2016

Bikei Cohomology and Counting Invariants

Jake L. Rosenfield
Claremont McKenna College

Claremont Mckenna College

Bikei Cohomology and Cocycle Enhancements

submitted to

Professor Sam Nelson

and

Dean Peter Uvin

by

Jake Rosenfield

for

Senior Thesis

Spring 2016

April 25, 2016

Abstract. This paper gives a brief introduction into the fundaments of *knot theory*: introducing knot diagrams, knot invariants, and two techniques to determine whether or not two knots are *ambient isotopic*. After discussing the basics of knot theory an algebraic coloring of knots knows as a *bikei* is introduced. The algebraic structure as well as the various axioms that define a bikei are defined. Furthermore, an extension between the Alexander polynomial of a knot and the Alexander Bikei is made. The remainder of the paper is devoted to reintroducing a modified homology and cohomology theory for involutory *biquandles* known as *bikei*, first introduced in [18]. The bikei 2-cocycles can be utilized to enhance the counting invariant for unoriented knots and links as well as unoriented and non-orienteable knotted surfaces in $\mathbb{R}^4$.

Contents

Acknowledgements. First and foremost, I have to thank my thesis advisor and professor Sam Nelson. Without his assistance and dedication throughout the entire process this paper would never have been possible. Thank you for guiding me on my journey–beginning in Calc I and culminating with the completion of this thesis. I owe the deepest amount of gratitude to my family: Andy, Betsy, Zak, Alex, and Lannie Rosenfield. You have been the greatest sources of inspiration in my life and I love you all very very much. Dad, I owe you the deepest amount of thanks. Without your continued support and mathematically stimulating conversations I never would have been a math major. Lovezels! Finally, I want to thank LFCDS for being the best school I ever went to, and for giving me the desire to pursue math. It has been the most challenging and wonderful journey of my life.

Chapter 1

Introduction

The modern day study of *knots* has been focused on discovering unique *knot invariants* and determining *ambient isotopic* behavior between knots. The behavior of knots can be studied through *planar isotopic* diagrams and a variety of distinct polynomials. Kurt Reidemeister, Alexander, and Briggs all proved that two knot diagrams K_0 and K_1 are ambient isotopic if and only if one knot can be continuously deformed into the other by a series of *planar isotopies* of three different types.

In [11], Joyce introduced an algebraic structure known as *quandles* which can be used to define computable invariants of oriented knots and links. For unoriented knots and links, a special case known as *involutory quandles* or *kei* (圭) has been studied going back to Takasaki [19]. In [9] quandles were generalized to *racks* and in [10] racks were generalized to *biracks*. In [2], the involutory case of biquandles was considered, now known as *bikei* (双圭).

In [10] a homology theory for racks and biracks was introduced in which the 2-cocycle condition corresponds to the Reidemeister III move for a certain way of associating 2-chains to crossings in an oriented rack-colored knot or link diagram. In [5] a subcomplex was defined corresponding to Reidemeister I moves in the quandle case, leading to the theory of quandle 2-cocycle invariants of knots and links. In [4] this construction was generalized to the biquandle case. In [8] the degenerate subcomplex was generalized for the case of non-quandle racks, in each case defining a new family of cocycle enhancements of counting invariants [18].

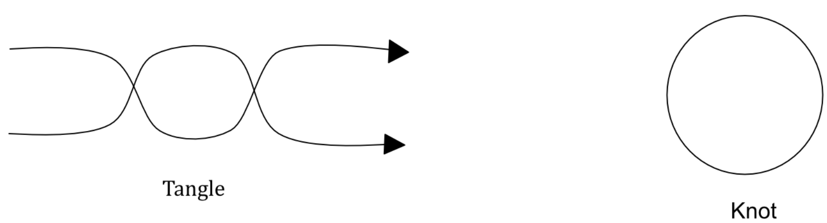
This paper reintroduces a biquandle homology to the case of bikei which will be called *bikei homology*. The bikei axioms are defined and proven. The paper then proceeds to give a definition of the bikei counting invariant and then establishes a bikei *cocycle enhancement*. The bikei counting invariant is then extended to non-orientable *knotted surfaces* in $\mathbb{R}^4$.

Chapter 2

Knot Theory

2.1 Knots

Knot theory refers to the study of mathematical objects know as *knots*. A knot is a simple closed curve which resides in three-dimensional space. Knots can therefore be thought of as the embedding of a a circle, S^1, into $\mathbb{R}^3$ [3]. The closure of the knot implies that no strand is left open on either side of the knot and the knot being simple means that it does not intersect itself at any point except for the closure [8]. Mathematical objects in which the strands are left open are known as *tangles*. Shoe laces and ribbons are common examples of tangles that people encounter daily, while bracelets and necklaces are real life examples of knots.

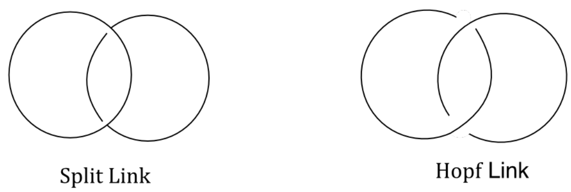


The tangle possesses no closure while the knot is connected everywhere.

2.2 Knot Diagrams

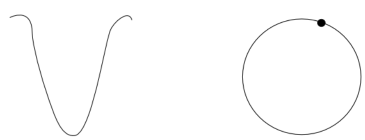
It is rather complicated to draw three-dimensional objects so mathematicians often use *knot diagrams* to simplify the construction of knots to two-dimensions. The diagram is considered to be an illustration of the shadow of a knot in two dimensions, where at each intersection point the under-strands are broken and the over-strands are solid [14].

The intersection points are therefore not actually composed of intersecting strands but rather strands passing over or under one another, at locations known as *crossings*. Knots with a finite number of crossings are known as *tame*, while knots with an infinite number of crossings are known as *wild*. Additionally, when various knots are knotted with one another they form what are known as *links*.

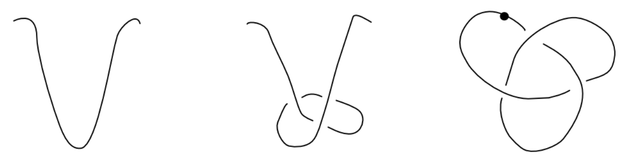


The *Split Link* on the left can be split to form two separate knots; however, the link on the right cannot be unlinked. The *Hopf Link* represented above is the simplest non-trivial link with more than one component.

Comprehending how knots are constructed is not an easy task. One way to help understand their formation is to create a knot. The simplest knot is created by taking a string and gluing the ends together. This type of simple closed curve is known as the *unknot* or the *trivial knot* in $\mathbb{R}^3$ [7]. The tied string can be transformed into a circle through bending and twisting. Therefore knots are preserved under continuous deformations and are considered as topological objects.

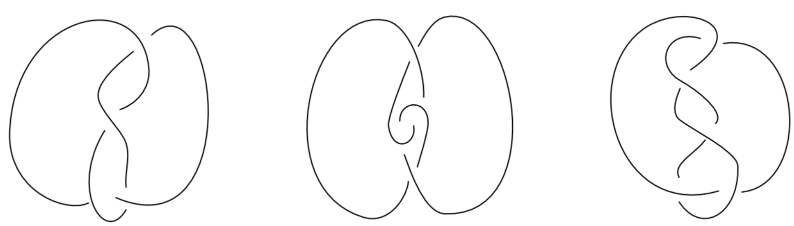


In [1], Colin Adams suggests creating a *knot* by taking a piece of string and tying a knot in it. Then proceeding to glue the ends together you have created a simple closed curve which cannot be untangled to form the unknot. In fact, the new knot that you have formed is called the *trefoil knot*. The trefoil knot is particularly interesting because it is the simplest knot that is non-trivial.



2.3 Ambient Isotopies

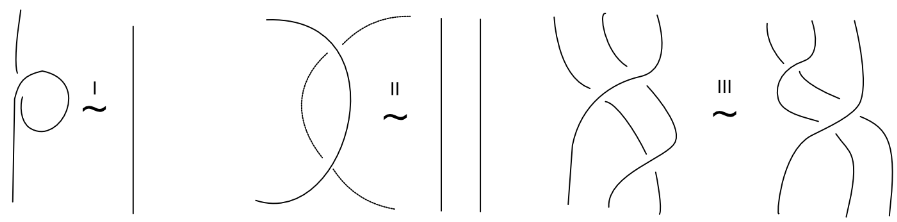
One of the preeminent question for knot theorists is: how can you determine if two distinct knot diagrams represent the same knot? For two knots to be the same it means that there is a way to transform one knot into the other with a series of moves that do not involve cutting or tearing. This sameness property of knots is known as the knots being *ambient isotopic*. The word "isotopy" implies that the knot is being deformed, and the word "ambient" means that the knot is being deformed in the same three-dimensional space that it resides in [1].



