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An Exposition and Calibration of the Ho-Lee Model of Interest Rates

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Abstract

The purpose of this paper is to create an easily understandable version of the Ho-Lee interest rate model. The first part analyzes the model in detail, and the second part calibrates it to demonstrate how it can be applied to real market data.

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1 Introduction

The goal of this paper is to present an exposition of the Ho-Lee [8] model. Many financial models are written with some barrier to entry as a reader because they are written with an expectation that the reader has a certain level of understanding of math and finance. When I first read tHo-Lee I understood it on the surface, but was confused about many facets and so I didn't feel like I really knew the model. In this paper I simplify and expand on the topics Ho and Lee merely brush over so that almost any reader may thoroughly understand the model. I do not actually add anything new to the model, I just rephrase and explain things in deeper detail in order to fill the background in the places where it is assumed the reader knows that background. Unfortunately the barrier to entry to add to the field of derivative and interest rate models is extremely high. Several of these more complex topics are developed fully in the appendices so the interested reader may read them at his or her discretion. I begin in Section 2 by discussing other financial models and how Ho-Lee is different from other interest rate models. In Section 3 I give a foreground to binomial pricing models with the original, the Cox-Ross-Rubinstein [5] model, and then give a background into stochastic processes. In Section 4 I interpret the Ho-Lee model and then in Section 5 I apply the model to real interest rates.

2 Option and Interest Rate Models

The first widely used mathematical model of derivatives was developed in 1973 through the publication of the Black-Scholes-Merton (Black-Scholes) [3] model, which priced stock options. The model was quickly followed by many derivative and interest rate models hoping to improve on the holes in the models that preceded them. One of the shortcomings of the Black-Scholes model is that they use a constant interest rate, something that is never seen in practice. In 1977 Vasicek [10] created a model specifically applicable to interest rate structures. Like Black-Scholes, and every subsequent model attempting to price derivatives, it is based on the assumption of no interest rate arbitrage. With no arbitrage it must be shown that all of the assets are priced appropriately so there is no opportunity for one to be able to lock in a risk-free profit. It follows then that nobody should be able to play one instrument off of another in order to lock in a risk-free profit. This model was followed by the Cox-Ingersoll-Ross (hereafter CIR) (1985) [4], Ho-Lee (1986) [8], Heath-Jarrow-Morton (hereafter HJM) (1987) [7], Hull-White(1990) [9], Black-Derman-Toy (hereafter BDT) (1990) [2], and a few other models. With many models all trying to create a way to model interest rate contingent claims, I propose to choose a model and analyze it deeper to understand where it breaks down when applied to real markets. Here I will focus on the Ho-Lee [8] model specifically so that it can be matched up and compared to the other similar models. This will help us understand the methods specific to Ho and Lee.

While Black-Scholes[1] may have been the first to model put and call options, only a few years later a model was written that was more useful for many real work applications. Cox-Ross-Rubinstein (CRR) (1979) [5] is a binomial options pricing model. This binomial lattice structure is the discrete time version of the continuos time Black-Scholes model[2] with a set number of periods. In each period leading up to the expiration date, the spot price could go up some predetermined amount or it could go down the predetermined amount. What made this model more useful compared to Black-Scholes was that, because of

[1]A mathematical model theoretically estimating European options, one that can only be exercised on the expiration date. The model assumes no arbitrage and depends on the volatility of the underlying asset, spot price, risk free rate, strike price, and time to maturity.

[2]While discrete time can only take certain values, continuous data can take any value within a certain range. Thus the probability of getting a certain value in discrete time may have a positive probability if it is in the probability space, yet in continuous time the probability of attaining any specific value is always zero.

its ability to model the underlying asset over a period of time, it did a much better job of valuing American options. Unlike European options, which can only be exercised at their maturity, American options can be exercised at any time leading up to it. The European option is a lot simpler because one must simply estimate the price on the expiration date; whereas, since the American option can be exercised at any time leading up to the expiration date, one must consider the value of exercising the option at all possible times. While the CRR model is a discrete time model with distinct number of periods, it allows an estimation of the value of an option by shrinking the size of the time period to as small as one desires. This versatility gives the CRR model a comparative advantage over Black-Scholes when valuing certain derivatives.

Ho-Lee [8] did a similar analysis to CRR and created a binomial pricing of interest rate contingent claims. This was a stochastic, normally distributed, single factor, short rate interest rate model theoretically applicable to all sorts of interest rate derivatives. The binomial lattice structure is based on the simple case that there is a chance that in the next period the price of the bond will move to the upstate and a chance that it will go to the downstate. If there is no risk in the economy then we can say for certain that the price of the bond should be the same in both states. Of course in the economy we know this to never be true. To account for the movement, some perturbation functions are defined for each of the two states that allow one to calculate the forward rate. We know the interest rate will deviate from our estimation; we just do not know by how much. These perturbation functions are what keep this model arbitrage free. In a no-risk environment with no arbitrage we can safely assume that the interest earned on a bond with maturity T + 1, r_{T+1}, is the same as the interest earned on a bond with maturity T, r_t, plus the interest earned on the one-year bond, r_1, or the risk free rate:

$$(1 + r_T) * (1 + r_1) = (1 + r_{T+1})$$

From here we can work backward using the lattice to price a bond with maturity T at any discrete time n. The important factor then is that there is an implied binomial probability π of moving either to the upstate or to the downstate, which is reflected in the price just as there is the implied volatility found in the Black-Scholes model [3].

Before arbitrage-free models all of the interest rate models were known as equilibrium models. The equilibrium model follows the idea of supply and demand. In the basic model we are taught in economics courses we graph the price vs. the quantity of a certain good. The result is an upward sloping supply curve and a downward sloping demand curve that meet at a point we call the equilibrium. With interest rates the equilibrium is the interest rate such that at that rate the total amount of money banks and other groups are willing to lend is equal to the total amount of money that people want to borrow. This equilibrium can either be stable; i.e. following a shock the interest rate will return to the equilibrium, or unstable; i.e. following a shock the interest rate will move away from the equilibrium. The largest problem with this model, and the reason that other tactics were used in subsequent models, is that it requires some knowledge about the preferences of the market participants. To figure out the supply and demand curves we would need to somehow get the risk preferences of both sides of the market. Another issue is that these models have a large number of parameters. The Ho-Lee model is much more flexible and allows calibration with observed market prices.

While Ho-Lee was the first arbitrage-free interest rate model[3], many more

[3]The Vasicek (1977), Cox, Ingersoll, and Ross (1985) and other previous equilibrium in-

have been created since; some of these models are just variations and others are quite different. In those models, Ho-Lee is mentioned as a starting point but then each one tries to fix one of the flaws found with the Ho-Lee model. One issue with Ho-Lee is that it is possible (i.e. there is a positive probability) that the model could produce a negative interest rate. In the real world the interest rate should never sink below zero because individuals always have the option to withdraw money and hold it in cash with an interest rate of zero. This happens because the short rate follows a normal distribution which, as time goes to ∞ will eventually give a negative result. BDT [2], another one-factor stochastic binomial model, corrects for this by using a lognormal distribution, instead of the normal distribution[4]. This means that instead of the short rate being normally distributed, the logarithm of the short rate is normally distributed. Another correction that BDT makes is that while both still have a constant interest rate, BDT allows for mean reversion. The idea is that while interest rates can be quite high and quite low, over time the interest rate should approach the mean. Hull-White is another example of a single factor, lattice model which is better for pricing future interest rates, giving it the ability to value financial instruments like interest rate swaptions, the option but not obligation to enter into an interest rate swap.

