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Model Updating by Adding Known Masses and Stiffnesses

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1 Abstract

New approaches are developed to update the analytical mass and stiffness matrices of a system. By adding known masses to the structure of interest, measuring the modes of vibration of this mass-modified system, and finally using this set of new data in conjunction with the initial modal survey, the mass matrix of the structure can be corrected. A similar approach can also be used to update the stiffness matrix of the system by attaching known stiffnesses. Manipulating the mass and stiffness correction matrices into vector forms, the connectivity information can be enforced, thereby preserving the physical configuration of the system, which enables successful identification of large structural models with relatively few measured modes.

2 Introduction

Highly accurate and detailed analytical models are required to analyze and predict the dynamical behavior of complex structures during analyses and design. With the advent of digital computers, new methods of analysis have been developed, in particular through the method of finite elements. Once the finite element model of a physical system is constructed, its accuracy is tested by comparing its analytical modes of vibration with those obtained from a modal survey. If the modal survey and the analytical predictions are in subjective agreement, then the analytical model can be used with confidence for future analysis. If the correlation between the two is unsatisfactory, then the analyst has the options of accepting the analysis, accepting the tests, modifying either or modifying both. The lack of correlation between the analytical predictions and the experimental results can be traced to either experimental or modeling errors, or a combination of both. These errors include inexact equipment calibration, excessive noise, equipment malfunction, inappropriate modeling assumptions, uncertainties in the material properties, insufficient modeling details, etc. The questions of improving testing methods or improving analysis procedures will not be addressed here. It will be assumed that the available test data are correct. Thus, when the analytical predictions do not match the test measurements, the finite element model must be corrected or updated such that the agreement between predictions and test results is improved. The above process is known as *model updating.*

In recent years various methods have been developed to improve the quality of the analytical finite element models using test data. A detailed discussion of every approach is beyond the scope of this paper, and interested readers are referred to the recent survey paper by Mottershead and Friswell [1]. In the following, we will only review some commonly used model updating techniques.

Berman [2] proposed a scheme based on the Lagrange multipliers formalism that uses the measured mode shapes to correct the mass matrix of a structure. This updating algorithm identifies a set of minimum changes in the analytical mass matrix so that the measured modes are orthogonal to the updated mass matrix of the system. Using an approach first proposed by Baruch and Bar Itzhack [3], Wei [4] developed an optimal method to update the stiffness matrix of a structure. He also employed the Lagrange multipliers formalism, subjected to the constraints of satisfying the generalized eigenvalue problem, the orthogonality condition of the measured mode shapes and the symmetry property of the stiffness matrix.

The Lagrange multipliers approaches to update the system matrices return fully populated mass and stiffness matrices that may not bear any resemblance to the physical system being analyzed. To preserve the physical load paths of the original analytical model, Kabe [5] assumed the analytical mass matrix to be correct and incorporated the readily available structural connectivity information in addition to the test data to optimally adjust the stiffness matrix. Thus, the zero and nonzero elements of the analytical model are preserved, and the adjusted model exactly reproduces the modes used in the identification. He also utilized a Lagrange multipliers formalism, so that the percentage change to each stiffness element is minimized. While Kabe's approach to updating the stiffness matrix is straightforward, the assumption that the actual mass matrix is identical to the analytical mass matrix remains debatable, since in practice there is significantly greater success using finite elements during static analyses (which use only the stiffness matrix) than during dynamic analyses (which use both the mass and stiffness matrix).

Using an approach based on matrix perturbation theory, Chen, Kuo and Garba [6] found the correction mass and stiffness matrices by enforcing the orthogonality conditions of the measured mode shapes with respect to the system matrices. Like the schemes proposed by Berman and Wei, however, the updating algorithms also return fully populated mass and stiffness matrices, thus failing to preserve the physical connectivity of the system. In addition, because the approach outlined in [6] is based on the perturbation theory, the updating algorithm can only be applied when the deviations of the actual parameters from the analytical values are small. Otherwise, the underlying assumption of asymptotic expansion used in the perturbation theory will be violated, and the perturbation updating scheme will yield erroneous results.