The three different knot diagrams above are all unique diagrams of the figure-eight knot. Thus, the three knots are all ambient isotopies of one another. More precisely, if two knots K_0 and K_1 are ambient isotopic then there exists a continuous function which sends K_0 to K_1. As defined in [8], the mapping $F : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ where $F(K_0, 0) = K_0$ and $F(K_0, 1) = K_1$ and $F(\vec{x}, t)$ is injective for all $t \in [0, 1]$ is the mapping of K_0 onto K_1. In the mapping F, the knot K_0 is being continuously deformed onto K_1 with respect to the time variable t. The notation for the existence of an ambient isotopy F taking K_0 to K_1 is $F : K_0 \to K_1$.

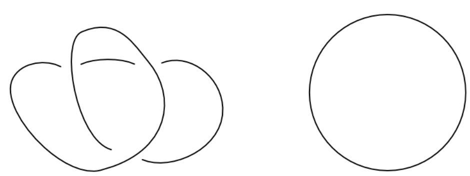
The first verified approach to determine whether or not two knot diagrams represent the same knot was discovered first by Kurt Reidemeister in 1926 and then later by J.W. Alexander and G.B. Briggs in 1927. As is the nature of mathematics, because Reidemeister was first to the discovery, he was bestowed with the honor of having the process named after him.

Definition 1. The three different mathematicians all proved that two knot diagrams K_0 and K_1 are ambient isotopic if and only if one knot can be continuously deformed into the other by a series of *planar isotopies* of the following three types, know as Reidemeister type I, II, and III moves [14]:

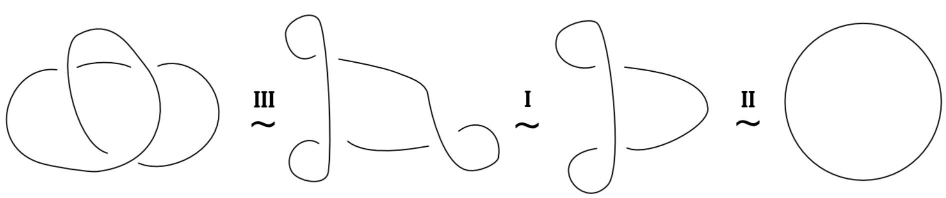


A planar isotopy is a deformation of the knot projection within the plane that it is being projected onto [1]. What this means is that at a single crossing the knot is manipulated in one of three ways such that the new knot is ambient isotopic to the old knot. The first of the three Reidemeister moves can be thought of as either creating a twist or untwist in the knot. The second of the moves allows for the addition or removal of two crossings, and the third move allows the movement of a strand from one side a crossing onto the other side of the crossing [1].

Example 1. Consider the two knots shown below. Is there a way to continuously transform the knot on the left into the knot on the right?



Making use of the three different Reidemeister moves it is clear that the two knots are ambient isotopies of one another.



By first moving one strand across the other then removing a crossing and finally untwisting yields the desired result. Therefore, comparing equivalence classes of knot diagrams is sufficient to determine whether or not two knots are planar isotopic.

2.4 Knot Invariants

Knots do not typically contain as few crossings as the ones presented earlier, and are often difficult to determine equivalence relations solely appealing to Reidemeister moves. Depending on the complexity of the knot there may be better alternatives to determine whether two knots are ambient isotopic.

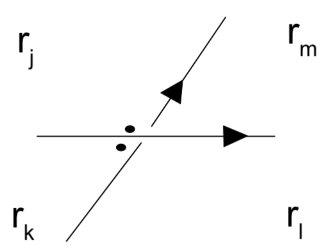
A *knot invariant* refers to a function $f : K \to A$ from the set composed of all possible knot diagrams of a knot K to the set A, such that after each Reidemeister move, the diagrams must satisfy,

$$f(K_1) = f(K_2)$$

where K_1 is the knot before the Reidemeister move and K_2 represents the knot after the move. If the knot invariant f exists, then when each knot diagram is evaluated, $f(K_1)$ must equal $f(K_2)$ [8].

2.5 The Alexander Polynomial

The development of knot theory into the modern era has been centrally focused on discovering new knot invariants. The *Alexander Polynomial*, discovered by J.W. Alexander in 1932, was the first discovered *knot polynomial.* The Alexander Polynomial is a polynomial invariant calculated directly from the diagram. The methodology for computing the polynomial invariant is quite easy to follow. By beginning with an oriented diagram D of a knot K with v crossing points and $c_1, c_2, \ldots, c_v$ crossings [15], and then appealing to Euler's theorem, the arcs in the knot diagram separate the plane into $v + 2$ regions (including the region surrounding the knot).



The two dots represented in the diagram are from Alexander's paper, where the location of the dots represents the left hand side of the under-crossing as the orientation of the knot is observed [15].

In the diagram, the intersection occurs at an arbitrary crossing c_i, and the the four regions are classified r_j, r_k, r_l and r_m where the labeling is observed by going around the knot diagram counterclockwise. After establishing the different regions an equation can be established for each crossing point. For crossing point c_i the equation is:

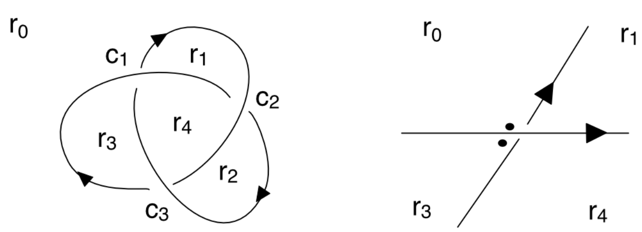
$$C_i(r) = tr_j - tr_k + r_l - r_m = 0.$$

The equation takes the alternating sum of the four regions in their cyclic order and multiplies the dotted regions by t.

Defining an equation at each crossing gives a system of linear equations which can be represented in a $v \times (v+2)$ matrix, M, where the entries are either $\pm t$, ± 1 or 0. The dimensions of the matrix implies that each row corresponds to a crossing point and each column corresponds to a region. After creating the matrix the first step in getting the Alexander Polynomial is to choose any two neighboring regions, r_j and r_k and delete their corresponding columns v_p, v_q. After removing the two columns the resulting matrix is a $v \times v$ square matrix, $M_{p,q}$, called the *Alexander matrix.* Let $\Delta_{p,q}(t)$ be the determinant of the matrix.

Given an equivalent knot diagram K the Polynomial $\Delta_{p,q}(t)$ differs only by a factor of $\pm t^k$ for some integer k. The common principle when expressing the diagram in terms of Alexander Polynomial is to set $\Delta_K(t) = t^n \Delta_{p,q}(t)$. This standardization of the polynomial ensures that the lowest degree term $\Delta_K(t)$ is a positive constant. The resulting normalized equation is called the Alexander Polynomial [15].

Example 2. Consider the oriented trefoil knot below.



The figure on the left gives an illustration of the oriented trefoil and the figure on the right gives Alexanders labeling at crossing c_1. Establishing a linear equation for each of the crossings yields the following three equations:

$$c_1(r) = tr_0 - tr_3 + r_4 - r_1 = 0$$

$$c_2(r) = tr_0 - tr_1 + r_4 - r_2 = 0$$

$$c_r(r) = tr_0 - tr_2 + r_4 - r_3 = 0$$

The resulting equations can be represented in a matrix.

$$M = \begin{bmatrix} t & -1 & 0 & -t & 1 \\ t & -t & -1 & 0 & 1 \\ t & 0 & -t & -1 & 1 \end{bmatrix}$$

Looking at the diagram it becomes clear that two regions that neighbor each other are r_3 and r_4. Therefore deleting their corresponding columns yields a square matrix $M_{3,4}$:

$$\Delta_{3,4}(t) = det(M_{3,4}) = \begin{vmatrix} t & -1 & 0 \\ t & -t & -1 \\ t & 0 & -t \end{vmatrix} = t \begin{vmatrix} -t & -1 \\ 0 & -t \end{vmatrix} + \begin{vmatrix} t & -1 \\ t & -t \end{vmatrix}$$

$$= t^3 - t^2 + t$$

$$= t(1 - t + t^2)$$

factoring out a t gives the normalized polynomial, which ensures the lowest degree term is a positive constant. Therefore,

$$\Delta_K(t) = 1 - t + t^2$$

Because this is the Alexander Polynomial of one construction of the trefoil knot,calculating Δ_K from any knot diagram will give the same equation for the trefoil knot [15].

Chapter 3

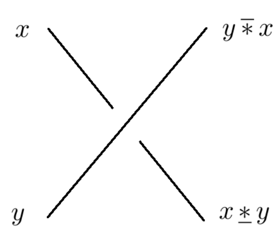
Bikei

3.1 Bikei

The knot diagrams illustrated in the proceeding sections can be given an algebraic structure that differentiates them from the typical knot diagram. The algebraic coloring of the knot that will be considered for the remainder of the paper is known as a *bikei*.

Definition 2. A bikei is a knot coloring that divides the knot at both the under- and over-crossings. At each *crossing point*, the *arcs* are split into *semiarcs*. When examining a knot diagram the semiarcs can be considered as the sections of the knot between where the flattened knot passes over or under itseld. [8].

Example 3. Every semiarc in the bikei construction is given a labeling.