Up until now all of the models mentioned have been single-factor models, based on the short rate. An issue with that is that volatility is, by nature, not fixed. Besides the general downward movement as the maturity approaches due to the fact that there is less time for the price of a bond to fluctuate extremely, there are event driven swings in interest rates that cannot be effectively calculated in the model with a constant rate. HJM[7] is a multi-factored model

terest rate models were the origin of arbitrage-free models, but fell short in that they require the preferences of all participants in the market in order to model.

[4]Whereas in the normal distribution we say that our random variable, X, is distributed normally, in the lognormal distribution the log of our random variable, $log(X)$, is distributed normally.

using many different Brownian[5] motions to model not just the short rate, but also volatility and other factors. This much more complicated model actually uses Ho-Lee as a starting point, but evolves their discrete time binomial process into a continuous distribution. Unlike Ho-Lee though, HJM is not a Markov chain. A Markov chain is a random process that is specifically characterized as memoryless, meaning that the following state only depends on the current state of the Markov chain and not any of the previous states:

$$P(X_{n+1} = j | X_0 = i_0, X_1 = i_1, ...X_n = i_n) = P(X_{n+1} = j | X_n = i_n)$$

Some assets are considered to be Markovian in that they take a random walk, while others are thought to have momentum from past movement and are thus non-Markovian. The problem with HJM being non-Markovian is that this means it is path dependent [7]. This becomes extremely complicated when attempting to model it because you have to keep track of all the previous data. Also, because it is in continuous time the state space is enormous so as the model grows over time as the previous data goes to infinity.

Research to improve upon these interest rate models is constantly being done. Unfortunately, unlike with many other areas of research, academia is not necessarily at the forefront. Instead, large financial firms like Goldman Sachs, Morgan Stanley, etc. have their own arms for financial research and development. It isn't until many years later that these models reach the public. For example, Black, Derman, and Toy all worked together at Goldman Sachs to create their model beginning in the early 1980's, but it was not published until 1990.

In 1993 Bjorn Flesaker tested a version of the Ho-Lee model, but in con-

[5] Brownian motion was originally discovered through the observation of the collisions between suspended particles in a liquid or gas. We can model Brownian motion via a Wiener process, W, which is a continuous random walk. Many economists equate the movement in stock prices to a type of Brownian motion.

tinuous time rather than its creation in discrete time [6]. He tested the simple case where there is just a single Brownian motion with a constant volatility. He found that the constant volatility inhibited the Eurodollar futures options he attempted to model. While the interest rate in the tested model follows a stochastic process, there are other factors in the model, like LIBOR[6] for example, Fleskar believes might follow a stochastic process as well [6]. What he and many economists have found is that while modeling one variable stochastically is viable, it rapidly gets quite complicated for someone to actually test a model with many such random variables. Even though Ho-Lee, HJM, BDT, and others were successful in advancing the financial world through their innovation in interest rate models, work is still being done to improve them and in this paper I attempt to examine why this process has not stopped. In a competitive industry where more accurate modeling means more profits it seems clear that there is no visible end or even "right" answer in the future.

3 Background

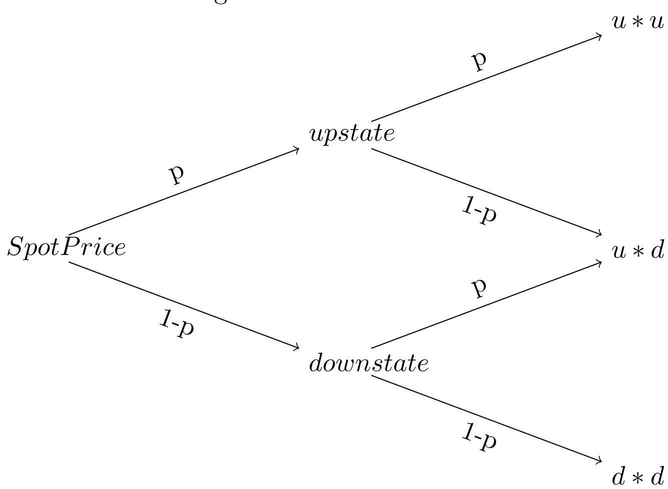
3.1 Cox-Ross-Rubinstein

The Cox-Ross-Rubinstein (CRR) model of options was the first binomial options pricing model. Originated in the late 1970's, this model simplified the process by assuming that the price of an option today could only be one of two values in the next period. The idea was that the spot price of the underlying asset would go up with probability p and down with probability $1 - p$ as seen in Figure 1 above.

This model also introduced no arbitrage pricing into options. The basic idea is that a riskless portfolio must just earn the risk-free rate. If a portfolio man-

[6]LIBOR stands for the London Interbank Offered Rate. Each morning the largest banks in London give the rate at which they would charge for another bank to borrow from them. These numbers are averaged and then published by Thomson Reuters.

Figure 1: Cox-Ross-Rubinstein Binomial Tree



ager could guarantee a risk-free profit then there must have been an arbitrage opportunity. We will see that this is also true in the Ho-Lee model, because of this no-arbitrage condition, so the price of an underlying asset after going to a downstate followed by an upstate must equal the price of that underlying asset after going to an upstate followed by a downstate. The simplicity of this model mathematically makes it easy to understand; yet it is actually quite accurate and still used extensively today.

3.2 Stochastic Processes

A stochastic process is a collection of random variables which are observed over time. In regular probability we might look at something like a coin flip which can either be heads or tails. If we flip a fair coin only once and have Y be the random variable equal to either heads or tails, then we say Y is distributed Bernoulli with parameter $p = .5$, $Y \sim Bernoulli(.5)$, because there is an equal chance of getting heads or tails. Now let us say that we flip that coin 100 times and let X be the random variable equal to the number of heads then

$X \sim Binomial(100, .5)$ where $n = 100$ and $p = .5$. While we are now flipping the coin many times, this is still not a stochastic process because it is not at all time dependent. We would get the same expected value of X, 50, if we flipped 5 coins now and 95 coins tomorrow. Instead, we can have a binomial process that we observe over time, called a Markov Chain.

A Markov Chain is a discrete time stochastic process such that the conditional distribution of the future state Y_{n+1} given all past states $Y_0, Y_1, Y_2, ...Y_{n-1}, Y_n$ depends only on the current state Y_n. As seen above, in the CRR binomial tree we can arrive at on of a number of nodes. Ending up in that node only depends on the state before it. For example, for us to end up at the node $u * d$ (an upstate followed by a downstate or a downstate followed by an upstate), it's not important what has happened in the periods before as long as in the previous period we are currently at the node "upstate" or the node "downstate". Markov chains are used quite frequently because of this nature of randomness. The difference between this stochastic process and the simple binomial case we presented is clear. Unlike in the example above with coins, it is now important when the coin is flipped and not just how many times.

4 Ho-Lee

4.1 Introducing the Model

The Ho-Lee model is an attempt to price interest rate contingent claims. Common ones include interest rate options, callable bonds (which is a bond that allows the issuer to retain the right to redeem the bond at a point prior to the maturity), floating rate notes (an instrument that is tied to a floating interest rate like the LIBOR), etc. In this section we look at the model from simply a mathematical point of view and see how it functions. To begin, we must account for the assumptions that are used in the model:

1. The theoretical market is frictionless. As a consequence we are assuming that there are no taxes, no transaction costs, and all securities must be perfectly divisible. This means that while our cash system can only be precise to the penny, the amount and price of these securities can be any real number.