In this paper, new model updating schemes to correct the system mass and stiffness matrices are proposed. The required solution techniques are introduced, and the minimum number of measured modes needed to ensure a sufficiently accurate updated model is given.

3 Proposed Model Updating Algorithms

Consider a structure whose analytical model, with N degrees-of-freedom, is given by

$$[K_o][X_o] = [M_o][X_o][\Lambda_o] \tag{1}$$

where $[M_o]$ and $[K_o]$ are the symmetric analytical mass and stiffness matrices of the system, $[X_o]$ is the $N \times N$ modal matrix of the analytical model, and $[\Lambda_o]$ is an $N \times N$ diagonal matrix whose elements correspond to the eigenvalues of the analytical model. Experimentally, it is difficult, if not impossible, to measure the same number of modes as the number of degrees-of-freedom of the analytical model. Thus, the measured data are said to be *incomplete*. Regardless, the modes of vibration of the physical system must satisfy the following generalized eigenvalue problem:

$$[K][X] = [M][X][\Lambda] \tag{2}$$

where $[M]$ and $[K]$ are the actual $N \times N$ symmetric mass and stiffness matrices of the physical system, $[X]$ is the measured $N \times N_e$ modal matrix (N_e denotes the number of measured modes; $N_e \leq N$), and $[\Lambda]$ is an $N_e \times N_e$ diagonal matrix whose elements correspond to the measured eigenvalues of the system. Knowing $[M_o]$, $[K_o]$, $[X]$ and $[\Lambda]$, the objective of model updating is to correct $[M_o]$ and $[K_o]$ such that the new analytical system matrices yield modes of vibration that are closer to the measured data than they were initially.

3.1 Analytical Formulation

To update the mass matrix of the analytical model, a known mass matrix, $[M_a]$, is added to the physical structure, at locations coincident with the nodes of the finite element model, so that the resulting system satisfies

$$[K][X_a] = ([M] + [M_a])\,[X_a][\Lambda_a] \tag{3}$$

where $[X_a]$ corresponds to the $N \times N_e$ measured modal matrix of the new system, and $[\Lambda_a]$ is an $N_e \times N_e$ diagonal matrix, whose elements are the measured eigenvalues of this mass-modified system.

Taking the transpose of Eq. (2) and postmultiplying the resulting matrix equation by $[X_a]$, we get

$$[X]^T[K][X_a] = [\Lambda][X]^T[M][X_a] \quad (4)$$

Premultiplying Eq. (3) by $[X]^T$, we have

$$[X]^T[K][X_a] = [X]^T([M]+[M_a])[X_a][\Lambda_a] \quad (5)$$

Subtracting Eq. (4) from Eq. (5), we obtain

$$[\Lambda][X]^T[M][X_a] = [X]^T([M]+[M_a])[X_a][\Lambda_a] \quad (6)$$

Introducing

$$\begin{cases} [Q] = [X]^T[M_a][X_a][\Lambda_a] \\ [P] = [X]^T[M][X_a] \end{cases} \quad (7)$$

then Eq. (6) simplifies to

$$[\Lambda][P] - [P][\Lambda_a] = [Q] \quad (8)$$

whose (i,j)th element is given by $(\lambda_i - \lambda_{aj})P_{ij} = Q_{ij}$, where λ_{aj} is the jth measured eigenvalue of the mass-modified system and $i,j = 1,\ldots,N_e$. Assuming the N_e measured eigenvalues of the original and the mass-modified systems do not coincide, we can solve for all of the unknowns P_{ij} and then construct matrix $[P]$. If any two measured eigenvalues of the original and the mass-modified systems are identical, we simply change the added masses or their locations to make the eigenvalues distinct.