The labeling is constructed by starting from the bottom left of the diagram and labeling the first arc y. Then, moving clockwise around the knot the next labeling is x because it represents a different arc. Proceeding clockwise, the next labeling occurs on the y arc so it is given the name $y \,\overline{*}\, x$, to signify that the y strand passes over the x strand. The final label occurs on the x strand, and is given the semiarc labeling $x \,\underline{*}\, y$ to demonstrate that the x arc passes under the y arc.

3.2 Bikei Axioms

Labeling the three distinct Reidemeister Diagrams with the bikei coloring gives a unique set of rules and axioms pertaining to their construction. The Reidemeister I move implies that there be *equal self-actions*, ie. $x \,\overline{*}\, x = x \,\underline{*}\, x$. The Reidemeister II move implies that rotating the diagram does not change the labeling of the over- and under-crossings. The Reidemeister III move implies that there are three separate conditions, or *exchange laws* that can be satisfied.

appealing to the definition provided in [18, 8] the various axioms can be easily verified.

Definition 3. A *bikei* is a set X with two binary operations $\underline{*}, \overline{*} : X \times X \to X$ such that for all $x, y, z \in X$

(i) $x \,\underline{*}\, x = x \,\overline{*}\, x$,

(ii)

$$\begin{aligned} (x \,\overline{*}\, y) \,\overline{*}\, y &= x && (ii.i) \\ (x \,\underline{*}\, y) \,\underline{*}\, y &= x && (ii.ii) \\ x \,\underline{*}\, (y \,\overline{*}\, x) &= x \,\underline{*}\, y && (ii.iii) \\ x \,\overline{*}\, (y \,\underline{*}\, x) &= x \,\overline{*}\, y && (ii.iv), \end{aligned}$$

(iii)

$$\begin{aligned} (x \,\overline{*}\, y) \,\overline{*}\, (x \,\overline{*}\, y) &= (x \,\overline{*}\, z) \,\overline{*}\, (y \,\underline{*}\, z) && (iii.i) \\ (x \,\underline{*}\, y) \,\overline{*}\, (x \,\underline{*}\, y) &= (x \,\overline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z) && (iii.ii) \\ (x \,\underline{*}\, y) \,\underline{*}\, (z \,\underline{*}\, y) &= (x \,\underline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z) && (iii.iii). \end{aligned}$$

Example 4. Let X be a set and $\sigma : X \to X$ represent any involution, i.e., any map such that $\sigma^2 = \mathrm{Id}_X$. Then X is a bikei with operations

$$x \,\underline{*}\, y = \sigma(x) = x \,\overline{*}\, y$$

known as a *constant action bikei.*

The proof for the constant action bikei can be completed using the three bikei axioms.

Proof. (i)

$$x \,\underline{*}\, x = \sigma(x) = x \,\overline{*}\, x$$

(ii)

$$\begin{aligned} x \,\underline{*}\, (y \,\overline{*}\, x) = \sigma(x) &= x \,\underline{*}\, y, \\ x \,\overline{*}\, (y \,\underline{*}\, x) = \sigma(x) &= x \,\overline{*}\, y, \\ (x \,\underline{*}\, y) \,\underline{*}\, y = \sigma^2(x) &= x, \\ (x \,\overline{*}\, y) \,\overline{*}\, y = \sigma^2(x) &= x, \end{aligned}$$

and

(iii)

$$\begin{aligned}(x \,\overline{*}\, y) \,\overline{*}\, (z \,\underline{*}\, y) = \sigma^2(x) = (x \,\overline{*}\, z) \,\overline{*}\, (y \,\overline{*}\, z),\\(x \,\overline{*}\, y) \,\overline{*}\, (z \,\underline{*}\, y) = \sigma^2(x) = (x \,\overline{*}\, z) \,\overline{*}\, (y \,\overline{*}\, z),\\(x \,\underline{*}\, y) \,\underline{*}\, (z \,\overline{*}\, y) = \sigma^2(x) = (x \,\underline{*}\, z) \,\underline{*}\, (y \,\underline{*}\, z),\end{aligned}$$

□

[8].

3.3 The Alexander Bikei

A nice extension from traditional knot theory into the study of bikei is the relationship between the Alexander polynomial of a knot and the Alexander bikei.

Example 5. Let $\Lambda = \mathbb{Z}[t,s]/(t^2, s^2, (t-1)(s-1))$ be the quotient of the ring of two-variable polynomials with integer coefficients such that $s^2 = t^2 = 1$ by the ideal generated by $(1-t)(1-s)$. Then any Λ-module X is a bikei with operations

$$x \,\underline{*}\, y = tx + (s-t)y, \quad x \,\overline{*}\, y = sx$$

known as an *Alexander bikei.* Verifying the different Bikei axioms:

(i) $x \,\underline{*}\, x = tx + (s-t)x = sx = x \,\overline{*}\, x$,

(ii)

$$\begin{aligned}(x \,\overline{*}\, y) \,\overline{*}\, y &= s^2 x\\&= x,\\(x \,\underline{*}\, y) \,\underline{*}\, y &= t(tx + (s-t)y) + (s-t)y\\&= t^2 x + (ts - t^2 + s - t)y\\&= x + (1-t)(1-s)y\\&= x,\\x \,\underline{*}\, (y \,\overline{*}\, x) &= tx + (s-t)(sy)\\&= tx + (s^2 - st)y\\&= tx + (1 - st)y\\&= tx + (1 - t^2 + s - t)y\\&= tx + (s-t)y\\&= x \,\underline{*}\, y \quad \text{and}\\x \,\overline{*}\, (y \,\underline{*}\, x) &= sx\\&= x \,\overline{*}\, y,\end{aligned}$$

and

(iii)

$$\begin{aligned}
(x \,\overline{*}\, y) \,\overline{*}\, (x \,\overline{*}\, y) &= s(sx) \\
&= (x \,\overline{*}\, z) \,\overline{*}\, (y \,\underline{*}\, z), \\
(x \,\underline{*}\, y) \,\overline{*}\, (x \,\underline{*}\, y) &= s(tx + (s-t)y) \\
&= t(sx) + (s-t)(sy) \\
&= (x \,\overline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z) \\
(x \,\underline{*}\, y) \,\underline{*}\, (z \,\underline{*}\, y) &= t(tx + (s-t)y) + (s-t)(tz + (s-t)y) \\
&= t^2x + t(s-t)y + t(s-t)z + (s-t)^2y \\
&= t^2x + t(s-t)z + (s-t)(t+s-t)y \\
&= t(tx + (s-t)z) + (s-t)(sy) \\
&= (x \,\underline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z)
\end{aligned}$$

[18]

Definition 4. A map $f : X \to Y$ between bikei is a *bikei homomorphism* if

$$f(x \,\underline{*}\, y) = f(x) \,\underline{*}\, f(y) \quad \text{and} \quad f(x \,\overline{*}\, y) = f(x) \,\overline{*}\, f(y)$$

for all $x, y \in X$. A bijective bikei homomorphism is a *bikei isomorphism* [18].

Example 6. Let $X = \{x_1, \ldots, x_n\}$ be a finite set. Any bikei structure on X with an $n \times 2n$ block matrix M, can be encoded using the two different bikei operations, $\underline{*}$ and $\overline{*}$ and setting $M_{j,k} = l$ and $M_{j,k+n} = m$ where $x_j \,\underline{*}\, x_k = x_l$ and $x_j \,\overline{*}\, x_k = x_m$ for $j, k \in \{1, \ldots, n\}$. For example, there are two nonisomorphic bikei on the set $X = \{x_1, x_2\}$, given by the matrices

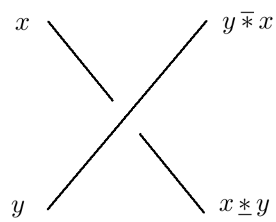
$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|cc} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right].$$

Example 7. Let D be an unoriented knot or link diagram representing an unoriented knot or link K and let G be a set of generators corresponding to semiarcs in D. The set W of *bikei words* in G is defined recursively by the rules

(i) $x \in G \Rightarrow x \in W$ and

(ii) $x, y \in W \Rightarrow x \,\underline{*}\, y, x \,\overline{*}\, y \in W$.

Then the *fundamental bikei* of D, denoted $\mathcal{BK}(D)$, is the set of equivalence classes of W under the equivalence relation generated by the bikei axioms and the *crossing relations* in D, i.e.



Such a bikei is expressed with a *bikei presentation*, i.e. an expression of the form

$$\mathcal{BK}(D) = \langle g_1, \dots, g_n \mid r_1, \dots, r_n \rangle$$

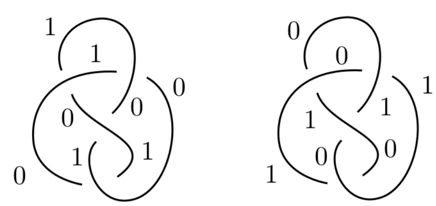
where $\{g_1, \dots, g_n\}$ are generators and $\{r_1, \dots, r_n\}$ are crossing relations, with the bikei axiom relations understood. It is easy to check that Reidemeister moves on D induce Tietze moves on presentations, and hence the isomorphism type of the fundamental bikei is an invariant of unoriented knots and links; hence, will generally be written $\mathcal{BK}(K)$ instead of $\mathcal{BK}(D)$.