2. The market clears. This means that the entire supply of the security is sold, at discrete points in time separated by constant intervals. In this case we focus on zero-coupon bonds with maturity T that pay the face value of $1 at the end of the Tth period.

3. This theoretical bond market must be complete. This means that for every maturity n (n = 0, 1, 2,...) there must be an available bond in the market.

4. At each of the maturity times n there are a finite number of states i, where i ranges from 0 to n. The equilibrium price of a bond in state i, at time n, with maturity T is represented here by $P_i^{(n)}(T)$. This is also known as the discount function.

We also know that since $P_i^{(n)}(T)$ is the value of an asset, $P_i^{(n)}(T) > 0$ for all values of T. Similarly, since the value of a bond that is purchased instantaneously before the maturity date must be equal to the face value, we have $P_i^n(0) = 1$ for all i and n. Lastly, as the maturity of a bond goes to infinity, the gain for the buyer would be insignificant:

$$\lim_{T \to +\infty} P_i^{(n)}(T) = 0 \text{ for all } i, n$$

Like the Cox-Ross-Rubinstein model of options, the Ho-Lee model is a binomial lattice. We begin here with the price of a bond at period $n = 0$ and state

Figure 2: Ho-Lee Binomial Pricing Tree

$$
\begin{tikzcd}
 & & P_2^{(2)}(T-2) \\
 & P_1^{(1)}(T-1) \arrow[ur] \arrow[dr] & \\
P_0^{(0)}(T) \arrow[ur] \arrow[dr] & & P_1^{(2)}(T-2) \\
 & P_0^{(1)}(T-1) \arrow[ur] \arrow[dr] & \\
 & & P_0^{(2)}(T-2)
\end{tikzcd}
$$

$i = 0$. One period from now, $n = 1$, we can either go to the upstate where $i = 1$, or to the downstate where $i = 0$. As we will see later, there is some probability p, $0 < p < 1$, that the price of the bond will go to the upstate and a similar probability $1 - p$ that it will go to the downstate. This follows into the future until the bond matures in T periods. During that period we will have $T + 1$ states ranging from $i = 0$ to $i = T$. Figure 2 below is a representation of what the binomial lattice would look through two full periods.

With this model we can see that the closer in time we are to maturity, the more certain we will be about the bond price as the number of potential future nodes shrinks. Conversely, the variation gets large quite quickly when we are far from our maturity. A bond with maturity N will have 2^N possible paths and the bond could end up in $\sum\limits_{i=1}^{N} i$ different nodes along the way.

4.2 Defining Ho-Lee as Arbitrage-Free

The specific bond structure used in the Ho-Lee model is what's known as an arbitrage-free model. As mentioned above this means that there must be no opportunities for an investor to lock in a risk-free profit. In this model this is combated by assuring that the price of a bond that goes to the upstate followed by the downstate will by the same price as a bond that goes to the downstate followed by the upstate. In the tree above this is why the node n = 2, i = 1 can be accessed by both period one nodes. If they were different then one could lock in a risk-free profit by borrowing money at one rate and lending money at the other rate. Thus we can can say,

$$P_i^{(n+1)}(T) = P_{i+1}^{(n+1)}(1) = \frac{P_i^{(n)}(T+1)}{P_i^{(n)}(1)} \tag{1}$$

The equation above is taken from theory in financial markets where there is no risk in the short rate. For example, if we are given an interest rate for the next nine years, r_9 and a short rate r_1, then we can theoretically equate that to the ten-year interest rate from the equation $(1+r_9)^9 * (1+r_1) = (1+r_{10})^{10}$. For this case specifically we look at the price of a bond in one time period after today. Remember that Ho-Lee is a short rate model; it attempts to describe the price of bonds related only to our knowledge of the one period rate. Equation (1) tells us that the price of any T period bond one period from now should be the price of a $T+1$ period bond one period longer, yet valued today, divided by a 1 period bond, valued today as well. This makes sense because theoretically, if we knew the future of the short rate, we should be just as equally content given a 1 period bond today and then given a T period bond tomorrow as just given a $T+1$ period bond. If we bring the 1 period bond from today to the following period, we see that it becomes a 0 period bond, or equal to 1 since $P_i^{(n)}(0) = 1$ for all i and n. If we brought forward the $T+1$ period bond to the next period, $P_i^{(n)}(T+1)$, we would see that $n \to n+1$ and $T+1 \to T$, as seen in Figure 2, and we would just be left with $P_i^{(n+1)}(T)$. In real markets, though, there will be some sort of discrepancy between the predicted future price and the actual future price. To combat this, Ho-Lee define two perturbation functions, unknown of course, that will account for this deviation, one for when the bond goes to the upstate, $h^*(t)$, and one for when the bond goes to the downstate $h(t)$. Thus the equation above will actually be:

$$P_{i+1}^{(n+1)}(T) = \frac{P_i^{(n)}(T+1)}{P_i^{(n)}(1)} h^*(T) \tag{2}$$

and

$$P_i^{(n+1)}(T) = \frac{P_i^{(n)}(T+1)}{P_i^{(n)}(1)} h(T) \tag{3}$$

In the actual modeling of Ho-Lee one must calibrate, as done later, to determine what these perturbation functions actually are. Earlier it is stated that for any bond, $P_i^{(n)}(0) = 1$ for all n and all i since that is the moment the bond matures. Substituting $T = 0$ into equations (2) and (3) above, we see that $P_i^{(n)}(0) = h(0)$ so

$$h(0) = h^*(0) = 1 \tag{4}$$

4.3 Recursive Structure

In order to use this model we must be able to determine the price of a bond at any node of a T period bond today. We can use equations (2) and (3) to recursively go backward until we have an equation. We could pick any n,i but let's look at the case where $n = 3$ and $i = 2$. This means that we are in period 3 and in state 2. Remember that in order for the model to be arbitrage free, it must be true that the results of the calculations along any of the paths on the tree that we take back from a node to today must be equal. It turns out that there are 3 different paths we could take going from today to period 3 state 2. We can either go to the upstate, followed by another upstate, followed by a downstate or an upstate, followed by a downstate, followed by an upstate, or lastly a downstate, followed by an upstate, followed by an upstate. All three of these paths are equivalent should this be a truly arbitrage-free market.

As we expand our tree to more and more periods, it becomes noticeable that there is a pattern with the number of ways to reach a certain node. Taking any node with period n and state i, starting from today we must go to the upstate i times and to the downstate $n-i$ times, but it is not important when due to no-arbitrage. The number of unique ways from today to the node is thus n choose i which is know in probability to be $\frac{n!}{i!(n-i)!}$. Another way to see this is to rotate the tree up 90^o and fill in Pascals triangle. In the case here of $n=3$ and $i=2$, we get $\frac{3!}{2!(3-2)!}=3$ unique paths. To achieve today's price though with the example of a 3 period tree, we simply work recursively:

$$
\begin{aligned}
P_2^{(3)}(T) &= \frac{P_2^{(2)}(T+1)}{P_2^{(2)}(1)} h^*(T) \\
P_2^{(2)}(T+1) &= \frac{P_1^{(1)}(T+2)}{P_1^{(1)}(1)} h(T+1) \\
P_1^{(1)}(T+2) &= \frac{P_0^{(0)}(T+3)}{P_0^{(0)}(1)} h(T+2)
\end{aligned}
$$

putting this together we get,

$$
P_2^{(3)}(T) = \frac{P_0^{(0)}(T+3)}{P_2^{(2)}(1) P_1^{(1)}(1) P_0^{(0)}(1)} h(T+2) h(T+1) h^*(T)
$$