The actual and the analytical mass matrices of the system are related as $[M] = [M_0] + [\delta M]$, where $[\delta M]$ represents the correction to the analytical mass matrix. Matrix $[P]$ of Eq. (7) can also be rewritten as

$$[X]^T[\delta M][X_a] = [P] - [X]^T[M_o][X_a] \quad (9)$$

Equation (10) can be manipulated so that $[\delta M]$ appears as an unknown column vector $\delta\mathbf{m}$, of length N^2, as follows

$$[A]\,\delta\mathbf{m} = \mathbf{r} \quad (10)$$

where elements of $[A]$, of size $N_e^2 \times N^2$, and $\mathbf{r}$, of length N_e^2, can be determined by expanding the left and right hand sides of Eq. (10), respectively.

To update the stiffness matrix of the analytical model, a known stiffness matrix, $[K_b]$, is added to the actual system, at locations coincident with the nodes of the finite element model, so that the stiffness-modified system satisfies

$$([K]+[K_b])[X_b] = [M][X_b][\Lambda_b] \quad (11)$$

where $[K] = [K_o] + [\delta K]$, $[X_b]$ corresponds to the $N \times N_e$ modal matrix of the new system, and $[\Lambda_b]$ is an $N_e \times N_e$ diagonal matrix whose elements are the eigenvalues of this stiffness-modified system. Both $[X_b]$ and $[\Lambda_b]$ are experimentally determined. Here, it is implicitly assumed that the additional stiffness matrix does not significantly affect the mass matrix of the system. Following the same steps used to update the mass matrix, an equivalent to Eq. (10) is obtained:

$$[X]^T[\delta K][X_b] = [U] - [X]^T[K_o][X_b] \quad (12)$$

where $[\delta K]$ represents the correction to the analytical stiffness matrix, and

$$[U] = [X]^T[K][X_b] \quad (13)$$

As before, Eq. (14) can be manipulated into the following form

$$[B]\,\delta\mathbf{k} = \mathbf{h} \quad (14)$$

where elements of $[B]$, of size $N_e^2 \times N^2$, and $\mathbf{h}$, of length N_e^2, can be determined by expanding the left and right hand sides of Eq. (14), respectively, and $\delta\mathbf{k}$ is an unknown column vector of length N^2.

3.2 Solution Technique

Equations (11) and (16) are of the form

$$[G]\,\mathbf{y} = \mathbf{z} \quad (15)$$

where matrix $[G]$ and vector $\mathbf{z}$ are both known and of size $N_e^2 \times N^2$ and length N_e^2, respectively. If $N_e < N$, Eq. (18) is known as an underdetermined (that is, the number of equations is less than the number of unknowns) least squares problem, which generally will have an infinite number of solutions. To render the solution unique, we seek a solution vector $\mathbf{y}$ such that the Euclidean norm of the residual vector $\|[G]\,\mathbf{y} - \mathbf{z}\|$ is minimized. The resulting solution is referred to as the least squares solution to Eq. (18). If the least squares problem has more than one solution, as in the case of an underdetermined system, the solution vector having the minimum Euclidean norm is called the minimum norm solution. Because the analytical and the actual system matrices are presumed to be close, then if there are infinite solutions, the minimum norm solutions of $\delta\mathbf{m}$ and $\delta\mathbf{k}$ can be used to update the analytical mass and stiffness matrices, respectively.

To preserve the physical configuration of the system and to reduce the size of the least squares problem that need to be solved, the optimal matrix storage scheme commonly used in finite elements [7] can be applied to pass along the readily available sparsity information of the analytical system. Mathematically, this can be achieved by eliminating all the known zero elements from $\mathbf{y}$ and by deleting all the corresponding columns in $[G]$. For example, if the mass matrix is diagonal, then $\delta m_{ij} = 0$ for $i \neq j$, and Eq. (11) reduces to

$$[A']\,\delta\mathbf{m}' = \mathbf{r} \tag{16}$$

where $[A']$ is obtained from $[A]$ by deleting all the columns that multiply by δm_{ij} for $i \neq j$, and $\delta\mathbf{m}'$ is an unknown vector of length N, which consists of δm_{ii} only. Thus, the initial problem of size $N_e^2 \times N^2$ is reduced to one of size $N_e^2 \times N$. The resulting least squares problem will be either overdetermined (that is, the number of equations is greater than or equal to the number of unknowns) or underdetermined, depending on whether $N_e^2 \geq N$ or $N_e^2 < N$, respectively.