Definition 5. Given an unoriented knot or link K represented by a diagram D and a finite bikei X, the *bikei counting invariant* $\Phi_X^{\mathbb{Z}}(K)$ is the cardinality of the set of bikei homomorphisms $f : \mathcal{BK}(K) \to X$, i.e.

$$\Phi_X^{\mathbb{Z}}(K) = |\mathrm{Hom}(\mathcal{BK}(K), X)|.$$

Every such homomorphism assigns an element of X to each generator of $\mathcal{BK}(K)$, which can be thought of as coloring the corresponding semiarc in D. Conversely, an assignment of elements of X to the semiarcs in D determines a bikei homomorphism $f : \mathcal{BK}(K) \to X$ only if it satisfies the crossing relations at every crossing. Hence, the bikei counting invariant of an unoriented knot or link can be computed by counting bikei colorings of any diagram of D which satisfy the crossing relations.

Example 8. Consider the bikei $X = \mathbb{Z}_2 = \{0, 1\}$ with $x \underline{*} y = x \overline{*} y = x + 1$. As a coloring rule, this says that each time going through a crossing either over or under, it is either switched from 0 to 1 or 1 to 0. Then for any knot, there are exactly two X-colorings, determined by our choice of starting color on a choice of semiarc.



The next section will enhance the bikei counting invariant with cocycles in a bikei homology theory to get a stronger invariant following [5, 4, 18] etc, but using bikei and unoriented diagrams.

Chapter 4

Bikei Cohomology

4.1 Homology and Cohomology

The purpose of homology and cohomology theory in mathematics is to provide algebraic solutions to geometric of topological questions [8]. The remainder of this paper will be devoted to summarizing the results presented in [18], where a modified homology and cohomology theory for involutory biquandles, also known as bikei, was introduced. The paper presents *bikei homology* and *cohomology* as as well as a bikei cocycle enhancement of the bikei counting invariant.

Definition 6. A *cell decomposition* of a subset $X \subset \mathbb{R}$ separates the structure, X, into cells of varying dimensions. Each cell in the geometric structure contains a *boundary* which is composed of lower dimensional cells.

Each cell can therefore be thought of as a linear combination of cells of one lower dimension. Thus, the entire set X can be described using a set of vector spaces that are generated by linear combinations of cells which encode the boundary maps of the figure. The big generalization of this rule is that the boundary map of a boundary map is empty, implying that the linear mapping of the composition of boundary transformations is the *zero map.*

Cohomology can also be found when generalizing the fundamental theorem of calculus to higher dimensions using *differential forms.*

Definition 7. A 0-form residing on a region in the xy plane is a scaler function of the type $f(x, y)$. A expression of the form $f(x, y)dx + g(x, y)dy$ is known as a differential 1-form. An expression of the type $F(x, y)dxdy$ is a differential 2-form. Therefore a k-form differential is a generalization of a differential to k-dimensions.

The d posses the property that $d(d\omega)$ is always zero. This property of differential gives rise to a branch of homology theorem known as *de Rahm cohomology* [8].

Given a field $\mathbb{F}$, let $C^0, C^1, C^2, \ldots C^n, \ldots$ be vector field spaces. Then, for each values of k, let $d^k : C^{k-1} \longrightarrow C^k$ be a linear transformation that can be illustrated through a matrix, A_k.

Definition 8. If for all values of $A_k A_{k-1}$ the product of the matrices is zero then the chain of vector spaces and linear transformations

$$\ldots \longleftarrow C^n \longleftarrow C^{n-1} \longleftarrow \ldots \longleftarrow C^2 \longleftarrow C^1 \longleftarrow C^0$$

is called a *cochain complex*.

It is important to recognize that for cohomology maps the arrows will always be pointing to the left. The cochain complex dictates that the column space of A_k is always a subspace of the null space of A_{k+1}. Therefore in regards to linear transformations it can be understood as $\mathrm{Im}(d^k) \subset \mathrm{Ker}(d^k + 1)$.

The vectors in C^k are knows as *k-cochains* and the various linear transformations d^k are known as the *coboundary maps* or the *differentials*. The column space in the matrix A_k is denoted B^k and the vectors in the column space are known as the *k-coboundaries*. Furthermore, the null space or the kernel of A_{k+1} is denoted Z^k, where all elements in A_{k+1} are known as *k-cocycles*. Therefore, for all chain complexes, it must be true that $B^k \subset Z^K$, and the quotient vector space

$$H^K = Z^k/B^k = Ker(d^{k+1})/Im(d^k)$$

is known as the k-th *cohomology space* of the chain complex [8].

Example 9. The sequence below illustrates a nontrivial example of cohomology.

$$0 \longleftarrow \mathbb{Z} \longleftarrow \mathbb{Z}^2 \longleftarrow \mathbb{Z}^2 \longleftarrow 0.$$

In the chain above, $C^0 = 0$, $C^1 = \mathbb{Z}^2$, $C^2 = \mathbb{Z}^2$, $C^3 = \mathbb{Z}$, and $C^4 = 0$. Additionally, $d^1 = 0$, d^2 is left multiplication by $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$, $d^3 = 0$ and $d^4 = 0$. It must be that, $d^1 \circ d^2{=}0$, $d^2 \circ d^3{=}0$, and finally $d^3 \circ d^4 = 0$. Therefore the sequence above forms a cochain complex. Therefore, the cocycles and coboundaries can now be determined. Starting with cocycles, it is evident that $Z^3 = \mathbb{Z}$, $Z^2 = \mathbb{Z}^2$. Finally, Z^1 can be determined by finding the null space of matrix A by row and column reduction. The domain is the integers so the resulting matrix must be placed in Smith normal form. Making use of row and column moves the matrix A becomes

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The resulting matrix is in Smith normal form. The kernel of Z^1 is isomorphic to $\mathbb{Z}_1 \oplus \mathbb{Z}_2 = \mathbb{Z}_2$. Transforming into coboundaries, it can be written $B^1 = 0$, $B^2 = \mathbb{Z}^2$, and $B^3 = 0$. Therefore, the cohomology groups, $H^1 = Z^1/B^1 = \mathbb{Z}^2$, $H^2 = \mathbb{Z}^2/\mathbb{Z}^2 = 0$, and the final cohomology group H^3 is $\mathbb{Z}/0 = \mathbb{Z}$.

Rather than completely redefining a new set of rules in order to define the homology space of a chain complex, simply changing to the cohomology map can give the desired result. If the k-th indicies are going down instead of up with respect to d then the *chain complex* has *homology spaces* rather than cohomology spaces. Therefore, when the coboundary maps are represented in matrix form, simply taking the transpose of each map will reverse the direction of the mapping, and change from cohomology to homology.

Definition 9. Let X be a bikei and set $C_n(X;A) = A[X^n]$ for an abelian group A. The *birack boundary map* $\partial_n : C_n(X;A) \to C_{n-1}(X;A)$ is defined on generators $\vec{x} = (x_1, \ldots, x_n)$ by

$$\partial(\vec{x}) = \sum_{k=1}^{n} (-1)^{k-1} \left(\partial_k^0(\vec{x}) - \partial_k^1(\vec{x})\right)$$

where

$$\begin{aligned} \partial_k^1(x_1, \ldots, x_n) &= (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \text{ and} \\ \partial_k^2(x_1, \ldots, x_n) &= (x_1 \,\underline{*}\, x_k, \ldots, x_{k-1} \,\underline{*}\, x_k, x_{k+1} \,\overline{*}\, x_k, \ldots, x_n \,\overline{*}\, x_k) \end{aligned}$$

and extended to $C_n(X;A)$ by linearity. The resulting homology is group of X in A is defined $H_n(X;A) = \text{Ker } \partial_n/\text{Im } \partial_{n-1}$ and the cohomology group of X in Z is defined $H^n = \text{Ker } \delta^{n+1}/\text{Im } \delta^n$ where

In the works of [5, 4, 6, 18], the subset $C_n^D(X;A)$ of $C_n(X;A)$ generated by elements $(x_1, \ldots, x_n)$ with $x_j = x_{j+1}$ for some $j = 1, \ldots, n-1$ was identified to be the *degenerate subcomplex*. The degenerate subcomplex allowed the results to be generalized to form the *biquandle homology and cohomology* groups, known as the *Yang-Baxter homology and cohomology* groups, to be defined as the cohomology groups of the quotient complex $C_n^B(X;A) = C_n(X;A)/C_n^D(X;A)$. In [18] the Yang-Baxter homology and cohomology groups are given a slight generalization to provide for the construction of the Bikei homology and cohomology groups.

Definition 10. Let X be a bikei. The *bikei degenerate* subgroup of $C_n(X;A)$, denoted $C_n^{BD}(X;A)$, is generated by chains of the forms

(0) $(x) - (x \,\underline{*}\, y)$ and $(x) - (x \,\overline{*}\, y)$,

(i) $(\ldots, x, x, \ldots)$,

(ii) $(x, \ldots, x, y, \ldots, y) - (x \,\underline{*}\, y, \ldots, x \,\underline{*}\, y, y \,\overline{*}\, x, \ldots, y \,\overline{*}\, x)$,

(iii) $(x,\dots,x,y,\dots,y)+(x,\dots,x,y\,\overline{*}\,x,\dots,y\,\overline{*}\,x)$, and

(iv) $(x,\dots,x,y,\dots,y)+(x\,\underline{*}\,y,\dots,x\,\underline{*}\,y,y,\dots,y)$.