To simplify, we define $P_0^{(0)}(T)$ as $P(T)$ and use the same equations (2) and (3) to simplify the bottom of our equation,

$$
\begin{aligned}
P_2^{(2)}(1) &= \frac{P_1^{(1)}(2)}{P_1^{(1)}(1)} h(1) \\
P_1^{(1)}(1) &= \frac{P_0^{(0)}(3)}{P_0^{(0)}(1)} h(2)
\end{aligned}
$$

giving us a final result of,

$$P_2^{(3)}(T) = \frac{P(T+3)}{P(3)} * \frac{h(T+2)h(T+1)h^*(T)}{h(2)h(1)} \tag{5}$$

Fortunately, with this result, we don't have to compute the other two paths recursively, we can just express them to be:

$$P_2^{(3)}(T) = \frac{P(T+3)}{P(3)} * \frac{h(T+2)h^*(T+1)h(T)}{h(2)h^*(1)} \tag{6}$$

$$P_2^{(3)}(T) = \frac{P(T+3)}{P(3)} * \frac{h^*(T+2)h(T+1)h(T)}{h^*(2)h(1)} \tag{7}$$

While which perturbation functions are for the upstate and which are for the downstate differ across these three, we know that all three equations must yield the same result. If we were to do this for nodes from other periods we would notice a pattern develop that can be expressed by the one equation.

$$P_i^{(n)}(T) = \frac{P(T+n)}{P(n)} * \frac{h^*(T+n-1)h^*(T+n-2)...h^*(T+i)h(T+i-1)...h(T)}{h^*(n-1)h^*(n-2)...h^*(i)h(i-1)...h(1)} \tag{8}$$

Since a perturbation of a negative number does not exist and $h(0) = 1$, we only include functions with h(T) where $T > 0$. This means that if $n-1 < i$ then we include no h^* functions and if $n-1 = i$ then we only include $\frac{h^*(T)}{h^*(1)}$. This works in a similar fashion for the h functions; thus if $i-1 \leq 0$ we include no h functions and only $\frac{h(T)}{h(1)}$ if $i-1 = 1$.

4.4 The Binomial Structure

A binomial distribution is a discrete distribution with parameters n and p, $X \sim Binomial(n,p)$. In a binomial structure n must be an positive integer, $n = 1, 2, 3, ...$, equal to the number of independent $Binomial(n,p)$ trials. The

Ho-Lee model uses n to determine the number of periods we have in the model. At every instant n in our binomial model there is a two-pronged event such that we have probability p of having one thing happen and probability $1-p$ of another thing happening. In this case p would be the probability of our bond going to the upstate while $1-p$ is the probability of the bond going to the downstate. By looking back at Figure 2 one can see that at each node there is a subsequent event determined by the probability p. It is important to note that each of these nodes, or events, are independent.

For simplicity, also due to our lack of knowledge about the market that also later assumed in the calibration, let's say that $p = .5$ so there is an equal chance of the bond going up or down. The events in the binomial process are independent because if at one node the bond price goes up, at the following node the bond will still go up with probability $p = .5$ and down with probability $1-p = .5$. The binomial distribution is known as a counting process. That is given $n = N$ we can count the number of times, X, that a certain event, goes up or down, happens. For the Ho-Lee model this number is very useful because it tells us what state our bond is in. If, for example, we have $n = 10$ and at the end $X = 4$, we know that our bond is in period 10 and state 4, or $P_4^{(10)}(T)$. If we were to use something as simple as a coin flip we would know the probability of it being either heads or tails, but when observing the market, one can only guess whether our bond will jump to the upstate or stay in the downstate. Thus, Ho-Lee introduce the implied probability, π, which is derived from a portfolio only including a one discount bond. The complete derivation of the following equation is shown in Appendix A of the Ho-Lee paper [8]:

$$\pi h(T) + (1-\pi)h^*(T) = 1 \;\; for\; n, i > 0 \tag{9}$$

The resulting equation relates the two perturbation functions by relating them to reward functions. This equation then makes sense intuitively because the probability, π, of getting the reward in the upstate times the reward plus the probability, $1-p$, of getting the reward in the downstate times the reward is equal to 1. This is true since the perturbation functions are both greater than zero and then the equation holds when one function is greater than or equal to one, $h(T) \geq 1$, and the other is less than or equal to one, $h^*(T) \leq 1$[7]. From the case above with $n = 3$, $i = 2$ we saw,

$$h(T+2)h^*(T+1)h(T)h^*(2)h(1) = h^*(T+2)h(T+1)h(T)h(2)h^*(1)$$

Here we can look at the simpler case of n = 2 and look at the the middle node in the third column of the tree in Figure 2. We know that it must not matter along which path we get to a single node so,

$$h(T+1)h^*(T)h^*(1) = h^*(T+1)h(T)h(1)$$

We can look back at equation (9) and solve for $h^*(T) = \frac{1-\pi h(T)}{(1-\pi)}$ and then plug in for $h^*(T)$ to eliminate all downstate perturbation functions. This leaves us with,

$$h(T+1)(1-\pi h(T))(1-\pi h(1)) = (1-\pi(h(T+1))h(T)h(1)(1-\pi) \tag{10}$$

The end goal is to solve for two equations for $h(T)$ and $h^*(T)$ in terms of only π, $h(1)$, and T. This is done completely in my Appendix A, not the one previously mentioned from Ho-Lee, in which the following expressions are derived:

$$h(T) = \frac{1}{\pi + \delta^T(1-\pi)} \tag{11}$$

[7]In the case of a recession though, when the yield curve is inverted, interest rates drop for longer period bonds, $h(T)$ and $h^*(T)$ would actually be reversed here.

$$h^*(T) = \frac{\delta^T}{\pi + \delta^T(1-\pi)} \tag{12}$$

Observe that equations (11) and (12) imply that $h^*(T) = h(T) * \delta^T$

With an expression for $h^*(T)$ in terms of $h(T)$ we can go back and simplify equation (8). Let's return to the example with $n = 3$ and look specifically at $i = 1$. Of course there are many different paths for us to take to this node, but because of the no-arbitrage condition it shouldn't matter which one we use. We can either solve this recursively or just use equation (8) to get,

$$P_1^{(3)}(T) = \frac{P(T+3)}{P(3)} * \frac{h^*(T+2)h(T+1)h(T)}{h^*(2)h(1)}$$

and then use equation (12) to get rid of any h^*,

$$\begin{aligned} P_1^{(3)}(T) &= \frac{P(T+3)}{P(3)} * \frac{h(T+2) * \delta^{T+2} h(T+1)h(T)}{h(2) * \delta^2 h(1)} \\ &= \frac{P(T+3)}{P(3)} * \frac{h(T+2)h(T+1)h(T)}{h(2)h(1)} * \delta^{T(3-1)} \end{aligned}$$

If we try this again with one more example we begin to discern the pattern. Now again let $n = 3$, but now have $i = 2$:

$$\begin{aligned} P_2^{(3)}(T) &= \frac{P(T+3)}{P(3)} * \frac{h^*(T+2)h^*(T+1)h(T)}{h^*(2)h^*(1)} \\ &= \frac{P(T+3)}{P(3)} * \frac{h(T+2) * \delta^{T+2} h(T+1) * \delta^{T+1} h(T)}{h(2) * \delta^2 h(1) * \delta^1} \\ &= \frac{P(T+3)}{P(3)} * \frac{h(T+2)h(T+1)h(T)}{h(2)h(1)} * \delta^{T(3-2)} \end{aligned}$$