Similarly, if the analytical stiffness matrix is tri-diagonal, then $\delta k_{ij} = 0$ for $|i - j| > 1$, and Eq. (16) reduces to

$$[B']\,\delta\mathbf{k}' = \mathbf{h} \tag{17}$$

where $[B']$ is obtained from $[B]$ by deleting all the columns that multiply by δk_{ij} for $|i - j| > 1$, and $\delta\mathbf{k}'$ is an unknown vector of length $3N - 2$, which consists of δk_{ij} for $|i - j| \leq 1$. Thus, the initial problem of size $N_e^2 \times N^2$ is reduced to one of size $N_e^2 \times (3N - 2)$. The resulting least squares problem will be either overdetermined or underdetermined, depending on whether $N_e^2 \geq 3N-2$ or $N_e^2 < 3N-2$, respectively.

4 Results

The proposed updating algorithms by adding knowing masses and stiffnesses are applied to the simple system of Fig. 1, whose mass matrix is diagonal and whose stiffness matrix is symmetric and tri-diagonal. Knowing the modes of vibration of the analytical model and the actual system, the goal is to correct the analytical system matrices. When solving a least squares problem, the CMLIB routine *sglss* was accessed, which is specialized to handle both underdetermined and overdetermined systems $[A]\mathbf{x} = \mathbf{b}$, where $[A]$ is an $m \times n$ matrix and $\mathbf{b}$ is a vector of length m. When the system is overdetermined ($m \geq n$), the least squares solution is computed by decomposing the matrix $[A]$ into the product of an orthogonal matrix $[Q]$ and an upper triangular matrix $[R]$ (QR factorization). When the system is underdetermined ($m < n$) , the minimum norm solution is computed by factoring the matrix $[A]$ into the product of a lower triangular matrix $[L]$ and an orthogonal matrix $[Q]$ (LQ factorization). If the matrix $[A]$ is determined to be rank deficient, that is the rank of $[A]$ is less than min(m,n), then the minimum norm least squares solution is computed.

Consider a system of 24 oscillators ($N = 24$). For now, the experimental data are assumed to be complete ($N_e = N = 24$). Table 1 shows the actual and the updated mass parameters. The masses of the analytical system are 2.00 kg. To apply the mass updating algorithm, lumped masses of magnitudes 0.20, 0.10, 0.25 and 0.22 (in kg) are added to masses (or nodes) 6, 12, 18 and 24, respectively. Note the excellent agreement between the actual and the updated masses. Table 2 shows the actual and the updated stiffness parameters. The stiffnesses of the analytical system are 10.00 N/m. Grounded stiffnesses of magnitudes 0.50, 0.25, 0.53, 0.45 and 0.37 (in N/m) are attached to masses (or nodes) 5, 11, 16, 20 and 24, respectively. Note the excellent agreement between the updated and the actual stiffnesses. Because the proposed updating algorithms allow the connectivity information to be enforced, the proposed mass and stiffness updating schemes return mass and stiffness matrices that are strictly diagonal and tri-diagonal, respectively, thus preserving the physical configuration of the structure.

The previous results are obtained for a complete set of measured modes. Due to physical, time and cost limitations, however, the set of measured modes is more likely to be incomplete in practice. Thus, of considerable interest is the effects of the number of measured modes, N_e, on the quality of the updated model. To quantify the accuracy of the mass updating algorithm, the following error parameter for the updated masses is introduced:

$$\epsilon_m = \frac{|\mathbf{m}_{\text{update}} - \mathbf{m}_{\text{actual}}|}{|\mathbf{m}_{\text{actual}}|} \tag{18}$$

where $\mathbf{m}_{\text{update}}$ and $\mathbf{m}_{\text{actual}}$ are vectors of length N

whose elements correspond to the updated and the actual lumped masses, respectively, and $|\mathbf{a}|$ represents the Euclidean norm of the vector $\mathbf{a}$. To illustrate the improvement of the updated masses over the initial analytical values, the following error parameter for the analytical masses is introduced:

$$(\epsilon_m)_o = \frac{|\mathbf{m}_{\text{analytical}} - \mathbf{m}_{\text{actual}}|}{|\mathbf{m}_{\text{actual}}|} \tag{19}$$

Similar expressions can also be defined for the stiffness error parameter. In order for an updated model to be judged more accurate than the initial analytical model, $\epsilon_m < (\epsilon_m)_o$ and $\epsilon_k < (\epsilon_k)_o$. For an updated model to be considered accurate, both ϵ_m and ϵ_k must be sufficiently small. Finally, the smaller the error parameters, the better the updated model.