For the (i) entry in the list the dots can represent arbitrary entries; however, in the (ii)-(iv) items the first k entries are equivalent and the last $n-k$ entries are equivalent.

Proposition 1. *For a bikei X, (C_n^{BD},∂_n) forms a subcomplex.*

Proof. The fact that $\partial_n((\dots,x,x,\dots))\in C_{n-1}^{BD}(X;A)$ is standard but easy to check: suppose the repeated entries x,x are in positions k and $k+1$. Then for $j<k$ and $j>k+1$, the terms in the boundary have repeated entries of the form x,x and $x\,\overline{*}\,x_j, x\,\overline{*}\,x_j$ and thus belong to $C_{n-1}^{BD}(X;A)$, so only the $j=k$ and $j=k+1$ cases need to be checked. Therefore,

$$\begin{aligned}
\partial_k^1((x_1,\dots,x,x,\dots,x_n)) &= (x_1,\dots,x_{k-1},x,x_{k+2},\dots,x_n)\\
\partial_{k+1}^1((x_1,\dots,x,x,\dots,x_n)) &= (x_1,\dots,x_{k-1},x,x_{k+2},\dots,x_n)\\
\partial_k^2((x_1,\dots,x,x,\dots,x_n)) &= (x_1\,\underline{*}\,x,\dots,x_{k-1},x\,\overline{*}\,x,x_{k+2}\,\overline{*}\,x,\dots,x_n\,\overline{*}\,x)\\
\partial_{k+1}^2((x_1,\dots,x,x,\dots,x_n)) &= (x_1\,\underline{*}\,x,\dots,x_{k-1},x\,\underline{*}\,x,x_{k+2}\,\overline{*}\,x,\dots,x_n\,\overline{*}\,x)
\end{aligned}$$

and since $x\,\underline{*}\,x=x\,\overline{*}\,x$, it follows that

$$(\partial_k^1-\partial_k^2-\partial_{k+1}^1+\partial_{k+1}^2)((\dots,x,x,\dots))=0.$$

Now, consider generators of type (ii); if $n>3$, then the boundary is a difference of chains of type (i), so only the $n=3$ case needs to be checked. There are two possibilities which are not already covered by condition (i): $(x,x,y)-(x\,\underline{*}\,y,x\,\underline{*}\,y,y\,\overline{*}\,x)$ and $(x,y,y)-(x\,\underline{*}\,y,y\,\overline{*}\,x,y\,\overline{*}\,x)$.

$$\begin{aligned}
\partial((x,x,y)-(x\,\underline{*}\,y,x\,\underline{*}\,y,y\,\overline{*}\,x)) &= (x,y)-(x\,\underline{*}\,y,y\,\overline{*}\,x)-(x\,\overline{*}\,x,y\,\overline{*}\,x)+((x\,\underline{*}\,y)\,\overline{*}\,(x\,\underline{*}\,y),(y\,\overline{*}\,x)\,\overline{*}\,(x\,\underline{*}\,y))\\
&\quad -(x,y)+(x\,\underline{*}\,y,y\,\overline{*}\,x)+(x\,\underline{*}\,x,y\,\overline{*}\,x)-((x\,\underline{*}\,y)\,\underline{*}\,(x\,\underline{*}\,y),(y\,\overline{*}\,x)\,\overline{*}\,(x\,\underline{*}\,y))\\
&\quad +(x,x)-(x\,\underline{*}\,y,x\,\underline{*}\,y)-(x\,\underline{*}\,y,x\,\underline{*}\,y)+((x\,\underline{*}\,y)\,\underline{*}\,(y\,\overline{*}\,x),(x\,\underline{*}\,y)\,\underline{*}\,(y\,\overline{*}\,x))\\
&= (x,x)-2(x\,\underline{*}\,y,x\,\underline{*}\,y)+((x\,\underline{*}\,y)\,\underline{*}\,(y\,\overline{*}\,x),(x\,\underline{*}\,y)\,\underline{*}\,(y\,\overline{*}\,x))\in C_2^{BD}(X)
\end{aligned}$$

and

$$\begin{aligned}
\partial((x,y,y)-(x\,\underline{*}\,y,y\,\overline{*}\,x,y\,\overline{*}\,x)) &= (y,y)-(y\,\overline{*}\,x,y\,\overline{*}\,x)-(y\,\overline{*}\,x,y\,\overline{*}\,x)+((y\,\overline{*}\,x)\,\overline{*}\,(x\,\underline{*}\,y),(y\,\overline{*}\,x)\,\overline{*}\,(x\,\underline{*}\,y))\\
&\quad -(x,y)+(x\,\underline{*}\,y,y\,\overline{*}\,y)+(x\,\underline{*}\,y,y\,\overline{*}\,x)-((x\,\underline{*}\,y)\,\underline{*}\,(y\,\overline{*}\,x),(y\,\overline{*}\,x)\,\overline{*}\,(y\,\overline{*}\,x))\\
&\quad +(x,y)-(x\,\underline{*}\,y,y\,\underline{*}\,y)-(x\,\underline{*}\,y,y\,\overline{*}\,x)+((x\,\underline{*}\,y)\,\underline{*}\,(y\,\overline{*}\,x),(y\,\overline{*}\,x)\,\underline{*}\,(y\,\overline{*}\,x))\\
&= (y,y)-2(y\,\overline{*}\,x,y\,\overline{*}\,x)+((y\,\overline{*}\,x)\,\overline{*}\,(x\,\underline{*}\,y),(y\,\overline{*}\,x)\,\overline{*}\,(x\,\underline{*}\,y))\in C_2^{BD}(X).
\end{aligned}$$

Next, consider generators of type (iii). There are two possibilities: $(x,x,y)+(x,x,y\,\overline{*}\,x)$ and $(x,y,y)+(x,y\,\overline{*}\,x,y\,\overline{*}\,x)$.

$$\begin{aligned}
\partial((x,x,y)+(x,x,y\,\overline{*}\,x)) &= (x,y)+(x,y\,\overline{*}\,x)-(x\,\overline{*}\,x,y\,\overline{*}\,x)-(x\,\overline{*}\,x,(y\,\overline{*}\,x)\,\overline{*}\,x)\\
&\quad -(x,y)-(x,y\,\overline{*}\,x)+(x\,\underline{*}\,x,y\,\overline{*}\,x)+(x\,\underline{*}\,x,(y\,\overline{*}\,x)\,\overline{*}\,x)\\
&\quad +(x,x)+(x,x)-(x\,\underline{*}\,y,x\,\underline{*}\,y)-(x\,\underline{*}\,(y\,\overline{*}\,x),x\,\overline{*}\,(y\,\overline{*}\,x)\\
&= 2(x,x)-2(x\,\underline{*}\,y,x\,\underline{*}\,y)\in C_2^{BD}(X)
\end{aligned}$$

and

$$\begin{aligned}
\partial((x,y,y)+(x,y\,\overline{*}\,x,y\,\overline{*}\,x)) &= (y,y)+(y\,\overline{*}\,x,y\,\overline{*}\,x)-(y\,\overline{*}\,x,y\,\overline{*}\,x)-((y\,\overline{*}\,x)\,\overline{*}\,x),(y\,\overline{*}\,x)\,\overline{*}\,x)\\
&\quad -(x,y)-(x,y\,\overline{*}\,x)+(x\,\underline{*}\,y,y\,\overline{*}\,y)+((x\,\underline{*}\,y)\,\overline{*}\,x),(y\,\overline{*}\,x)\,\overline{*}\,(y\,\overline{*}\,x))\\
&\quad +(x,y)+(x,y\,\overline{*}\,x)-(x\,\underline{*}\,y,y\,\underline{*}\,y)-(x\,\underline{*}\,(y\,\overline{*}\,x),(y\,\overline{*}\,x)\,\underline{*}\,(y\,\overline{*}\,x))\\
&= 0.
\end{aligned}$$

Next, consider the generators of type (iv). There are two possibilities: $(x,x,y)+(x\,\underline{*}\,y,x\,\underline{*}\,y,y)$ and $(x,y,y)+(x\,\underline{*}\,y,y,y)$.