What we end up with is a complete interpretation of any bond at any period, only using the h perturbation function:

$$P_i^{(n)}(T) = \frac{P(T+n)}{P(n)} * \frac{h(T+n)h(T+n-1)(h(T+n-2)h(T+n-3).....h(T)}{h(n)h(n-1)h(n-2)h(n-3).....h(1)} * \delta^{T(n-i)} \tag{13}$$

[8]This works just like equation (8) so we can't have a perturbation of a maturity less than T. Thus a perturbation function $h(T-1)$ or $h(-1)$, and those similar, cannot be part of the expression in equation (13)

The Ho-Lee model is a stochastic short rate model. So far we have looked at using a binomial lattice to price different bonds, but we have yet to look at the interest rate r for a single period, the short rate. A yield curve is a line that plots interest rates of bonds, here we think of them all as U.S. treasury bills, based on their time to maturity. This curve is quite important because it typically ranges from the short 3-month bonds to the long 30-year bonds and so it can tell us a lot about the state of the economy. Typically a normal yield curve slopes upwards, meaning that short term interest rates are low and the longer the term structure, the higher the interest rate. This should make sense because if you are going to lend someone money, you might be more worried they won't pay you back the longer they borrow it, so you charge them a higher rate. This is called normal because in a good economy this is the yield curve one might expect. An inverted yield curve is one where the short term interest rates are much higher than the longer term interest rates. An example of when this happened was the beginning of most recent recession in 2008. This is why the yield curve is known as a leading indicator for the state of the economy. The yield of a bond is the return on investment which comes from interest payments and dividends and is typically given as a percentage of the value of the bond itself. So the yield of a bond with maturity T is the total interest rate payment discounted to today.

The yield of a T period bond y with no coupon, like the U.S. treasury bills, is defined as $Y = (\frac{F}{PV})^{1/T} - 1$ where F and PV are the face value and the present value of the bond respectively. In this model F is always 1 and PV is

[8]There is a small typo here in the *Journal of Finance's* version. It has $\delta^{T(n-1)}$ instead of $\delta^{T(n-i)}$.

then a percentage of F. Here we will assume that all interest rate payments are continuously compounded so the present value is $PV = Fe^{-rT}$. The present value, PV, is what someone would be willing to pay to receive the interest payments so PV = P(T). If we again set F = 1 then we get $P(T) = e^{-rT}$. Taking the log of both sides and then dividing by -T we get a function for the interest rate,

$$r(T) = \frac{-lnP(T)}{T} \tag{14}$$

The Ho-Lee stochastic short rate model attempts to price interest rate contingent claims by estimating the interest rate, which we can derive if we know the price of any bond. The short rate though is just the one period interest rate, which is the most realistic to predict given its short future. Using equation (13) we can plug in the price of a one period bond to solve for the short rate at any period n or state i:

$$\begin{aligned} r_i^{(n)}(1) &= -ln\Big[\frac{P(1+n)h(n)\delta^{n-i}}{P(n)}\Big] \\ &= -ln\Big[\frac{P(1+n)}{P(n)} * \frac{\delta^{n-i}}{\pi + (1-\pi)\delta^n}\Big] \\ &= -ln\Big[\frac{P(1+n)}{P(n)}\Big] + ln[\delta^{n-i}] - ln[\pi + (1-\pi)\delta^n] \end{aligned}$$

Earlier we defined π, $0 \leq \pi \leq 1$, as the implied binomial probability. This was what the market believes the chance of going to an upstate is versus the chance of going to a downstate. There has yet to be defined a probability p for our binomial distribution. Inherently different from the implied probability, p is the actual probability of going to an upstate versus a downstate. Thus the state i at which our model is distributed $i \sim Binomial(n, p)$ and the expected value is $E(i) = np$. Plugging this back into our short rate, $r(T)$ we are then given the expected value of our one period interest rate:

$$E[r(1)] = -ln\Big[\frac{P(1+n)}{P(n)}\Big] + ln[\delta^{n-np}] - ln[\pi + (1-\pi)\delta^{n}] \tag{15}$$

In order to compute the variance it is easiest to look at the 1st and 2nd moments of a simple case in which $n = 1$. Given that we start at any node there is some probability p that we will go to the upstate and have payoff of δ which is defined by the uncertainty of how interest rates will move. If $\delta = 1$ then there is no uncertainty since $ln(1) = 0$. There is also a chance, with probability $1 - p$, that we will go to the downstate and receive a payoff of $-\delta$. The present value of these payoffs, assuming continuous compounding, are $ln(\delta)$ and $-ln(\delta)$. In order to find the variance we then compute $var = E(X^2) - E(X)^2$. We end with the variance:

$$var = np(1-p) * ln(\delta)^2 \tag{16}$$

The full derivation of the variance, equation (16), is found in Appendix B. The variance of a binomial distribution is defined as $np(1 - p)$. The difference in our special case here compared to a normal binomial distribution is that there is some payoff added to the equation. With this we now have a complete model for the short rate. This short rate from any period can be applied to any sort of interest rate contingent claim in the market or simply used to estimate where rates might go on a floating rate loan.

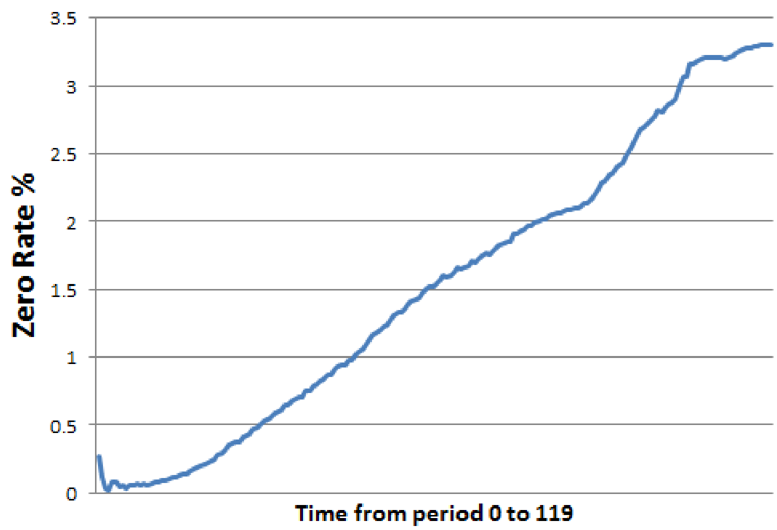
5 Calibrating Ho-Lee

The problem with many models in economics and finance is that while theoretically they work, many of them either fall short when applied or simply can't be applied. Ho-Lee is an exception because with just a few simplifications, we can actually calibrate it to real interest rates. It won't look exactly the same as the Ho-Lee model has been described so far, but it follows the binomial tree quite closely. Appendix C is a complete Visual Basic for Applications (VBA)[9] Excel code written out for potential use. In order to test this model we must first decide on the period. The continuos time version, while possibly the most realistic version, of this would have a period of $\frac{1}{\infty} = 0$, so it is an impossible task. Here I have decided to make the short rate the interest rate for a 3-month treasury bill, so the period is .25 years. In order to find the market rates for each three month period we must bootstrap real bonds found on Bloomberg. When taking information, it is important to obtain all of the data for all of the bonds from a single day to realistically create the yield curve because the next day will have it's own respective yield curve. The 3-month, 6-month, 12-month, 2-year, 5-year, 10-year, and 30-year bonds are the U.S. Treasury Securities currently traded in markets. We will use the bootstrapping method of taking the current yields, prices, coupons, and maturities of these bonds, turning them into zero-coupon bonds, and then creating a yield curve. By using the Financial Toolbox package in Matlab this can be done with a function called *zbtyield*. Once we have these zero rates for each of our 120 periods, we can calibrate Ho-Lee to fit this curve. The yield curve pictured in Figure 3, is not perfectly smooth, especially in the quite volatile first year when the spread between the bid-yield and the ask-yield is quite large, but it does, with its slope, accurately depict how the interest rate market looks today.