Figure 2 shows the variations of ϵ_m and ϵ_k as a function of N_e. Also shown are the corresponding $(\epsilon_m)_o$ and $(\epsilon_k)_o$ (given by the horizontal lines), which are independent of N_e. Note that the error parameters decrease as N_e increases. The observed results are consistent with physical intuition: the more information that can be gathered about the physical system, the better the updated model becomes. Note also that for $N_e \geq 4$, $\epsilon_m > (\epsilon_m)_o$ and $\epsilon_k > (\epsilon_k)_o$. Thus, for the system parameters of Tables 1 and 2, as long as $N_e \geq 4$, the proposed updating algorithms return a better updated model than the initial analytical model.

From various numerical experiments, it is observed that more accurate updated system matrices are obtained when the system is overdetermined as opposed to underdetermined. To make the problem overdetermined when updating the masses, at least $N_e^2 \geq N$ measured modes are needed. To have an overdetermined problem when updating the stiffnesses, at least $N_e^2 \geq 3N - 2$ measured modes must be obtained. Thus, for $N = 24$, to induce overdeterminancy requires at least 9 measured modes. For $N_e \geq 9$, the mass and stiffness error parameters are $\epsilon_m < 0.034\%$ and $\epsilon_k < 6.249\%$, respectively. The criterion regarding the minimum number of measured modes needed to perform the update is purely empirical. Thus, depending on the system parameters and on the desired level of accuracy, sometimes a few more and other times a few less measured modes than the predicted minimum number may be required. Nevertheless, using the heuristic criterion regarding the smallest N_e needed to perform the update, we obtain adjusted system matrices whose modes of vibration are substantially closer to the measured data than they were initially.

5 Conclusion

New mass and stiffness updating algorithms are developed. Using the original test data and the newly acquired mass-modified and stiffness-modified modes of vibration, the mass and stiffness matrices of the analytical model can be accurately corrected. By manipulating the matrix equations so that the unknown correction mass and stiffness matrices appear as column vectors, the connectivity information can be easily implemented, thus preserving the physical configuration of the system and reducing the amount of computational efforts required to correct the analytical model. In addition, the determinancy of the least squares problems reveals the minimum number of modes that we would need to measure to ensure a sufficiently accurate updated model. The proposed mass and stiffness updating algorithms require additional work, because the modes of vibration for the mass-modified and stiffness-modified systems need to be obtained. The additional effort, however, is a relatively small price to pay for the ability to correct the analytical model accurately.

References

[1] J. E. Mottershead and M. I. Friswell. "Model Updating in Structural Dynamics: A Survey." Journal of Sound and Vibration. Vol. 167. No. 2. Pp. 347–375. 1993.

[2] A. Berman. "Mass Matrix Correction Using an Incomplete Set of Measured Modes." AIAA Journal. Vol. 17. No. 10. Pp. 1147–1148. 1979.

[3] M. Baruch and I. Y. Bar Itzhack. "Optimal Weighted Orthogonalization of Measured Modes." AIAA Journal. Vol. 16. No. 4. Pp. 346-351. 1978

[4] F. S. Wei. "Stiffness Matrix Correction from Incomplete Test Data." AIAA Journal. Vol. 18. No. 10. Pp. 1274–1275. 1980.

[5] A. M. Kabe. "Stiffness Matrix Adjustment Using Mode Data." AIAA Journal. Vol. 23. No. 9. Pp. 1431–1436. 1985.