$$\begin{aligned}
\partial((x,x,y)+(x\,\underline{*}\,y,x\,\underline{*}\,y,y)) &= (x,y)+(x\,\underline{*}\,y,y)-(x\,\overline{*}\,x,y\,\overline{*}\,x)-((x\,\underline{*}\,y)\,\overline{*}\,(x\,\underline{*}\,y),y\,\overline{*}\,(x\,\underline{*}\,y))\\
&\quad -(x,y)-(x\,\underline{*}\,y,y)+(x\,\underline{*}\,x,y\,\overline{*}\,x)+((x\,\underline{*}\,y)\,\underline{*}\,y,(x\,\underline{*}\,y)\,\underline{*}\,y))\\
&\quad +(x,x)+(x\,\underline{*}\,y,x\,\underline{*}\,y)-(x\,\underline{*}\,y,x\,\underline{*}\,y)-((x\,\underline{*}\,y)\,\underline{*}\,y,(x\,\underline{*}\,y)\,\underline{*}\,y)\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\partial((x,y,y)+(x\,\underline{*}\,y,y,y)) &= (y,y)+(y,y)-(y\,\overline{*}\,x,y\,\overline{*}\,x)-(y\,\overline{*}\,(x\,\underline{*}\,y),y\,\overline{*}\,(x\,\underline{*}\,y)\\
&\quad -(x,y)-(x\,\underline{*}\,y,y)+(x\,\underline{*}\,y,y\,\overline{*}\,y)+((x\,\underline{*}\,y)\,\underline{*}\,y,y\,\overline{*}\,y)\\
&\quad +(x,y)+(x\,\underline{*}\,y,y)-(x\,\underline{*}\,y,y\,\underline{*}\,y)-((x\,\underline{*}\,y)\,\underline{*}\,y),y\,\underline{*}\,y)\\
&= 2(y,y)-2(y\,\overline{*}\,x,y\,\overline{*}\,x)\in C_2^{BD}(X).
\end{aligned}$$

Finally, when $n=2$ the degenerate chains (x,x), $(x,y)+(x\,\underline{*}\,y,y)$ and $(x,y)+(x,y\,\overline{*}\,x)$ have boundary

$$\partial((x,x))=(x)-(x\,\overline{*}\,x)-(x)+(x\,\underline{*}\,x)=0,$$

$$\begin{aligned}
\partial((x,y)+(x\,\underline{*}\,y,y)) &= [(y)-(y\,\overline{*}\,x)-(x)+(x\,\underline{*}\,y)]\\
&\quad +[(y)-(y\,\overline{*}\,(x\,\underline{*}\,y))-(x\,\underline{*}\,y)+((x\,\underline{*}\,y)\,\underline{*}\,y]\\
&= [(y)-(y\,\overline{*}\,x)-(x)+(x\,\underline{*}\,y)]+[(y\,\overline{*}\,x)-(y)-(x\,\underline{*}\,y)+(x)]\\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}\partial((x,y)+(x,y\,\overline{*}\,x)) &= [(y)-(y\,\overline{*}\,x)-(x)+(x\,\underline{*}\,y)] \\ &\quad +[(y\,\overline{*}\,x)-((y\,\overline{*}\,x)\,\overline{*}\,x)-(x)+(x\,\underline{*}\,(y\,\overline{*}\,x))] \\ &= [(y)-(y\,\overline{*}\,x)-(x)+(x\,\underline{*}\,y)]+[(y\,\overline{*}\,x)-(y)-(x)+(x)] \\ &= 0\end{aligned}$$

while the chains $(x,y)-(x\,\underline{*}\,y, y\,\overline{*}\,x)$ have boundary

$$\begin{aligned}\partial((x,y)-(x\,\underline{*}\,y,y\,\overline{*}\,x)) &= [(y)-(y\,\overline{*}\,x)-(x)+(x\,\underline{*}\,y)] \\ &\quad -[(y\,\overline{*}\,x)-((y\,\overline{*}\,x)\,\overline{*}\,(x\,\underline{*}\,y))-(x\,\underline{*}\,y)+((x\,\underline{*}\,y)\,\underline{*}\,(y\,\overline{*}\,x)] \\ &= [(y)-(y\,\overline{*}\,x)-(x)+(x\,\underline{*}\,y)]-[(y\,\overline{*}\,x)-(y)-(x\,\underline{*}\,y)+(x)] \\ &= 2[(y)-(y\,\overline{*}\,x)]-2[(x)-(x\,\underline{*}\,y)]\in C_1^{BD}(X;A).\end{aligned}$$

Hence, $\partial_n(C_n^{BD}(X;A))\subset C_{n-1}^{BD}(X;A)$ and $(\partial, C_n^{BD}(X;A))$ is a subcomplex. □

Definition 11. Let X be a bikei. Then for each $n\geq 2$, set $C_n^{BK}(X;A)=C_n(X;A)/C_n^{BD}(X;A)$. The resulting homology and cohomology groups $H_n^{BK}(X;A)$ and $H^n_{BK}(X;A)$ are the *bikei homology and cohomology* groups of X.

Since $C_n^{BD}(X;A)=C_n^{D}(X;A)$ for $n\geq 3$, it must be true:

Theorem 2. *For $n>3$, the biquandle and bikei homology and cohomology groups for a bikei X coincide, i.e. $H_n^B(X;A)=H_n^{BK}(X;A)$ and $H^n_B(X;A)=H^n_{BK}(X;A)$.*

Proposition 3. *If X is a bikei in which $x\,\overline{*}\,y=x$ for all x,y (that is, a kei) or a bikei in which $x\,\underline{*}\,y=x$ for all x,y, then $H^2_{BK}(X;\mathbb{F})=\{0\}$ for any field $\mathbb{F}$.*

Proof. In either of the listed cases, the degenerate group includes all cochains of the form $\phi(x,y)+\phi(x,y)=2\phi(x,y)$, so it must be true that $2\phi(x,y)=0$. Then if our coefficients belong to a field, it must be that $\phi(x,y)=0$ for all $x,y\in X$. □

Further calculations lead [18] to propose the following conjecture:

Conjecture 1. *The free part of $H^2_{BK}(X;\mathbb{Z})$ is $\{0\}$ for all finite bikei X.*

At first, it may seem like bikei homology is completely trivial, but it turns to have nontrivial torsion part in at least some cases.

Proposition 4. *Let X be an Alexander bikei structure on an abelian group A, i.e., a choice of units $t, s \in A$ such that $(1-s)(1-t) = 0$ defining operations*

$$x \,\underline{*}\, y = tx + (s-t)y \quad \text{and} \quad x \,\overline{*}\, y = sx.$$

Then a linear map $\phi(x,y) = ax + by$ defines a bikei cocycle in $H^2_{BK}(X;A)$ if and only if

$$b = -a \quad \text{and} \quad 2a = a(1+s) = a(1+t) = a(1-t) = a(s-t-2) = 0.$$

Such a cocycle is called a *linear Mochizuki bikei cocycle* since it is similar to Mochizuki cocycles for Alexander quandles [16].

Proof. Begin by checking the bikei cocycle condition. First, $\phi(x,x) = ax + bx = (a+b)x = 0$ requires $b = -a$. Then setting $\phi(x,y) = a(x-y)$, the other degeneracy conditions yield

$$\begin{aligned}
\phi(x,y) - \phi(x \,\underline{*}\, y, y \,\overline{*}\, x) &= 0 \\
a(x-y) - a(tx + (s-t)y - sy) &= 0 \\
a(1-t)x - a(-1+s-t-s)y &= 0 \\
a(1-t)x + a(1+t)y &= 0
\end{aligned}$$

and it must be $a(1-t) = a(1+t) = 0$,

$$\begin{aligned}
\phi(x,y) + \phi(x \,\underline{*}\, y, y) &= 0 \\
a(x-y) + a(tx + (s-t)y - y) &= 0 \\
a(1+t)x + a(-1+s-t-1)y &= 0 \\
a(1+t)x - a(s-t-2)y &= 0
\end{aligned}$$

therefore it is necessary that $a(s-t-2) = 0$, and

$$\begin{aligned}
\phi(x,y) + \phi(x, y \,\overline{*}\, x) &= 0 \\
a(x-y) + a(x-sy) &= 0 \\
a(2)x + a(-1-s)y &= 0 \\
2ax - a(1+s)y &= 0
\end{aligned}$$

so it must be $2a = a(1+s) = 0$. Finally, checking the cocycle condition:

$$\begin{aligned}
\phi(x,y) + \phi(y,z) + \phi(x \,\underline{*}\, y, z \,\overline{*}\, y) &= \phi(x,z) + \phi(x \,\underline{*}\, z, y \,\underline{*}\, z) + \phi(y \,\overline{*}\, x, z \,\overline{*}\, x) \\
a(x-y) + a(y-z) + a(tx + (s-t)y - sz) &= a(x-z) + a(tx + (s-t)z - ty - (s-t)z) + a(sy - sz) \\
a(1-t)x + a(s-t)y - a(1+s)z &= a(1+t)x + a(s-t)y - a(1+s)z
\end{aligned}$$

so no further conditions are required. Hence the given list of conditions is necessary and sufficient. □

Example 10. Let $X = \mathbb{Z}_8$ and set $s = 3, t = 1$ and $a = 4$. Then, $(1 - s)(1 - t) = 2(0) = 0$, so X is a bikei with operations

$$x \,\underline{*}\, y = x + 2y \quad \text{and} \quad x \,\overline{*}\, y = 3y.$$

It can be quickly verified,

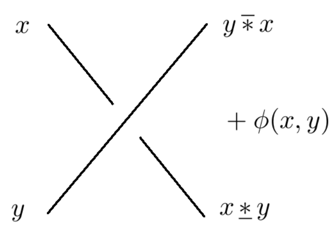
$$2(4) = 4(1 + 3) = 4(1 + 1) = 4(1 - 1) = 4(3 - 1 - 2) = 0$$

and $\phi(x, y) = 4x - 4y$ is a nonzero bikei cocycle.

