\phi(z(t_0)) = 0$ for some $t_0 > 0$, then $\phi(z(t)) \leqslant 0$ for all $t \geqslant t_0$. Hence $z(t) \equiv u$ for all $t \geqslant t_0$. So in this case $\lim_{t\to\infty} z(t) = u$.

If $\phi(z(t)) > 0$ for all $t \geqslant 0$, then from (7) we have $((\phi(z(t)))'/(\phi(z(t)))^2) \leqslant -c^2$. Hence

$$\phi(z(t)) \leqslant \phi(z(0))/(1 + c^2\phi(z(0))t),\ t \geqslant 0.$$

Therefore $\lim_{t\to\infty} \phi(z(t)) = 0$ and so, using (ii), $\lim_{t\to\infty} z(t) = u$, and the theorem is proved.

THEOREM 2. *Suppose* F *is a locally Lipschitzian function from* H *to* K. *Let* $\phi(x) = \|F(x)\|^2/2$ *for* $x \in H$. *Suppose also that if* $r > 0$ *there is* $c > 0$ *such that* $\|\partial^o\phi(x)\| \geqslant c\|F(x)\|$ *for* $\|x\| \leqslant r$.

Assertion. *There exists* $g: H \to H$, *locally Lipschitzian on* $H - \{u; F(u) = 0\}$, *such that if* $z'(t) = -g(z(t))$, $t \in [0,\infty)$, *and the range of* z *is bounded then* $\lim_{t\to\infty} z(t)$ *exists and* $F(\lim_{t\to\infty} z(t)) = 0$.

SKETCH OF PROOF. The function g is constructed in the same manner we build g in the previous theorem. Using that such a g satisfies (6) and the arguments of the proof of Theorem 4 of [4] it is shown that $\lim_{t \to \infty} z(t)$ exists and that $F(\lim_{t \to \infty} (z(t))) = 0$.

The same methods lead to the proof of the following version of Theorem 5 of [4].

THEOREM 3. *Suppose that* $\phi: H \to [0,\infty)$ *is a locally Lipschitzian function. Then there exists* $g: H \to H$, *locally Lipschitzian on* $\{u; g(u) \neq 0\}$, *such that if* $z'(t) = -g(z(t))$, $t \geqslant 0$, *and for some* $t_0 \geqslant 0$, $\delta > 0$, $\Sigma_{n=1}^{\infty} (\phi(z(t_0 + n\delta)) - \phi(z(t_0 + (n+1)\delta)))^{1/2}$ *converges, then* $u \equiv \lim_{t \to \infty} z(t)$ *exists, and* $\partial^o \phi(u) = 0$.

REFERENCES

1. K. Chang, *Variational methods for non-differentiable functionals and their applications to partial differential equations,* J. Math. Anal. Appl., 80(1981), 102-129.
2. F. H. Clarke, *A new approach to Lagrange multipliers,* Math. Oper. Res., 1(1976), 165-174.
3. D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications,* Academic Press, 1980.
4. J. W. Neuberger, *Some global steepest descent results for nonlinear systems,* preprint.
5. ———, *A type-independent method for systems of nonlinear partial differential equations: application to the problem of transonic flow,* Computers and Math. Appl., 6(1980), 67-68.
6. ———, *Steepest descent method for systems of nonlinear partial differential equations,* Oak Ridge National Laboratory, CSD TM-161, 1981.

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