[6] J. C. Chen, C. P. Kuo, and J. A. Garba. "Direct Structural Parameter Identification by Modal Test Results." 24th AIAA/ASME/ASCE/AHS Structural Dynamics and Materials Conference. Pp. 44–49. 1983.

[7] K. J. Bathe. Finite Element Procedures in Engineering Analysis. Prentice-Hall, Inc., New Jersey. 1982

Actual	Updated	Actual	Updated
m_1=2.510	2.510	m_{13}=1.329	1.329
m_2=1.435	1.434	m_{14}=2.019	2.019
m_3=2.146	2.146	m_{15}=2.538	2.538
m_4=1.449	1.449	m_{16}=1.468	1.468
m_5=2.526	2.526	m_{17}=2.534	2.534
m_6=2.350	2.350	m_{18}=1.512	1.512
m_7=1.812	1.812	m_{19}=1.891	1.891
m_8=2.680	2.680	m_{20}=1.698	1.698
m_9=1.839	1.839	m_{21}=1.326	1.326
m_{10}=1.823	1.823	m_{22}=1.315	1.315
m_{11}=2.607	2.607	m_{23}=1.855	1.855
m_{12}=2.098	2.098	m_{24}=1.386	1.386

Table 1. The actual and the updated masses (in kg), for $N_e = N = 24$. The analytical masses are 2.000 kg.

Actual	Updated	Actual	Updated
k_1= 8.624	8.626	k_{13}= 7.337	7.337
k_2=13.008	13.002	k_{14}=11.580	11.581
k_3=10.968	10.960	k_{15}=10.796	10.796
k_4=12.417	12.412	k_{16}=11.517	11.518
k_5= 6.695	6.693	k_{17}=10.051	10.051
k_6=13.412	13.412	k_{18}=12.177	12.177
k_7= 7.291	7.291	k_{19}=12.036	12.036
k_8= 7.278	7.278	k_{20}=13.292	13.291
k_9=11.923	11.923	k_{21}=12.564	12.564
k_{10}=12.624	12.625	k_{22}= 9.892	9.892
k_{11}=11.518	11.519	k_{23}=12.229	12.229
k_{12}=12.112	12.112	k_{24}=11.281	11.281

Table 2. The actual and the updated system stiffnesses (in N/m), for $N_e = N = 24$. The analytical stiffnesses are 10.000 N/m.

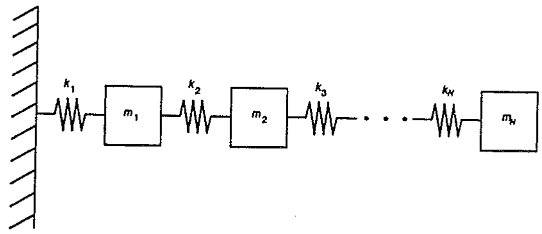

Figure 1: Simple chain of coupled oscillators.

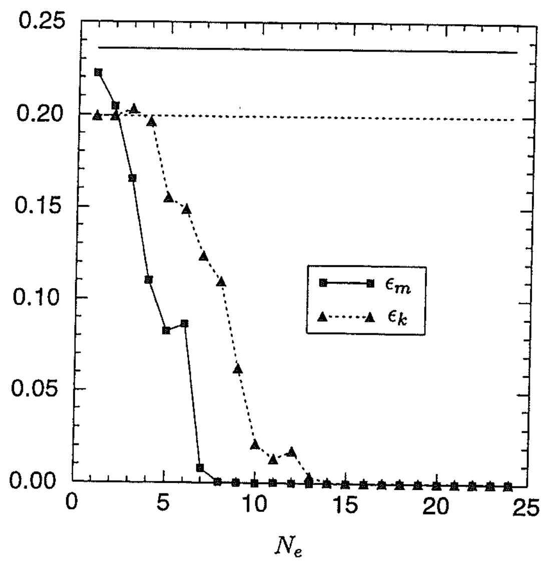

Figure 2: Variations of ϵ_m and ϵ_k as a function of N_e. The horizontal lines are $(\epsilon_m)_o$ and $(\epsilon_k)_o$.