4.2 Cocycle Enhancements

One motivation for bikei homology comes from the desire to extend cocycle enhancements of the bikei counting invariant to unoriented knots and links, and in particular to non-orientable knotted surfaces in $\mathbb{R}^4$.

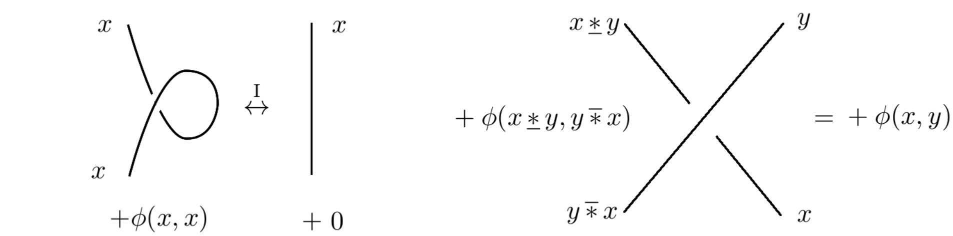
Let $\phi \in H^2_{BK}(X; A)$ and let D be an unoriented knot or link diagram representing a knot or link K. For any bikei homomorphism $f : \mathcal{BK}(K) \to X$, let D_f denoted the X-coloring of D determined by f. Then at each crossing a *Boltzmann weight* $\phi(x, y)$ can be assigned where x and y are the bikei colors on the under and over crossing semiarcs when the crossing is positioned as depicted, with the overstrand going from upper right to lower left.

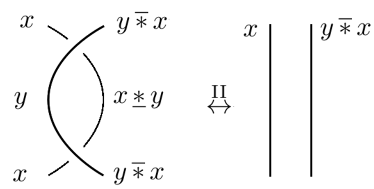


Then the Boltzmann weight for the bikei coloring f is the sum of the Boltzmann weights $\phi(x, y)$ at each crossing C in the set $\mathcal{C}(D_f)$ of crossings in diagram D_f,

$$BW(f) = \sum_{C \in \mathcal{C}(D_f)} \phi(x, y).$$

The bikei 2-cocycle conditions are precisely the conditions required to ensure that the Boltzmann weight is unchanged by Reidemeister moves. The degeneracy conditions insure that the Boltzmann weight is well-defined and give invariance under Reidemeister I and II moves:



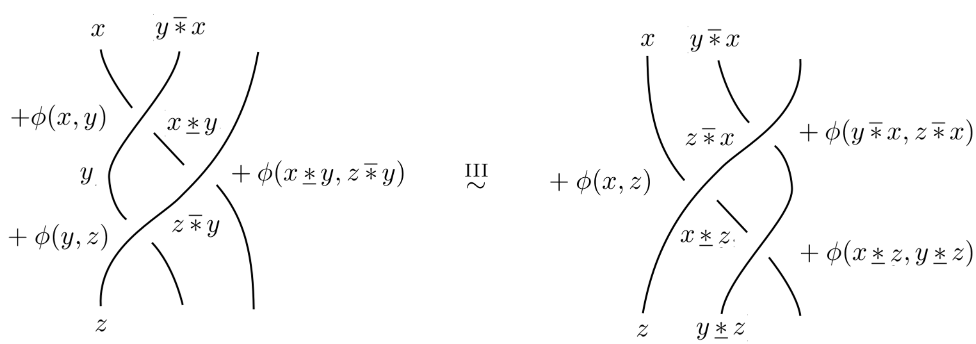


$$+\,\phi(x,y) + \phi(x, y \overline{*} x) \;=\; +\,0$$
$$+\,\phi(x,y) + \phi(x \underline{*} y, y) \;=\; +\,0$$

The 2-cocycle condition

$$\begin{aligned}
\delta^2(\phi(x,y,z)) &= \phi(\partial_2(x,y,z)) \\
&= \phi(-(y,z) + (x \underline{*} y, z \overline{*} y) + (x,z) - (y \overline{*} x, z \overline{*} x) - (x,y) + (x \underline{*} y, y \underline{*} z)) \\
&= 0
\end{aligned}$$

guarantees equivalence under Reidemeister III moves:



Thus,

Definition 12. Let X be a finite bikei and $\phi \in H^2_{BK}(X)$. Then for any unoriented knot or link K represented by a diagram D, the *2-cocycle enhanced bikei counting invariant* of K is the multiset

$$\Phi_X^{\phi,M}(D) = \{BW(D_f) \mid f \in \mathrm{Hom}(\mathcal{BK}(K), X)\}$$

or its generating function

$$\Phi_X^{\phi}(D) = \sum_{f \in \mathrm{Hom}(\mathcal{BK}(K),X)} u^{BW(D_f)}$$

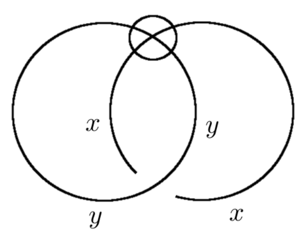
By construction (and also see [4] etc.),

Theorem 5. *If X is a finite bikei, $\phi \in H^2_{BK}(X)$ and D and D' are unoriented knot or link diagrams related by Reidemeister moves, then*

$$\Phi_X^{\phi,M}(D) = \Phi_X^{\phi,M}(D') \quad \text{and} \quad \Phi_X^{\phi}(D) = \Phi_X^{\phi}(D').$$

As with many knot invariants, the case of Φ_X^{ϕ} can be extended to the case of virtual knots and links by simply ignoring the virtual crossings.

Example 11. Let X be the Alexander bikei from example 10 and consider the virtual Hopf link VH below.



This gives a system of coloring equations

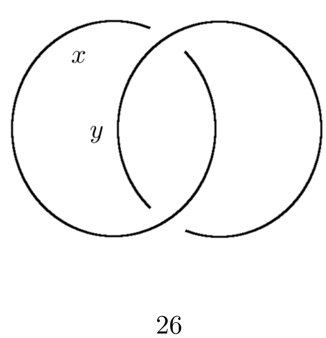
$$\begin{array}{rcl} x \underline{*} y & = & x \\ y \overline{*} x & = & y \end{array} \Rightarrow \begin{array}{rcl} x+2y & = & x \\ 3y & = & y \end{array} \Rightarrow 2y = 0$$

so a pair (x, y) yields a valid coloring for $y \in \{0, 4\}$ and no further conditions on $x \in \mathbb{Z}_8$, so there are 16 X-colorings. Each coloring has a Boltzmann weight of

$$\phi(x,y) = 4(x-y) = \begin{cases} 4 & x \text{ odd} \\ 0 & x \text{ even} \end{cases}$$

at the single crossing, so the invariant is $\Phi_X^{\phi}(VH) = 8 + 8u^4$.

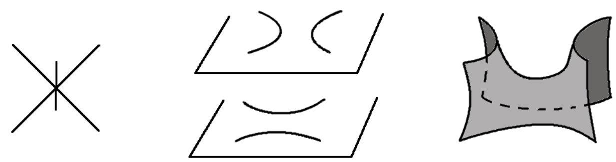
Comparing this with the case of the usual Hopf link



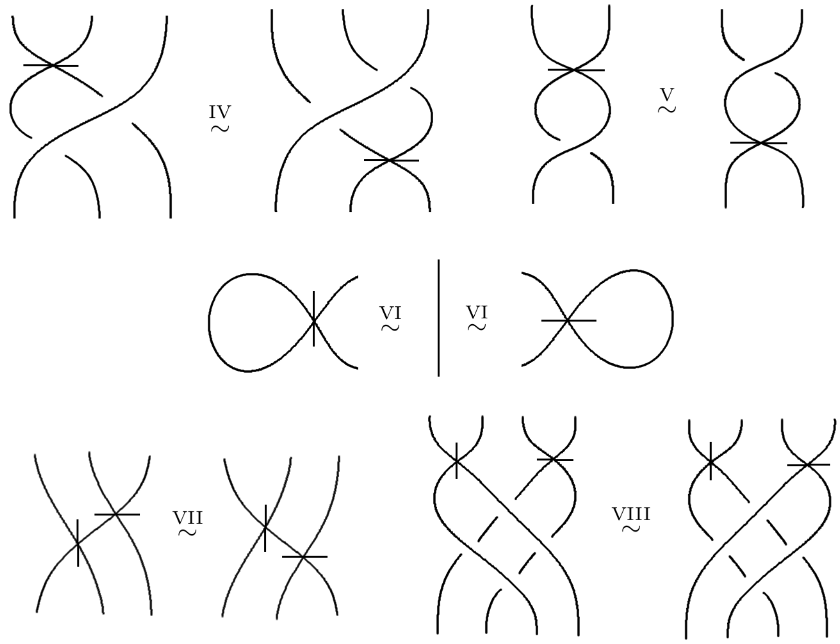
the coloring equations $3x = x + 2y$ and $3y = y + 2x$ can be obtained, both of which reduce to $2x = 2y$. Thus, there are sixteen colorings, and each has Boltzmann weight $\phi(x,y)+\phi(y,x) = 4(x-y)+4(y-x) = 0$, yielding an invariant value of $\Phi_X^\phi(H) = 16$. In particular, this demonstrates that Φ_X^ϕ is not determined by the counting invariant and hence is a proper enhancement.