For my process I followed the lecture notes from a course by Backus and Zin (1999) [1]. When the Ho-Lee model is applied, the perturbation functions $h(T)$ and $h^*(T)$ need to be interpreted somehow for each period. This is where the calibration comes in. The goal for this calibration is to have a complete

[9] VBA is a language that is specific to the program Microsoft Excel. It allows users to create their own more complicated functions by using the many already available Excel functions. In order for the code to run successfully though one must also be using the matching Excel .xlsm file making it less advantageous for open source code.

Figure 3: Yield Curve



understanding, for the next 30 years at least, of the expected value of what a dollar n periods from now will be worth today. To find this, we will need the short rate for every node (n, i). Unfortunately, the best method for this is to simply guess and check, which explains the many looping functions I have created below. Instead of working recursively, like it was did in the theoretical model, for the calibration we start today, at the node $n = 0$, $i = 0$, and work forward. To start, we can begin in the first node where $n = 0$ and $i = 0$. Clearly a dollar today is worth a dollar today. For the next period we will have an upstate $n = 1$, $i = 1$, and a downstate, $n = 1$, $i = 0$. The applicable formula given from the lecture notes gives the next period's zero-rates as functions of the previous period's zero-rate, the drift u_{t+1}, the standard deviation σ, of the change in interest rates, the length of the period, and lateral movement of the bond [1]. It is written as:

$$r_{t+1} = r_t + u_{t+1} + L^{.5}\sigma\nu_{t+1} \tag{17}$$

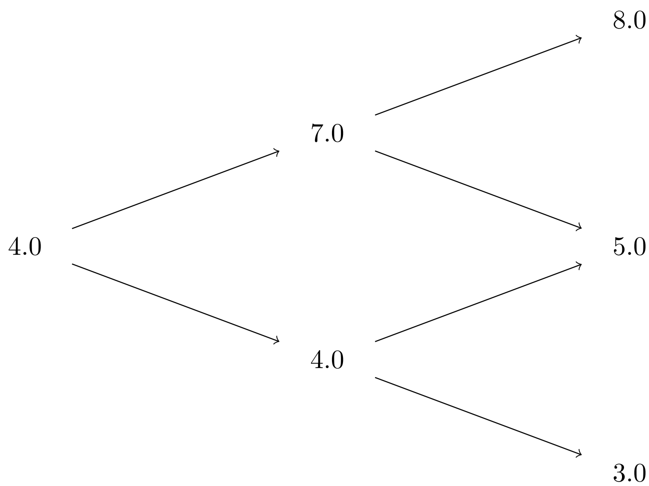
Where L is the length of the periods, and ν_{t+1} is either -1 or 1 depending on whether we are moving to a downstate or an upstate respectively. In our case we have set $L = .25$. From bootstrapping we've already calculated the zero rate for today. In my particular case I found the zero-rate for today to be about .036%. To find σ, I took 3-month treasury bill interest rates for the past month and took the standard deviation and found it to be about .18%. With this we have everything for equation (17) except for the drift parameter, u_{t+1}. The drift works such that there is a single drift for each period, but there are not necessarily any periods with the same drift. To calculate the drift for the first period we guess, calculate the state prices, then compare our answer to the true zero-rate.

The state prices are from where we will eventually derive the discount factor. State prices are named such because they are particular to the state and period the binomial tree is in. The state price is simply the dollar value today of a dollar at that node. So we begin with a dollar at that node, (n, i), and then compute the dollar values for the period before recursively until today. One thing that we must determine here is the probability of going to the upstate vs. the downstate, denoted as p above. Since we have no other information about the market, we must assume the simplest case where $p = 1 - p = .5$. Computing the state prices is then done using another formula given to us by Backus and Zin [1]:

$$S(n, i) = .5\beta(n+1, i+1) * S(n+1, i+1) + .5\beta(n+1, i) * S(n+1, i) \quad (18)$$

Where $S(n, i)$ is the state price of that specific node and $\beta(n, i)$ is the discount factor for that specific node. The same can be said for the nodes $(n+1, i+1)$ and $(n+1, i)$, the upstate and the downstate in the following period, $n+1$, respectively. The discount factor for a node is simply the continuous compounded

Figure 4: Zero-Rates



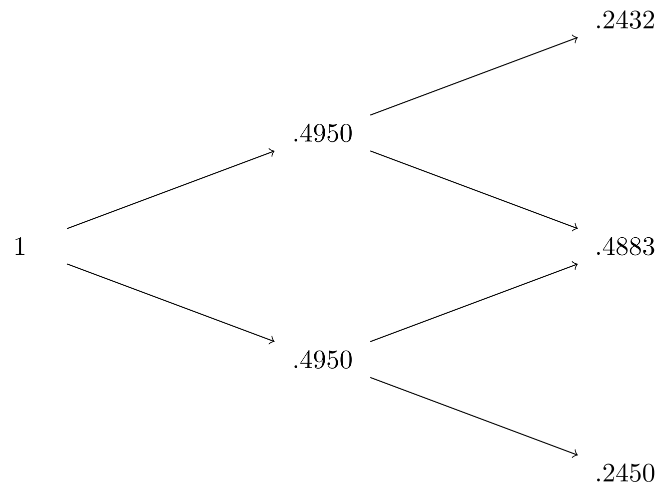
returns at the interest rate of that node by the equation:

$$\beta = e^{-rL/100} \tag{19}$$

Note that equation (19) is very similar to equation (14) which also took into account continuous compounding. Now we can apply these equations and create a tree of state prices. Given two periods, we can find the state price for the node (2,1) by applying a 1 to that node and working backwards. Below I have created an example case starting with zero-rates for each node. Then by calculating equation (18) for $S(2,1)$ and using equation (19) to find the discount rates β using the interest rates from figure 4 we get that $S(1,1) = \frac{.9826}{2} * (0+1)$. Similarly $S(1,0) = \frac{.9900}{2}(1+0)$. Finally, $S(0,0) = \frac{.9900}{2}(.4913 + .4950) = .4883$. $S(0,0)$ is the state price for the node $(2,1)$ and we can do the exact same thing for the other four nodes to fill out the tree. In Figure 5 I have filled out all the nodes.

It is clearly noticeable that while the two nodes in the first period are equal, the nodes in the second period vary heavily with the middle one being weighted

Figure 5: State Prices



heavier. This happens because, while there is only one path to get to the nodes $(2,2)$ and $(2,0)$, there are two paths to get to the node $(2,1)$ which get factored in. With our state prices we can find the discount rate for each period simply by summing all of the state prices in that period. Finally, to find the expected zero-rate for each period we simply reverse equation (19) so that we are solving for r,

$$r = -ln(\beta) * \frac{100}{L} \tag{20}$$

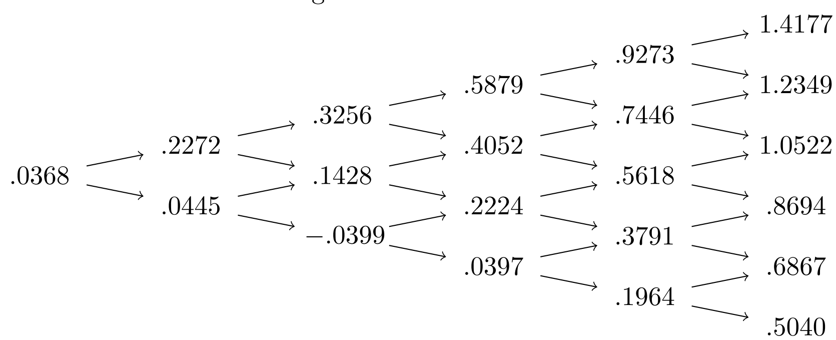
For example this gives us a 3-month short rate of, $r = -ln(.9765) * \frac{100}{.5} = 4.756\%$ for period 2.