4.3 Invariants of Knotted Surfaces

Abikei cocycle invariants for knotted unoriented (including non-orientable) surfaces in $\mathbb{R}^4$ can be defined in the same way. Recall that a *marked graph diagram*, also called a *marked vertex diagram*, is a diagram with ordinary crossings together with *saddle crossings*



representing saddle point. More precisely, given a knotted surface $\Sigma \subset \mathbb{R}^4$, first move the maxima in the x_4 direction to the hyperplane $x_4 = 1$, the minima to $x_4 = -1$ and the saddle points to $x_4 = 0$. Then the intersection of Σ with $x_4 = 0$ is a link diagram with singularities at the saddle points; the direction of the saddle is indicated with a small bar. Such a diagram represents a knots closed surface if both resolutions of the saddle yield unlinks; otherwise, the diagram represents a cobordism between the links represented by the smoothed diagrams. Two such diagrams represent ambient isotopic knotted surfaces if and only if they are related by a sequence of the Reidemeister moves together with the *Yoshikawa moves*



See for instance [12, 13] for more.

In [17], bikei colorings and counting invariants of marked graph diagrams were considered. Specifically, all of the semiarcs meeting at a saddle crossing determine the same generator of $\mathcal{BK}(\Sigma)$ and must have the same color. It is now observable that the bikei counting invariant can be enhanced with bikei 2-cocycles in the same way as knots and links in $\mathbb{R}^3$. Specifically,

Definition 13. Let X be a finite bikei and $\phi \in H^2_{BK}(X)$. Then for any unoriented knotted surface Σ represented by a marked vertex diagram D, the *2-cocycle enhanced bikei counting invariant* of Σ is the multiset

$$\Phi_X^{\phi,M}(D) = \{BW(D_f) \mid f \in \mathrm{Hom}(\mathcal{BK}(\Sigma), X)\}$$

or its generating function

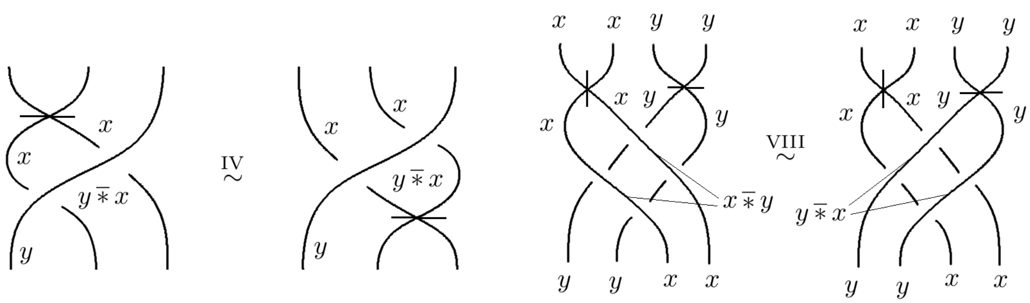
$$\Phi_X^{\phi}(D) = \sum_{f\in \mathrm{Hom}(\mathcal{BK}(\Sigma),X)} u^{BW(D_f)}$$

Therefore

Theorem 6. *If X is a finite bikei, $\phi \in H^2_{BK}(X)$ and D and D' are unoriented marked vertex diagrams related by Yoshikawa moves, then*

$$\Phi_X^{\phi,M}(D) = \Phi_X^{\phi,M}(D) \quad \text{and} \quad \Phi_X^{\phi}(D) = \Phi_X^{\phi}(D).$$

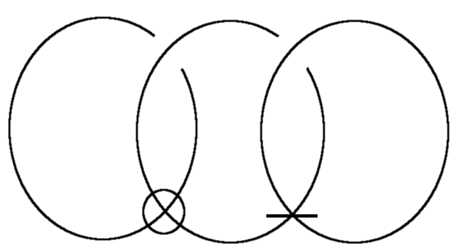
Proof. This is a matter of verifying that the Yoshikawa moves do not change the Boltzmann weight of a bikei colored marked vertex diagram. Moves VI and VII do not involve non-saddle crossings, so these cannot change the Boltzmann weight, and in move V all semiarc colors are the same, so both sides of the move contribute $\phi(x,x)=0$. For moves IV and VIII, both sides of the move contribute degenerate chains: $\phi(x,y)+\phi(x,y\,\overline{*}\,x)$ on both sides of move IV, and $2\phi(y,x)+2\phi(y,x\,\overline{*}\,y)$ on the left and $2\phi(x,y)+2\phi(x,y\,\overline{*}\,x)$ on the right of move VIII.



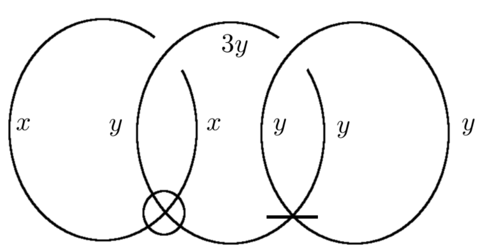
□

Analogously to the case of knotted and linked curves, including virtual crossings in marked vertex diagram yields virtual knotted surface diagrams, with the rule that two such diagrams are equivalent if related by Reidemeister moves, Yoshikawa moves and the detour move, i.e., redrawing an arc with only virtual crossings as another arc with only virtual crossings and the same endpoints.

Example 12. Let X again be the Alexander bikei from example 10 and consider the virtual marked vertex diagram D below, representing a virtual linked surface with one sphere component and one projective plane component.



X-labelings of D are given by pairs $(x,y)\in(\mathbb{Z}_8)^2$ satisfying $3y=y$, i.e. $2y=0$, with $x=x+2y$ imposing no further conditions on x:



Hence there are16 X-colorings similarly to example 11; each coloring has Boltzmann weight $4(x-y)+4(y-y)$, so colorings with odd x contribute u^4 while colorings with even x contribute 1 to the invariant, and again the result is $\Phi_X^{\phi}(K)=8+8u^4$.

Bibliography

[1] C. C. Adams. *The knot book*. W. H. Freeman and Company, New York, 1994. An elementary introduction to the mathematical theory of knots.

[2] S. Aksoy and S. Nelson. Bikei, involutory biracks and unoriented link invariants. *J. Knot Theory Ramifications*, 21(6):1250045, 13, 2012.

[3] G. Burde and H. Zieschang. *Knots*, volume 5 of Student Mathematical Society of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2003.

[4] J. S. Carter, M. Elhamdadi, and M. Saito. Homology theory for the set-theoretic Yang-Baxter equation and knot invariants from generalizations of quandles. *Fund. Math.*, 184:31–54, 2004.

[5] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, and M. Saito. Quandle cohomology and state-sum invariants of knotted curves and surfaces. *Trans. Amer. Math. Soc.*, 355(10):3947–3989, 2003.

[6] J. Ceniceros, M. Elhamdadi, M. Green, and S. Nelson. Augmented biracks and their homology. *Internat. J. Math.*, 25(9):1450087, 19, 2014.

[7] R. H. Crowell and R. H. Fox. *Introduction to knot theory*. Springer-Verlag, New York-Heidelberg, 1977. Reprint of the 1963 original, Graduate Texts in Mathematics, No. 57.

[8] M. Elhamdadi and S. Nelson. *Quandles—an introduction to the algebra of knots*, volume 74 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2015.

[9] R. Fenn and C. Rourke. Racks and links in codimension two. *J. Knot Theory Ramifications*, 1(4):343–406, 1992.

[10] R. Fenn, C. Rourke, and B. Sanderson. Trunks and classifying spaces. *Appl. Categ. Structures*, 3(4):321–356, 1995.

[11] D. Joyce. A classifying invariant of knots, the knot quandle. *J. Pure Appl. Algebra*, 23(1):37–65, 1982.

[12] J. Kim, Y. Joung, and S. Y. Lee. On the alexander biquandles of oriented surface-links via marked graph diagrams. *J. Knot Theory Ramifications*, 23(7):1460007, 26, 2014.

[13] J. Kim, Y. Joung, and S. Y. Lee. On generating sets of yoshikawa moves for marked graph diagrams of surface-links. *J. Knot Theory Ramifications*, 24(4):1550018, 21, 2015.

[14] W. B. R. Lickorish. *An introduction to knot theory*, volume 175 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.

[15] E. Long. Topological invariants of knots. 2005.

[16] T. Mochizuki. Some calculations of cohomology groups of finite Alexander quandles. *J. Pure Appl. Algebra*, 179(3):287–330, 2003.

[17] S. Nelson and P. Rivera. Bikei invariants and gauss diagrams for virtual knotted surfaces. *Preprint*.

[18] S. Nelson and J. Rosenfield. Bikei homology. *arXiv preprint arXiv:1603.02719*, 2016.

[19] M. Takasaki. Abstraction of symmetric transformations. *Tohoku Math. J*, 49(145-207):43, 1942.