So far we have ignored the drift parameter, denoted as u, which is what we are actually trying to find by calibrating. With this process though we can begin our guess and check. We begin with the first period and estimate some u and then calculate both interest rates for the period with equation (17). We use these rates and equations (18) and (19) to find our discount rate and then zero-rate. If that zero-rate is the same as the zero-rate we found by bootstrapping bonds then we are finished and can do this with the next period and on until

Figure 6: Results

Maturity	0.25	0.50	0.75	1.00	1.25	1.50
Zero-Rates	0.0368%	0.0622%	0.1051%	0.1574%	0.2381%	0.3634%
Discount Rates	0.9877	0.9996	0.9992	0.9984	0.9970	0.9946

Figure 7: Interest Rate Tree



period 119.

In actual application, it is impossible to perfectly mimic the observed zero-rates so we set some small $\varepsilon > 0$ such that the absolute value of the difference between my calculated zero-rate and the observed zero-rate is less than ε. One can adjust the accuracy of this model by keeping ε as small as possible and by minimizing the difference between each guess. Of course the smaller these parameters are, the longer the program will take to run. Figure 6 is a chart with the discount factors and zero-rates that were calculated for the first 5 periods of the complete 119 that are calibrated. Figure 7 is the full accompanying short rate interest rate tree for each period n and state i. This process now allows one to take bond yield data from any single day and calibrate their own version.

6 Conclusion

From here I believe an exploration into how this model prices different derivatives is the logical next step. Theoretically the model is made to price bond options, bond futures, swaptions, interest rate caps, and many other interest rate contingent claims. Given the complete 119 period tree that can be created with the bootstrapped bonds and the code in Appendix C below, one can apply the discount factors to the market and test their accuracy. This model was the foreground for arbitrage-free interest rate models, but due to some of its shortcomings it has been criticized and attemped to be improved upon. Black-Derman-Toy [2] for example, eliminated the possibility of a negative interest rate, it added mean reversion, and it has a log normal distribution. For these reasons the BDT model is used more frequently to model complex interest rate contingent claims. In this paper just the Ho-Lee model was expositioned and calibrated, but in order to really analyze interest rate models as a whole one should calibrate each of them and compare the results to test their relative accuracies. Each of these models are useful in order to understand interest rate contingent claims and in the future I would enjoy exploring all of them.

Appendix A Derivation of Equations (11) and (12)

We begin where we left off above with equation (10),

$$(T+1)(1-\pi h(T))(1-\pi h(1)) = (1-\pi h(T+1))h(T)h(1)(1-\pi)$$

First we divide by $h(T)h(T+1)$ in order to isolate each on their own sides and get,

$$\Big[\frac{1}{h(T)} - \pi\Big](1-\pi h(1)) = \Big[\frac{1}{h(T+1)} - \pi\Big]h(1)(1-\pi)$$

We can then isolate $\frac{1}{h(T+1)}$ on the left side and have,

$$\frac{1}{h(T+1)} = \pi + \frac{1-\pi h(1)}{h(1)(1-\pi)}\frac{1}{h(T)} - \pi\frac{1-\pi h(1)}{h(1)(1-\pi)}$$

Now we rearrange to get our equation in the form of a line such that $\frac{1}{h(T)}$ is our only non constant. This will give us,

$$\begin{aligned}
\frac{1}{h(T+1)} &= \frac{1-\pi h(1)}{h(1)(1-\pi)}\frac{1}{h(T)} + \pi\Big[1 - \frac{1-\pi h(1)}{h(1)(1-\pi)}\Big] \\
&= \frac{1-\pi h(1)}{h(1)(1-\pi)}\frac{1}{h(T)} + \pi\Big[\frac{h(1)(1-\pi)-(1-\pi h(1))}{h(1)(1-\pi)}\Big] \\
&= \frac{1-\pi h(1)}{h(1)(1-\pi)}\frac{1}{h(T)} + \frac{\pi(h(1)-1)}{(h(1)(1-\pi)}
\end{aligned}$$

In order to follow the notation of Ho-Lee[8] we set,

$$\delta = \frac{1-\pi h(1)}{h(1)(1-\pi)} \tag{A.1}$$

and

$$\gamma = \frac{\pi(h(1)-1)}{(h(1)(1-\pi)} \tag{A.2}$$

If we plug back in equations (A.1) and (A.2) to our function of $\frac{1}{h(T+1)}$ we get,

$$\frac{1}{h(T+1)} = \frac{\delta}{h(T)} + \gamma \tag{A.3}$$

Now let us look at how equation (A.3) looks over time and solve it as a first-order linear difference equation. First let $f(T) = \frac{1}{h(T)}$ to make it easier visually. Next we create what is called a difference equation. Essentially we look at the first view values of T until we see a pattern and then create a function for $f(T)$ that works for any T. Equation (A.3) then becomes,

$$f(T+1) = \delta f(T) + \gamma$$

The equation is then solved recursively as follows,

$$f(2) = f(1+1) = \delta f(1) + \gamma \tag{A.4}$$

$$f(3) = f(2+1) = \delta f(2) + \gamma \tag{A.5}$$

Putting equations (A.4) and (A.5) together we get,

$$f(3) = \delta^2 f(1) + \delta\gamma + \gamma$$

Similarly,

$$f(4) = \delta^2 f(1) + \delta^2\gamma + \delta\gamma + \gamma$$

Thus in general we see that for any $T \geq 2$,

$$f(T) = \delta^{T-1} f(1) + \gamma \sum_{k=0}^{T-2} \delta^k$$

$$f(T) = \delta^{T-1} f(1) + \gamma \frac{1-\delta^{T-1}}{1-\delta} \tag{A.6}$$

as long as $\delta \neq 1$, which is the case when we are completely confident that bond prices will not adjust in the next period, thus the interest rate will remain constant. We can rearrange equation (A.6) to get,

$$\begin{aligned} f(T) &= \frac{f(1)}{\delta}\delta^T + \frac{\gamma}{1-\delta} - \frac{\gamma}{\delta(1-\delta)}\delta^T \\ &= \frac{\gamma}{1-\delta} + \left[\frac{f(1)}{\delta} - \frac{\gamma}{\delta(1-\delta)}\right]\delta^T \end{aligned}$$

Now let's create some constant C and let it be,

$$C = \frac{f(1)}{\delta} - \frac{\gamma}{\delta(1-\delta)}$$

then note that, by the definitions of δ and γ, given by equations (A.1) and (A.2), we know that,

$$\frac{\gamma}{1-\delta} = \pi$$

Therefore $f(T) = \pi + C\delta^T$, but since we let $f(T) = \frac{1}{h(T)}$, we get,

$$h(T) = \frac{1}{\pi + C\delta^T} \tag{A.7}$$

Earlier in the paper, in equation (4) we defined $h(0) = 1$, so we can use this as our initial condition in order to solve for C.

$$\begin{aligned} h(0) &= \frac{1}{\pi + C\delta^0} \\ 1 &= \frac{1}{\pi + C} \\ C &= 1 - \pi \end{aligned}$$

Plugging this back into equation (A.7) we get as our function of $h(T)$,

$$h(T) = \frac{1}{\pi + \delta^T(1-\pi)} \tag{A.8}$$

By plugging this into equation (10) we get the $h^*(T)$ version,

$$h^*(T) = \frac{\delta^T}{\pi + \delta^T(1-\pi)} \tag{A.9}$$

Equations (A.8) and (A.9) are what we have previously defined in the body of the paper as equations (11) and (12).

Appendix B Derivation of the Variance (16)

The variance is defined as the second moment, $E(X^2)$, minus the first moment, $E(X)$, squared. Let's begin by solving for the expected value. In our simple case with n = 1, this is the probability of going to the upstate, p, times the payoff added to the probability of going to the downstate, $1-p$, times the payoff. The payoff in this case is determined by δ, which is the difference between the two perturbation functions, $h(T)$ and $h^*(T)$. Instead of looking at the difference one period from now, in order to find the true expected value, we must discount it to today. Using the formula for continuous compounding, $\delta = e^{-rT}$, where r is the interest rate and T is the time. Since this is just a one period case, we set $T = 1$. We can then solve for our interest rate $r = -ln\delta$. Since δ is the difference between the perturbation functions we can then let our payoff in the upstate be half the total, $-\frac{1}{2}ln\delta$ and the payoff in the downstate to be the other half, $\frac{1}{2}ln\delta$. Note that $0 < \delta \leq 1$.

$$
\begin{aligned}
E(X) &= p\frac{1}{2}(-ln(\delta)) + (1-p)(\frac{1}{2}ln(\delta)) \\
&= -\frac{1}{2}pln(\delta) + \frac{1}{2}ln(\delta) - \frac{1}{2}pln(\delta) \\
&= \frac{1}{2}ln(\delta) - pln(\delta)
\end{aligned}
$$

For the second moment we can do the same thing but our payoff is now $(\frac{1}{2}ln(\delta))^2$

$$
\begin{aligned}
E(X^2) &= p(-\frac{1}{2}ln(\delta))^2 + (1-p)(\frac{1}{2}ln(\delta))^2 \\
&= \frac{1}{4}pln(\delta)^2 + \frac{1}{4}ln(\delta)^2 - \frac{1}{4}pln(\delta)^2 \\
&= \frac{1}{4}ln(\delta)^2
\end{aligned}
$$

Now for the variance we have,

$$\begin{aligned}
E(X^2) - E(X)^2 &= \frac{1}{4}ln(\delta)^2 - (\frac{1}{2}ln(\delta) - pln(\delta))^2 \\
&= \frac{1}{4}ln(\delta)^2 - \frac{1}{4}ln(\delta)^2 + pln(\delta)^2 - p^2 ln(\delta)^2 \\
&= pln(\delta)^2 - p^2 ln(\delta)^2 \\
&= p(1-p) * ln(\delta)^2
\end{aligned}$$

This equation actually has an invisible n, which is the number of periods in our case, since $n = 1$. This gives us a variance of,

$$var = np(1-p) * ln(\delta)^2 \tag{B.1}$$

Equation (B.1) is what we have previously defined in the body of the paper as equation (16)

Appendix C VBA Code

Unfortunately, the problem with VBA is that it works congruently with an Excel document. This means that this code itself will not be entirely useful without the document. If you are interested in the document itself I would be happy to provide it to you. The accuracy, held in the variables j and epsilon, can always be adjusted though to get a closer approximation of the true spot rates, although it will just take even longer to run. It should be noted that in some cases, in order to fit the full lines of code, in the cases of "Sum" and "Ln" I omitted "WorksheetFunction" which is part of the correct syntax but unnecessary in the understudying of the code. After finishing and running this code, I do regret using VBA. The main shortcoming of Excel is that it is a very weak processor. It is very slow running the large loops that other programs run much faster. Regrettably I only recently started using Matlab[10], in order to bootstrap the yield curve, but given another chance I would have written all the code in Matlab since it would have had no trouble calculating the entire tree.

```
Sub StatePrices(Time)
'Takes the period number and creates state prices
For Period = 0 To Time
  For State = 0 To Period
    Row = 495
    Col = 2
    For c = 254 To 373  'Sets cells to a function of the interest rate for each state/period
      For d = 2 To 121
      Cells(c, d) = Exp(-Cells(c - 244, d) * Cells(2, 2) / 100)
      Next d
    Next c
    Range(Cells(376, Col), Cells(Row, Col + 119)) = 0
    'Resets everything then gives target period price of 1
    Cells(Row - State, Col + Period) = 1
```

[10]A programming language and computing environment excellent for working with large amounts of data and very useful for applications in Economics and Finance

```
    Node = 0
    If Period > 0 Then 'Calculates state prices
      i = 1
      For b = Period To 0 Step -1
        For a = b To 0 Step -1
        If Row - (a - 1) > 495 Then Exit For
        'This is equation 18 as defined above
        Cells(Row - (a - 1), Col + (Period - i)) = ((Cells(Row - a, Col + (Period - i + 1)) + _
        Cells(Row - (a - 1), Col + (Period - i + 1))) _
        * Cells(Row - (a - 1) - 122, Col + (Period - i)) / 2)
        Node = (Cells(Row - (a - 1), Col + (Period - i)))
        Next a
        i = i + 1
      Next b
    End If
    aRow = 251
    If Node <> 0 Then
      Cells(aRow - State, 2 + Period) = Node
    End If
  Next State
  aRow = aRow + 1
Next Period
End Sub

Private Sub DriftRates_Click()
'Click button command which calibrates model
epsilon = 0.0001 'Parameter to decide how accurate calibration needs to be
For h = 1 To 119 'Runs from period 1 to 119. We already know for 0
  For j = -0.004 To 0.004 Step 0.00001 'Parameter for the accuracy
    For Period = h To h
      b = 3
      Row = 129
      For a = 0 To Period 'Give a guess for each state in period
        'All state but 0 are upstate from period n. (n+1,0) is downstate of (n,0)
        If b Mod 2 <> 0 Then
          k = -1
```

```
            i = 0
          Else:
            k = 1
            i = 1
          End If
          'Calls upon the cells that give previous state rate, period = .25, and StDev
          Guess = Cells(Row + i, 1 + Period) / 100 + j + (Cells(2, 2) ^ 0.5) * Cells(7, 3) * k
          'Sets cell in tree to the guess
          Cells(129 - a, 2 + Period) = Guess * 100
          b = b * 2
          Row = Row - 1
                  Next a
              Next Period
      StatePrices (Period) 'Calls function to gives state prices
      'Calculates Discount Factor
      Cells(6, Period + 1) = Sum(Range(Cells(132, Period + 2), Cells(251, Period + 2)))
      DiscountFactor = Cells(6, Period + 1)
      'Calculates Zero Rate
      Cells(5, Period + 1) = -(100 / (Cells(1, Period + 2) * Cells(2, 2)) * Ln(DiscountFactor))
      CalibratedRate = Cells(5, Period + 1)
      BootstrappedRate = Cells(3, Period + 1)
      If Abs(CalibratedRate - BootstrappedRate) <= epsilon Then
        'Checks if the Calibrated rate is close enough to the real rate
        Exit For
      End If
  Next j
Next h
End Sub
```

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